ISM Unit 2

Contents

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Unit 2

Unit 2 contains the following lessons

- the divergence test
- the integral test
- the alternating series test
- the ratio test
- a review of convergence tests

2.1 The Divergence Test

What This Lesson Covers

In this short lesson we will only introduce the divergence test.

Learning Objectives

After reading this lesson and completing a sufficient number of exercices on paper, you should be able to use the divergence test to determine if a given infinite series diverges.

Topics

1. Introduction To The Divergence Test
2. A Useful Theorem
3. The Divergence Test
4. A Divergence Test Flowchart
5. A Simple Divergence Test Example
6. Final Thoughts on the Divergence Test

Additional Resources for This Lesson

- Problems from the list of recommended exercises
- Additional Example 1: Divergence Test with Square Roots
- Additional Example 2: Divergence Test with arctan(x)
- Video Examples

2.1.01 Introduction to The Divergence Test

In previous lessons, we defined the concept of the convergence of an infinite series: the infinite series

$$\sum_{k=1}^{\infty} a_k$$

converges if and only if the limit

exists. As was mentioned, it can be very difficult to apply this limit directly, leading us to the question: **is there an easier way to determine if an infinite series converges or diverges**?

Examples

Consider for example

$$\sum_{k=1}^{\infty} \arctan(k),$$

or perhaps

$$\sum_{k=1}^{\infty} \frac{k}{\sqrt{1+k^2}}.$$

There might be a way of finding an explicit formula for $s_n$for these series, and with these formula we may be able to use the definition of convergence directly to establish whether or not these series converge.

But, there is an easier way that quickly tells us that they do not. We can use the divergence test.

2.1.02 A Useful Theorem

The following theorem will yield the divergence test.

Theorem 1
If the infinite series $\sum_{k=1}^{\infty} a_k$ is convergent, then $\lim_{k\to\infty} a_k = 0.$

Proof of Theorem 1

The proof of this theorem can be found in most introductory calculus textbooks that cover the divergence test and is supplied here for convenience. Let the partial sum $s_n$be

Then

$$s_{n-1} = \sum_{k=1}^{n-1} a_k = a_1 + a_2 + a_3 + \ldots + a_{n-1}$$

and $s_n - s_{n-1} = a_n$.

By assumption, a_n is convergent, so the sequence $\{s_n\}$ is convergent (using the definition of a convergent infinite series). Let the number S be given by

$$S = \lim_{n\to\infty} s_n.$$

Since n-1 also tends to infinity as n tends to infinity, we also have

$$S = \lim_{n\to\infty} s_{n-1}.$$

Finally,

Thus, if

$$\sum_{k=1}^{\infty} a_k$$

is convergent, then

$$\lim_{k\to\infty} a_k = 0,$$

as required.

2.1.03 The Divergence Test

Theorem 1 immediately yields the divergence test.

Theorem: The Divergence Test
Given the infinite series, $\sum_{k=1}^{\infty} a_k$ if the following limit $\lim_{k\to\infty} a_k$ does not exist or is not equal to zero, then the infinite series $\sum_{k=1}^{\infty} a_k$ must be divergent.

No proof of this result is necessary: the Divergence Test is equivalent to Theorem 1.

If it seems confusing as to why this would be the case, the reader may want to review the appendix on the divergence test and the contrapositive.

Beware: The Converse is Not Necessarily True

Observe that **the converse of Theorem 1 is not true in general**. If

$$\lim_{k\to\infty} a_k = 0$$

we cannot conclude that the infinite series is convergent! For example, the harmonic series has the property that

