$$d\tau^2 = \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)dr^2 - r^2\left(d\theta^2 + \sin\theta\, d\phi^2\right)$$

$$u^\alpha = \begin{bmatrix} \frac{dt}{d\tau} \\ \frac{dr}{d\tau} \\ \frac{d\theta}{d\tau} \\ \frac{d\phi}{d\tau} \end{bmatrix} = \begin{bmatrix} u^t \\ 0 \\ 0 \\ \frac{d\phi}{dt}\frac{dt}{d\tau} \end{bmatrix} = \begin{bmatrix} u^t \\ 0 \\ 0 \\ \Omega u^t \end{bmatrix}$$

$$1 = \left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - r^2\left(\frac{d\phi}{d\tau}\right)^2$$

$$1 = \left(1-\frac{2M}{r}\right)(u^t)^2 - r^2\Omega^2(u^t)^2$$

$$\left[\left(1-\frac{2M}{r}\right) - r^2\Omega^2\right]^{-1} = (u^t)^2$$

$$\frac{du_\gamma}{d\tau} - \frac{1}{2}u^\alpha u^\beta g_{\alpha\beta,\gamma} = 0$$

Want $\frac{du_r}{d\tau} = 0$

$$u^t u^t g_{tt,r} + u^\phi u^\phi g_{\phi\phi,r} = 0$$

$$u^t u^t g_{tt,r} = -u^\phi u^\phi g_{\phi\phi,r}$$

$$\frac{u^\phi u^\phi}{u^t u^t} = -\frac{g_{tt,r}}{g_{\phi\phi,r}} \rightarrow \Omega^2 = -\frac{\frac{2M}{r^2}}{-2r} = \frac{M}{r^3} \quad \boxed{\Omega^2 r^3 = M}$$

$$u_t = g_{tt}u^t = \frac{1-\frac{2M}{r}}{\sqrt{\left(1-\frac{2M}{r}\right) - r^2\Omega^2}}$$

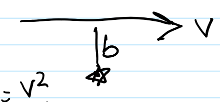
$$\approx \left[1-\frac{2M}{r}\right]\left[1+\frac{M}{r}+\frac{1}{2}r^2\Omega^2\right]$$

$$\approx 1-\frac{M}{r}+\frac{1}{2}r^2\Omega^2 = 1-\frac{M}{r}+\frac{1}{2}\frac{M}{r} = 1-\frac{1}{2}\frac{M}{r}$$

There is a problem!

* imagine material is moving relative to the star

* $l = vb$



$$a_{\text{centripetal}} = \frac{v^2}{r}$$

$$= \frac{(vb)^2}{r^3} = \frac{GM}{r^2}$$

$$r = \frac{(vb)^2}{GM} = 10^{-3}\,\text{AU}\left(\frac{v}{1\,\text{km/s}}\frac{b}{1\,\text{AU}}\right)\frac{M_\odot}{M}$$

Spherical flow $\longrightarrow$ disk accretion

Key quantity is angular momentum

$$u_\phi = g_{\phi\phi}u^\phi = g_{\phi\phi}\Omega u^t$$

$$= \Omega r^2\left(1-\frac{2M}{r} - r^2\Omega^2\right)^{-1/2}$$

$$\Omega^2 = \frac{M}{}$$

$$= \Omega r^2 \left(1 - \frac{2M}{r} - r^2\Omega^2 \right)$$

$$\Omega^2 = \frac{M}{r^3}$$

$$U_\phi = (Mr)^{1/2}\left(1 - \frac{2M}{r} - \frac{M}{r}\right)^{-1/2}$$

$$U_\phi = \left(\frac{Mr}{1 - \frac{3M}{r}}\right)^{\frac{1}{2}} \rightarrow \ell = (GMr)^{1/2}$$

angular momentum per mass

what happens at $r = 3M$?

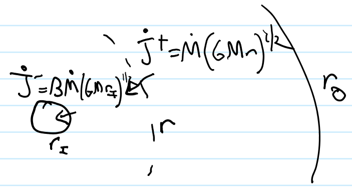
$$d\tau^2 = \left(1 - \frac{2}{3}\right)dt^2 - 9M^2 d\phi^2$$

$$\frac{d\tau^2}{dt^2} = \left(1 - \frac{2}{3}\right) - 9M^2 \frac{M}{(3M)^3}$$

$$= 1 - \frac{2}{3} - \frac{1}{3} = 0 \ !$$

$$\frac{u_\phi}{u_t} = \frac{g_{\phi\phi}\,\Omega\, u^t}{g_{tt}\, u^t} = \frac{r^2\Omega}{1 - \frac{2M}{r}}$$

