[illegible]

[illegible]

- Classical mechanics gained sway because it could explain and predict the motion of the planets.
- Although this now seems rather pedagogical, at the time it had great practical importance.
- After this week, you will be able to
  - Derive the shapes of the orbits for an isotropic harmonic oscillator and gravity
  - Derive Kepler's Three Laws of Planetary Motion.
  - Use cross-sections to calculate rates.

DISASTER LUNACY

INFLUENZA

SUNDAY

MONDAY LUNES LUNDI

TUESDAY MARTES MARDI

WEDNESDAY MIÉRCOLES MERCREDI

THURSDAY JUEVES JEUDI

FRIDAY VIERNES VENDREDI

SATURDAY

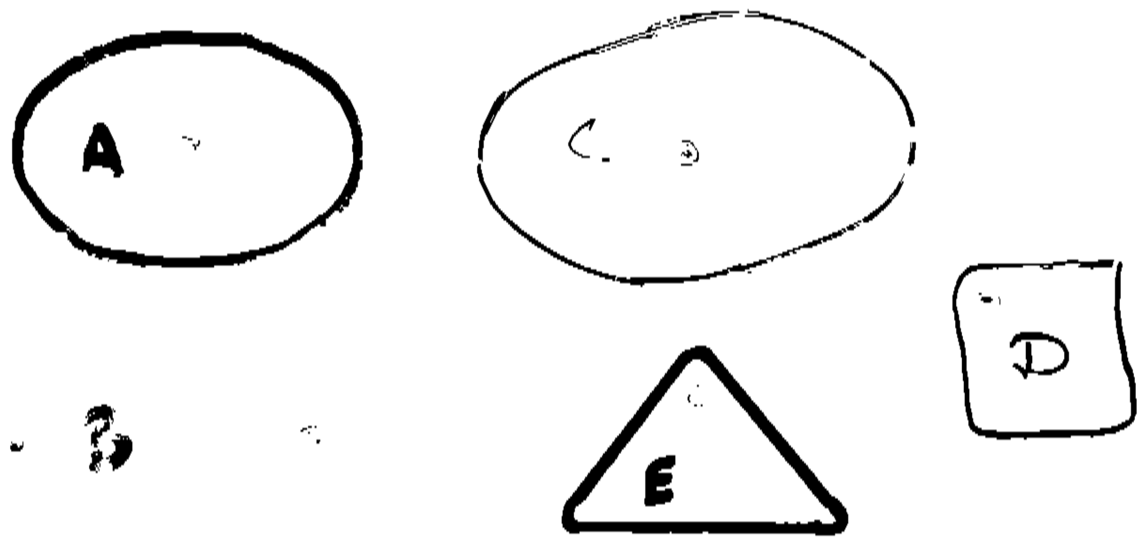
BY JOVE, MARTIAL, JOVIAL

STAR-CROSSED

ASTROLOGY

ASTRAL

# WHAT DOES A COMET'S PATH AROUND THE SUN LOOK LIKE?



CENTRAL FORCES #2.

## [illegible]

- Let's start with the question I posed to you.

- What is conserved in a $1/r^2$ central force?

a) $V(r) = -\frac{GMm}{r}$ → Energy

b) $\vec{\tau} = \vec{r} \times \vec{F} = 0$ → Angular momentum

Anything else?

CENTRAL FORCES

Q3

# THERE IS A HIDDEN SYMMETRY

- The Newtonian Law of Gravitation has a hidden symmetry that gives another conserved quantity

$$\vec{C} = \hat{r} + \frac{\vec{J}\times\vec{p}}{GMm^2}$$

It is pretty well hidden.

$$\frac{d\vec{C}}{dt} = \frac{d}{dt}\left(\frac{\vec{r}}{\sqrt{\vec{r}\cdot\vec{r}}}\right) + \frac{\vec{J}}{GMm^2}\times\frac{d\vec{p}}{dt}\Bigg)$$

$$= \frac{\vec{v}}{r} - \frac{\vec{r}}{r^3}\,\vec{v}\cdot\vec{r} - \frac{m(\vec{r}\times\vec{v})}{GMm^2}\times\left(\frac{GMm}{r^3}\vec{r}\right)$$

$$= \frac{\vec{v}}{r} - \frac{\vec{r}}{r^3}\,\vec{v}\cdot\vec{r} - \frac{(\vec{r}\times\vec{v})\times\vec{r}}{r^3} = 0$$

# CENTRAL FORCES



[illegible]

What do we already know about the solution?

