1 Solutions to assignment 3, due May 31

1. Problem 9.11 If we set $f(a) = r$ for each $a \in A$, then this function is not surjective (there is no point $b \in A$ so that $f(b) = s$, for example), and it is not injective, as $f(w) = f(x)$, but $w \neq x$.

2. Problem 9.13 The function $f(n) = 2n + 1$ is injective, but not surjective. Is it not surjective as its range is the set of all odd integers. It is injective, since if $f(n) = f(m)$ we have that

$$\begin{aligned} 2n + 1 &= 2m + 1 \\ 2n &= 2m \\ n &= m \end{aligned}$$

and so it is injective.

3. Problem 9.14 The function $f(n) = n - 3$ is injective and surjective. Injectivity is similar to the previous problem. As for surjectivity, let $m \in \mathbb{Z}$. Then $f(m + 3) = (m + 3) - 3 = m$, and so the map is surjective.

4. Problem 9.15 The function $f(n) = 5n + 2$ is injective, but not surjective. Injectivity is the same as before, and the lack of surjectivity can be seen due to the fact that the images are all congruent to 2 mod 5, and so the number 5 (for example) is not in the range.

5. Problem 9.18 There is such a function. Every odd degree polynomial is surjective since we have either

$$\lim_{x\to\infty} p(x) = +\infty \qquad \text{and} \qquad \lim_{x\to-\infty} p(x) = -\infty$$

or

$$\lim_{x\to\infty} p(x) = -\infty \qquad \text{and} \qquad \lim_{x\to-\infty} p(x) = +\infty.$$

Thus if we consider, say, $p(x) = x^3 - x$ then $p(-1) = p(0) = p(1) = 0$, and so this is not injective.

6. Problem 9.19 Possible examples are:

   (a) $f(n) = n$.

   (b) $f(n) = 2n$.

   (c) $f(n) = \begin{cases} n - 1 & n > 1 \\ 1 & n = 1 \end{cases}$. This is not injective, since $f(1) = f(2)$, but it is surjective.

   (d) $f(n) = 372$. This is clearly neither.

7. Problem 9.20 Injectivity is the same in this case as it was in problem 9.13. As for surjectivity (which is now different), we see this by simply inverting the function. So let $r \in \mathbb{R}$. We want to find some x such that $f(x) = r$. So if we let $x = \frac{r+2}{7}$, we see that

$$f\Big(\frac{r+2}{7}\Big) = 7\Big(\frac{r+2}{7}\Big) - 2 = (r+2) - 2 = r$$

and so the function is surjective.

8. Problem 9.21 We could show injectivity and surjectivity by hand, but we will do this via a much shorter route. Consider the function $g(x) = \frac{-2x-1}{-x+5}$. (This function is obtained by inverting the relationship $y = \frac{5x+1}{x-2}$). Then

$$\begin{aligned} f \circ g(x) &= \frac{5\frac{-2x-1}{-x+5} + 1}{\frac{-2x-1}{-x+5} - 2} \\ &= \frac{5(-2x-1) + (-x+5)}{-2x-1-2(-x+5)} \\ &= \frac{-10x-5-x+5}{-2x-1+2x-10} = \frac{-11x}{-11} = x \end{aligned}$$

and a similar computation shows that $g \circ f(x) = x$. Thus these two functions are inverses of each other (i.e. $g = f^{-1}$, and so they are both bijective.

9. Problem 9.24 Consider the two function $f_1(x) = x^2$ and $f_2(x) = \sqrt{x}$. Then these are both increasing (and continuous!) functions (and thus are injective) which satisfy $f_i(0) = 0$ and $f_i(1) = 1$. Thus they are (by the intermediate value theorem) surjective, i.e. bijective.

10. Problem 9.25 This follows immediately due to the fact that f is its own inverse, but we will prove it directly.

    We want to first show that f is injective. So suppose that $f(x_1) = f(x_2)$. Applying f to both sides of the equation (and noting that $f \circ f = id_A$ implies that $f \circ f(x) = x$) we find that

$$\begin{aligned} f\big(f(x_1)\big) &= f\big(f(x_2)\big) \\ x_1 &= x_2 \end{aligned}$$

    and thus the function is injective.

    As for surjectivity, let $x \in A$. We want to show that there is some $y \in A$ such that $f(y) = x$. So let $y = f(x)$. Then $f(y) = f\big(f(x)\big) = x$ as desired.

11. Problem 9.29

    (a) This is true. This is because the corresponding statement is true for both injective and surjective functions, as we showed in class.

(b) This is false. Consider $A = \{0\}$, and $B = C = \{a, b\}$. Let $f(0) = a$, and let $g : B \to C$ be the identity function. Then $g \circ f$ is not surjective, even though g is.

(c) This is false. Consider $A = \{a, b\}$, and consider $B = C = \{0\}$. If we let $f : A \to B$ be given by $f(a) = 0$, $f(b) = 0$, and if we let g be the identity function, then the composition is not injective, even though g is.

(d) This is true. Let $A = C = \{0\}$, and let $B = \{a, b\}$. Then if $f : A \to B$ is given by $f(0) = a$, and $g : B \to C$ is given by $g(a) = 0, g(b) = 0$, then the composition is onto even though f is not.

(e) This is false. If f is not injective, then there are $a \neq b, \in A$ with $f(a) = f(b)$. However, we would then have $g \circ f(a) = g \circ f(b)$, which contradicts the injectivity of $g \circ f$.