Let's imagine a portion of the disk



torque = (force along $\hat\phi$ / area) $\times$ (area) $\times$ (r) = $\dot J^+ - \dot J^-$

viscous stress (units of pressure)

$$(f_\phi)(2\pi r\, 2h)(r) = \dot M\left[(GMr)^{1/2} - \beta(GMr_I)^{1/2}\right]$$

also

$$f_\phi = -\eta \frac{d\Omega}{d\ln r} = -\eta r \frac{d}{dr}\left(\sqrt{GM}\, r^{-3/2}\right) = \frac{3}{2}\eta\Omega$$

$$\eta = \frac{\dot M}{6\pi r^2 h \Omega}\left[(GMr)^{1/2} - \beta(GMr_I)^{1/2}\right]$$

$\hookrightarrow$ larger η smaller h

power per area

$$2hQ \approx 2h\frac{(f_\phi)^2}{\eta} = \frac{9}{2}\Omega^2 h\eta$$

$$2hQ = \frac{3\dot M}{4\pi r^2}\frac{GM}{r}\left[1 - \beta\left(\frac{r_I}{r}\right)^{1/2}\right] \quad \boxed{\text{Does not depend on } \eta}$$

Luminosity

$$L = \int_{r_I}^{\infty} 2hQ(2\pi r)\, dr = \left(\frac{3}{2} - \beta\right)\frac{GM\dot M}{r_I}$$

<u>Accretion</u>

<u>Spherical</u>:

$$\nabla\cdot(\rho v) + \frac{\partial\rho}{\partial t} = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho v) = 0$$

continuity

$$v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{GM}{r^2} = 0$$

$$\frac{\partial P}{\partial r} = \frac{\partial P}{\partial \rho}\frac{\partial \rho}{\partial r} = c_s^2\frac{\partial\rho}{\partial r}$$

$$\cancel{\frac{1}{r^2}}\left[\frac{\partial \rho}{\partial r}(r^2 v) + \rho\frac{\partial(r^2 v)}{\partial r}\right] = 0$$

$$\boxed{\frac{1}{\rho}\frac{\partial \rho}{\partial r}} + \frac{1}{r^2 v}\frac{\partial(r^2 v)}{\partial r} = 0$$

$$v\frac{\partial v}{\partial r} - \frac{c_s^2}{r^2 v}\frac{\partial}{\partial r}(r^2 v) + \frac{GM}{r^2} = 0$$

$$\frac{1}{2}\frac{\partial v^2}{\partial r} - \frac{c_s^2}{r^2 v}\left[2rv + r^2\frac{\partial v}{\partial r}\right] + \frac{GM}{r^2} = 0$$

$$\frac{1}{2}\frac{\partial v^2}{\partial r} - c_s^2\left[\frac{2}{r} - \frac{1}{2}\frac{1}{v^2}\frac{\partial v^2}{\partial r}\right] + \frac{GM}{r^2} = 0$$

$$\boxed{\frac{1}{2}\left(1 - \frac{c_s^2}{v^2}\right)\frac{\partial v^2}{\partial r} = -\frac{GM}{r^2}\left(1 - \frac{2c_s^2 r}{GM}\right)}$$

$$\frac{2c_s^2 r}{GM} = 1$$

$$r_c = \frac{GM}{2c_s^2(r_c)} \approx 7.5\times10^{13}\left(\frac{T}{10^4 K}\right)^{-1}\left(\frac{M}{M_\odot}\right) \text{ cm}$$

Draw the phase diagram

$$\frac{1}{\rho}\frac{\partial \rho}{\partial r} = \gamma\frac{K\rho^{\gamma-1}}{\rho}\frac{\partial \rho}{\partial r} = \gamma K\rho^{\gamma-2}\frac{\partial \rho}{\partial r}$$

$P = K\rho^\gamma$

$$= \frac{\partial}{\partial r}\left[\frac{\gamma K\rho^{\gamma-1}}{\gamma-1}\right] = \frac{\partial}{\partial r}\left[\frac{\gamma P}{\rho}\frac{1}{\gamma-1}\right]$$

$$= \frac{c_s^2}{\gamma-1}$$

Bernoulli

$$\frac{v^2}{2} + \frac{c_s^2}{\gamma-1} - \frac{GM}{r} = \text{constant}$$

$r \to \infty$, $v \to 0$

$$\frac{v^2}{2} + \frac{c_s^2 - c_s^2(\infty)}{\gamma-1} - \frac{GM}{r} = 0$$

At r_c $\quad v^2 = c_s^2 \quad \frac{GM}{r_c} = 2c_s^2$

$$\frac{c_s^2(r_c)}{2} + \frac{c_s^2(r_c) - c_s^2(\infty)}{\gamma-1} - 2c_s^2(r_c) = 0$$

$$c_s^2(r_c) = c_s^2(\infty)\left[\frac{2}{5-3\gamma}\right]$$

$$r_c = \frac{GM}{c_s^2(\infty)}\frac{5-3\gamma}{4}$$

$$\rho(r_c) = \rho(\infty)\left[\frac{2}{5-3\gamma}\right]^{\frac{1}{\gamma-1}}$$

$$\dot{M} = 4\pi r_c^2\rho(r_c)c_s(r_c)$$

$$= \pi G M^2\frac{\rho(\infty)}{c_s^3(\infty)}\left(\frac{2}{5-3\gamma}\right)^{\frac{5-3\gamma}{2(\gamma-1)}}$$

$$= 1.4\times10^{11}\,\text{g s}^{-1}\left(\frac{M}{M_\odot}\right)^2\left[\frac{\rho(\infty)}{10^{-24}\,\text{g cm}^{-3}}\right]\left[\frac{c_s(\infty)}{10\,\text{km/s}}\right]^{-3}$$