(1) The motion is restricted to a plane. $3 \to 2$

(2) The orbit sweeps out equal area in equal time. $2 \to 1$

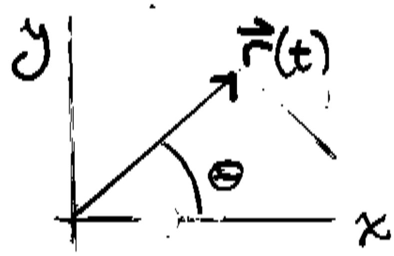
Although the motion is 3-D, we only have to worry about the radial motion!

# CENTRAL FORCES



- Because the force is central, we have conservation of angular momentum

$$J = mr^2\dot{\theta}$$




so once we know $r(t)$ we can solve for $\theta(t)$ by integration.

$$\frac{d\theta}{dt} = \frac{J}{mr^2}$$

$$\theta - \theta_0 = \int_{t_0}^{t} \frac{J}{m} \frac{dt}{r^2(t)}$$

# RADIAL MOTION

- We have a <u>formal solution</u> to the angular motion. Now let's look at the radial motion.

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$$

$$m\ddot{r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r}$$

N.B. $|\vec{J}| = mr^2\dot{\theta}$ so

$$\boxed{m\ddot{r} = \frac{\vec{J}^2}{mr^3} - \frac{\partial V(r)}{\partial r}}$$

Radial equation of motion

# CENTRAL FORCES



- We can also get an equation for the radial velocity from the equation for energy

$$E = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r)$$

$$E = \frac{m}{2}\dot{r}^2 + \frac{\vec{J}^2}{2mr^2} + V(r)$$

$$\dot{r}^2 = \frac{2}{m}\left(E - \frac{\vec{J}^2}{2mr^2} - V(r)\right)$$

gravity ↙ ↘ spring

$$\dot{r}^2 = \frac{2}{m}\left(E - \frac{\vec{J}^2}{2mr^2} + \frac{GMm}{r}\right) \qquad \dot{r}^2 = \frac{2}{m}\left(E - \frac{J^2}{2mr^2} - \frac{1}{2}kr^2\right)$$

# CENTRAL FORCES



- Solving for the orbit requires a bit of inspiration.

For gravity we have

$$\dot{r}^2 = \frac{2E}{m} - \frac{\vec{J}^2}{m^2 r^2} + \frac{2GM}{r}$$

Lot's of stuff downstairs!

What about $r(\theta)$ instead of $r(t)$?

$$\frac{dr(\theta)}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\frac{J}{mr^2}$$

Now the 2nd trick. Let $u = \frac{1}{r(\theta)}$

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} \Rightarrow \dot{r} = -\frac{du}{d\theta}\frac{J}{m}$$

$$\frac{J^2}{m^2}\left(\frac{du}{d\theta}\right)^2 = \frac{2E}{m} - \frac{J^2}{m^2}u^2 + 2GMu$$

## CENTRAL FORCES



### THE [illegible]

- Let's clean things up. $\quad \frac{2}{\ell}$

$$\left(\frac{du}{d\theta}\right)^2 = -u^2 + \frac{2Em}{J^2} + 2\frac{GMm^2}{J^2}u$$

Now the 3rd trick $\quad v = u - \frac{1}{\ell}$

$$\left(\frac{dv}{d\theta}\right)^2 = -\left(v+\frac{1}{\ell}\right)^2 + \frac{2Em}{J^2} + \frac{2}{\ell}\left(v+\frac{1}{\ell}\right)$$

$$\left(\frac{dv}{d\theta}\right)^2 = -v^2 - \frac{2}{\ell}v - \frac{1}{\ell^2} + \frac{2v}{\ell} + \frac{2}{\ell^2} + \frac{2Em}{J^2}$$

$$v^2 + \left(\frac{dv}{d\theta}\right)^2 = \frac{2Em}{J^2} + \frac{1}{\ell^2} \qquad e$$

$$v = \sqrt{\frac{2Em}{J^2} + \frac{1}{\ell^2}}\cos\theta = \frac{1}{\ell}\boxed{\sqrt{\frac{2Em\ell^2}{J^2}+1}}\cos\theta$$

$$\frac{1}{r} - \frac{1}{\ell} = \frac{e}{\ell}\cos\theta$$

$$\frac{1}{r} = \frac{e\cos\theta + 1}{\ell} \qquad \boxed{r = \frac{\ell}{1+e\cos\theta}}$$

# CENTRAL FORCES



- Only one other potential admits an analytic solution. Do you know what it is?

Newton was rather lucky that $V(r) = -\frac{GMm}{r}$ was solvable.

- Using the results for one-D motion we can still calculate various quantities.

# CENTRAL FORCES

- If we look at the radial motion there is an effective potential.

$$V_{eff}(r) = \frac{\vec{J}^2}{2mr^2} + V(r)$$