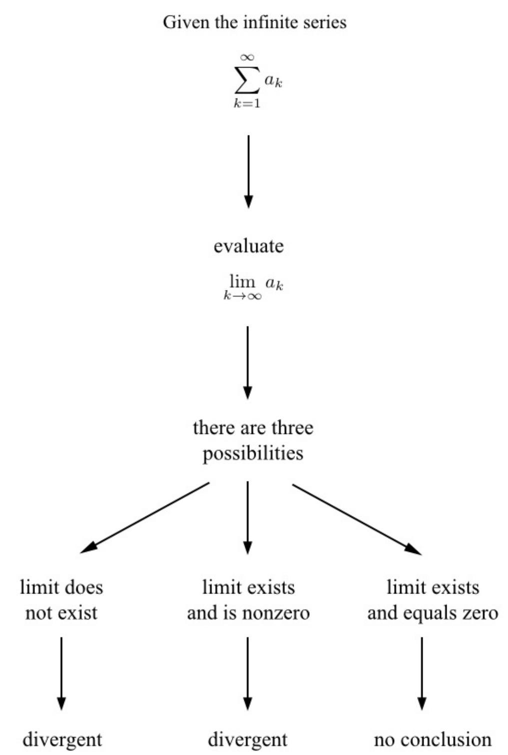
$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k} = 0.$$

But the harmonic series is **not** a convergent series, as was shown in a an earlier section in the lesson on the harmonic and telescoping series.

Therefore, **if the limit is equal to zero, the Divergence Test yields no conclusion**: the infinite series may or may not converge. In this case, other convergence tests can be used to try to determine whether or not the series converges, if required.

2.1.04 A Divergence Test Flowchart

The steps involved in applying the divergence test to an infinite series are given in the flowchart below.



This diagram shows the steps involved in conducting the divergence test. Observe that in the case that the limit equals zero, the test does not tell us if the series is convergent or divergent (other convergence tests can be employed in this case). This is an area that is a common source of confusion among calculus students.

2.1.05 Simple Divergence Test Example

Problem

Using only the divergence test, determine whether or not the following series diverges

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{1+k^2}}.$$

Complete Solution

Applying the divergence test yields

$$\begin{aligned}\lim_{k\to\infty} \frac{1}{\sqrt{1+k^2}} &= \lim_{k\to\infty} \frac{1}{\sqrt{1+k^2}} \cdot \frac{1/k}{1/k} && (1)\\ &= \lim_{k\to\infty} \frac{1/k}{\sqrt{1/k^2+1}} && (2)\\ &= 0.\end{aligned}$$

Since the limit equals zero, the divergence test yields no conclusion.

Explanation of Each Step

Step (1)

To apply the divergence test, we replace our sigma with a limit.

To apply our limit, a little algebraic manipulation will help: we may divide both numerator and denominator by the highest power of k that we have. Taking the radical into account, the highest power of k is 1, so we divide both numerator and denominator by $k^1 = k$.

Step (2)

The algebra in the denominator may be a little tricky. Here is the above derivation with two extra lines of math:

$$\begin{aligned}\lim_{k\to\infty} \frac{1}{\sqrt{1+k^2}} &= \lim_{k\to\infty} \frac{1}{\sqrt{1+k^2}} \cdot \frac{1/k}{1/k}\\ &= \lim_{k\to\infty} \frac{1/k}{\frac{1}{k}\sqrt{1+k^2}}\\ &= \lim_{k\to\infty} \frac{1/k}{+\sqrt{\frac{1}{k^2}}\sqrt{1+k^2}}\\ &= \lim_{k\to\infty} \frac{1/k}{\sqrt{1/k^2+1}}\\ &= 0.\end{aligned}$$

Potential Challenge Areas

Drawing the Wrong Conclusion

As mentioned earlier, one mistake students could make either one of two mistakes when the limit equals zero:

- that the series converges
- that the convergence of the given series cannot be established

However, when the limit equals zero, the test yields no conclusion, and it could be that the convergence of the given series could be established with a different test. Be careful to not make either of these mistakes.

2.1.06 Divergence Test with Square Roots

Question

Use the divergence test to determine whether the infinite series

$$\sum_{k=1}^{\infty} \frac{\sqrt{k}}{\sqrt{1+k^2}}$$

converges or diverges, or if the test yields no conclusion.