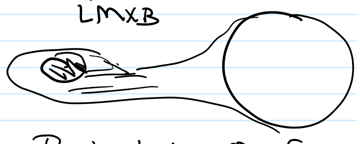
Accretion onto neutron stars and white dwarfs

What is different?

- The inner boundary condition?
- magnetic fields
- The outer boundary condition

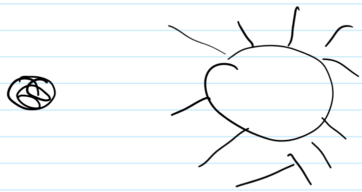
Neutron Star/BH binaries

Companion object is less massive LMXB



Roche Lobe Overflow

Companion is more massive HMXB



Wind accretion

White Dwarfs

- Cataclysmic variables

- magnetic field at large distance is determined by magnetic moment

$\mu = B_p R^3$

B_p 0 ~ 1GG

R 3000-10000 km

$\mu \sim 0 - 10^{36}$ G cm^3

c.f. NS B_p up to 10^{15} G

$\mu \sim 10^{33}$ G cm^3

typically 10^{30} G cm^3

How does magnetic field affect the flow?

$$\frac{B^2(r_A)}{8\pi} \simeq \frac{1}{2}\rho(r_A)v^2(r_A)$$

magnetic energy density — kinetic energy of flow

- We assume that the flow velocity is free fall

↳ remember circular velocity is $\frac{1}{\sqrt{2}}v_{ff}$

$v(r) \approx v_{ff} = \left(\frac{2GM}{r}\right)^{1/2}$

$\rho(r) \simeq \rho_{ff} = \frac{\dot{M}}{}$

$v(r) \approx v_{ff} = \left(\frac{2GM}{r}\right)^{1/2}$

$\rho(r) \Rightarrow \rho_{ff} = \frac{\dot{M}}{4\pi r^2 v_{ff}}$

$r_A = \left(\frac{\mu^4}{2GM\dot{M}^2}\right)^{1/7} = 3.2\times10^8\, \dot{M}_{17}^{-2/7} \mu_{30}^{4/7} \left(\frac{M}{M_\odot}\right)^{-1/7}$ cm

c.f. $R_{WD} \sim 10^9$ cm

$R_{NS} \sim 10^6$ cm

so if $r_A < R_*$ accretion in disk or free fall to surface

in WDs we have 3 possibilities:

$r_A < R_*$ regular CV

$R_* < r_A < r_{outer}$ intermediate polar, DQ Her

$r_A > r_{outer}$ polar, AM Her

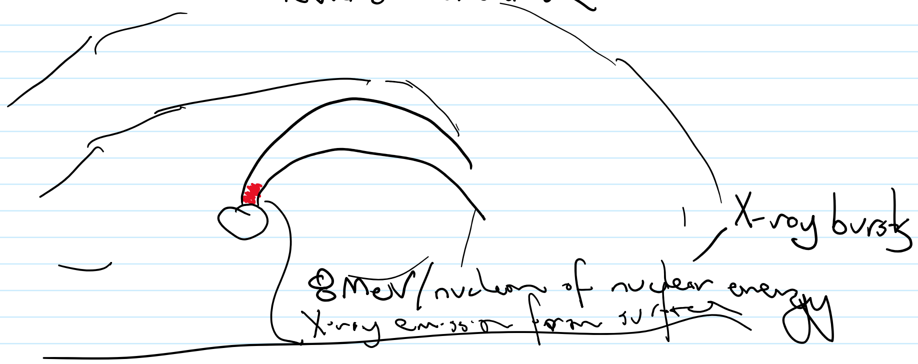
In neutron stars we have

$R_* < r_A$ Normal accreting X-ray pulsars — Usually wind accretion

$r_A < R_*$ LMXBs with disk to surface

If $r_A > R$ the disk ends above the neutron star surface. If the spin rate of the neutron star (ω) exceeds Ω at the inner edge of the disk $\dot{\omega} < 0$

Otherwise $\dot{\omega} > 0$. → so neutron stars evolve toward corotation



Accretion on to white dwarfs

- also 8 MeV/nucleon can be released which exceeds the binding energy nova!
- accretion shocks forms above the surface

↳ $T_{surf} \sim 5\times10^5$ K

$T_{shock} \sim T_{ff} \sim 10^9$ K