DIHEDRAL GROUP

1. Dihedral Group D_n

Definition 1. *For each $n \geq 3$, a group G is called Dihedral group of order n if it is generated by two elements a, b such that $o(a) = n, o(b) = 2$ and $ba = a^{-1}b$.*

It can be shown that for each $n \geq 3$, D_n is a group of order $2n$ and is unique up to isomorphism. It can be thought as a group of rotations and reflections of a regular n-gons with n vertices where a is the rotation through angle $2\pi/n$ and b is the reflection. In this note, we will study the group D_4 and we will classify all non-abelian group of order 8.

2. The Group D_4.

The dihedral group of order 4 is the following group:

$$G =< a, b > \text{ with } o(a) = 4, o(b) = 2, \text{ and } ba = a^3 b$$

Example 1. *The subgroup of invertible 2×2 matrices over $\mathbb{R}$ generated by A= $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a D_4 group.*

Theorem 1 (Properties of D_4 .). *If G is a D_4 group then G is non-commutative group of order 8 where each element of D_4 is of the form $a^i b^j, 0 \leq i \leq 3, 0 \leq j \leq 1$.*

Proof. $G =< a, b >= \{a^{i_1} b^{j_1} a^{i_2} b^{i_2} ... a^{i_n} b^{i_n} : i_t, j_t \in \mathbb{Z}\}$. Since $ba = a^3 b$ hence elements of G are of the form $a^n b^m$ where $n, m \in \mathbb{Z}$. Since $a^4 = b^2 = e$ and $a^{-1} = a^3, b^{-1} = b$, every elements of G is of the form $a^i b^j, 0 \leq i \leq 3, 0 \leq j \leq 1$.(Hence G has at most 8 elements.)
Now since $o(a) = 4$, e, a, a^2, a^3 are all distinct and b, ba, ba^2, ba^3 are all distinct.Since $a \neq b \neq e, a^{-1} = a^3, b^{-1} = b$, we see that

$$\{e, a, a^2, a^3\} \cap \{b, ba, ba^2, ba^3\} = \emptyset$$

Hence $G = \{e, a, a^2, a^3, b, ba, ba^2, ba^3\}$ has exactly 8 elements.
Finally, suppose $ab = ba$ then $ab = a^3 b$ then $a^2 = e$ which is a contradiction. Hence G is noncommutative. □

It is now clear that the group D_4 is unique up to isomorphism. Now we consider all subgroups of D_4. By Lagrange's Theorem, its proper nontrivial subgroups can have order 2 or 4. In D_4 ,

$$o(a) = 4, o(a^2) = 2, o(a^3) = 4, o(b) = 2$$

$$(ab)^2 = abab = aa^3 bb = e.$$

$$(a^2 b)^2 = a^2 ba^2 b = a^2 (a^3 b) ab = abab = e.$$

$$(a^3 b)^2 = a^3 ba^3 b = a^3 (a^3 b) a^2 b = a^2 ba^2 b = e.$$

- $H_1 = \{e, a^2\}, H_2 = \{e, b\}, H_3 = \{e, ab\}, H_4 = \{e, a^2b\}, H_5 = \{e, a^3b\}$ are subgroups of order 2.
- $T_1 = \{e, a, a^2, a^3\}, T_2 = \{e, a^2, b, a^2b\}, T_3 = \{e, ab, a^2, a^3b\}$ are subgroups of order 4.

Exercise 1. *Prove that these subgroups together with $\{e\}, D_4$ are all subgroups of D_4. Draw the subgroup lattice of D_4. Prove that H_5 is a normal subgroup of T_3 and T_3 is a normal subgroup of d_4 but H_5 is not a normal subgroup of D_4.*

Exercise 2. *Prove that the center of $D_4, Z(D_4)$, is H_1.(Hint: the center of a group is a normal subgroup.)*

Exercise 3. *Prove that $D_4/Z(D_4)$ is isomorphic to the Klein 4 group.*

3. The Quaternion Group Q_8

Definition 2. *A group G is called quaternion group if*

$$G =< a, b >, o(a) = 4, a^2 = b^2, ba = a^3b$$

Example 2. *The subgroup of invertible 2×2 matrices over $\mathbb{C}$ generated by A= $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ is a quaternion group.*

Exercise 4. *Prove that a quaternion group is a noncommutative group of order 8 whose elements are of the form $a^ib^j, 0 \leq i \leq 3, 0 \leq j \leq 1$.*

Now it is easy to see that the quaternion group is unique up to isomorphism. We denote it by Q_8

$$Q_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$$

Exercise 5. *Verify that $o(a^2) = 2, o(a^3) = 4, o(ab) = o(a^2b) = o(a^3b) = 4$. Verify that $H_0 = \{e\}, H_1 = \{e, a^2\}, H_2 = \{e, a, a^2, a^3\}, H_3 = \{e, ab, a^2, a^3b\}, H_4 = \{e, b, a^2, a^2b\}, Q_8$ are all subgroups of Q_8. Verify that every proper subgroup of Q_8 is cyclic.*

Exercise 6. *By considering elements of order 4, explain why D_4 is not isomorphic to Q_8.*

Theorem 2. *There are only 2 noncommutative groups of order 8 :D_4 and Q_8.*

Proof. Let G be a noncommutative group of order 8. Since $|G|$ is even , there is an element $u \neq e$ such that $u = u^{-1}$ i.e. $u^2 = e$. If $x^2 = e$ for all $x \in G$ then G is commutative, contradiction. Let $a \in G$ be such that $a^2 \neq e$. Since $o(a)|8, o(a) = 4$ or 8. If $o(a) = 8$ then G is cyclic, contradiction. Hence $o(a) = 4$. Let $H = \{e, a, a^2, a^3\}$ then H is a normal subgroup of G (why?).Let $b \in G$ be such that $b \notin H$ then $G = H \cup Hb, H \cap Hb = \emptyset$ hence

$$G = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\} =< a, b >$$

Since H is normal, $bab^{-1} \in H$. If $bab^{-1} = e$ then $a = e$,contradiction. If $bab^{-1} = a$ then $ab = ba$ then G is commutative(why?), contradiction. If $bab^{-1} = a^2$ then $ba^2b^{-1} = (bab^{-1})^2 = a^4 = e$ then $a^2 = e$, contradiction. Hence $bab^{-1} = a^3$. and so $ba = a^3b$. Since $|G/H| = 2, b \notin H$ hence $o(Hb) = 2$ and $b^2 \in H$. If $b^2 = a$ or a^3 then b has order 8, which impies that G is commutative, contradiction. Hence $b^2 = e$(i.e. b has order 2, $G \cong D_4$) or $b^2 = a^2$(i.e. $G \cong Q_8$.) ∎