Solution

As always, we apply the divergence theorem by evaluating a limit as k tends to infinity. In this case we find

$$\begin{aligned} \lim_{k\to\infty} \frac{\sqrt{k}}{\sqrt{1+k^2}} &= \lim_{k\to\infty} \frac{\sqrt{k^{-1}}}{\sqrt{k^{-2}+1}} && (1) \\ &= \frac{\sqrt{0}}{0+1} && (2) \\ &= 0. \end{aligned}$$

Therefore, because the above limit equals zero, the divergence test yields no conclusion.

Discussion of Each Step

Step (1)

Essentially, we replaced the sigma

$$\sum_{k=1}^{\infty}$$

in the given series with a limit

$$\lim_{k\to\infty}$$

to obtain the left-hand-side of (1). To obtain the right-hand-side we used a common trick for evaluating limits: diving numerator and denominator by the highest power of k.

$$\lim_{k\to\infty} \frac{\sqrt{k}}{\sqrt{1+k^2}} = \lim_{k\to\infty} \frac{1/k}{1/k} \frac{\sqrt{k}}{\sqrt{1+k^2}} = \lim_{k\to\infty} \frac{\sqrt{k^{-1}}}{\sqrt{k^{-2}+1}}$$

Step (2)

In the second step, we took the limits inside the radical signs:

$$\begin{aligned}\lim_{k\to\infty} \frac{\sqrt{k^{-1}}}{\sqrt{k^{-2}+1}} &= \frac{\sqrt{\lim_{k\to\infty} k^{-1}}}{\sqrt{\lim_{k\to\infty} k^{-2} + \lim_{k\to\infty} 1}} \\ &= \frac{\sqrt{0}}{0+1} = 0.\end{aligned}$$

Potential Challenge Areas

Getting Started

Since the question asks for the divergence test to be used, getting started would imply that

$$\lim_{k\to\infty} \frac{\sqrt{k}}{\sqrt{1+k^2}}$$

needs to be evaluated.

Coming to the Wrong Conclusion

When the limit yields zero, the divergence test yields no conclusion. Other tests might be used to determine whether this limit diverges or converges. In this particular example, this might be difficult using only the tests we cover in the ISM. In the last lesson of this unit, we will address what one could do if all our tests fail (*link to be added here*).

2.1.07 Divergence Test with arctan

Question

Use the divergence test to determine whether the infinite series

$$\sum_{k=1}^{\infty} \arctan(k)$$

diverges.

Complete Solution

As always, we apply the divergence theorem by evaluating a limit as k tends to infinity. In this case we find

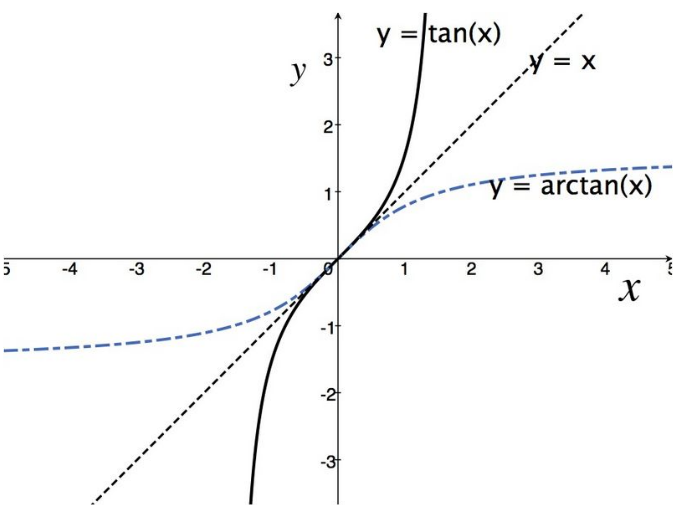
$$\lim_{k\to\infty} \arctan(k) = \infty$$

Therefore, because $arctan(x)$ tends to infinity as x tends to infinity, the divergence test tells us that the infinite series diverges.

Potential Challenge Areas

Remembering What arctan Looks Like

The arctan function is the inverse of the tan function. One way of remembering what it looks like is to remember that the graph of the inverse of a function can be obtained by reflecting it through the straight line $y = x$. The two functions are shown in the figure below.



The graph of arctan(x) (the blue dashed line) can be obtained by reflecting tan(x) (the solid black line) through the line y = x (the dashed black line).

2.1.08 Videos on The Divergence Test

Patrick JMT: Showing a Series Diverges Using its Partial Sums
In this video, Patrick JMT explores an example of what can be done when the divergence test fails. In this case, using partial sums can tell us whether a series diverges.
More Patrick JMT videos available here [1].

Patrick JMT: Geometric Series and the Test for Divergence (Part 1)
In this video, Patrick JMT discusses two geometric series examples. The example at the very end is completed in the next video.
More Patrick JMT videos available here [1].

Patrick JMT: Geometric Series and the Test for Divergence (Part 2)
In this video, Patrick JMT finishes the example he started in the previous video.
More Patrick JMT videos available here [1].

References

[1] http://patrickjmt.com/

2.1.09 Final Thoughts on the Divergence Test

Recall our Learning Objectives

In this lesson we covered only one topic:

- the divergence test.

After reading this lesson and completing a suitable number of exercises, you should be able to, given an infinite series

- apply the divergence to determine whether or not the infinite series diverges

Note that we **cannot** use the divergence test to determine whether or not an infinite series **converges**. If the divergence test fails to provide any information, we can only resort to other tests to establish convergence.

If the Limit Equals Zero, Do We Give Up?

Of course not! In the event that the limit equals zero, the divergence test yields no conclusion, and we can try any of the other tests we explore in the ISM. For example, in the next lesson we introduce the integral test, which can determine whether or not an infinite series converges or diverges provided our infinite series meets certain conditions.

2.2 The Integral Test

What This Lesson Covers

In this lesson we will introduce the integral test and the p-series.

Learning Objectives

After reading this lesson and completing a sufficient number of exercices on paper, you should be able to

- determine if a given series is a p-series
- determine if a p-series converges
- determine whether the integral test can be applied to a given infinite series
- apply the integral test to determine if it converges or diverges

Topics

1. A Motivating Problem for the Integral Test
2. A Second Motivating Problem for the Integral Test
3. Theorem: The Integral Test
4. An Integral Test Flowchart
5. Integral Test Example
6. Integral Test Example with Logarithm
7. The *p*-Series
8. Videos on The Integral Test
9. Final Thoughts on the Integral Test

2.2.01 A Motivating Problem for The Integral Test

Example

Does the following series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

converge? The method of determining whether this series converges only requires a few steps, but first let's explore this problem using the tools we already have to see why the integral test is needed.

In practice, determining whether an infinite series converges is not a matter of simply applying an algorithm that can be memorized. Rather, we must try different approaches that we know and sometimes develop a new approach to solve a problem.

Can We Use the Divergence Test?

Realistically, when students are working on infinite series problems on their own, they often encounter situations where the test they want to apply does not provide the information they are looking for. For example, applying the divergence test on this series, we find that the limit

$$\lim_{k\to\infty} \frac{1}{k^2}$$

equals zero. So the divergence test yields no information, and **we must use another test**.

In this lesson we will introduce a method that can be used to establish that it does using the Integral Test.

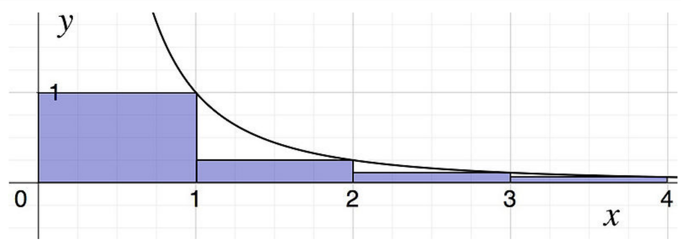
Relating Our Series to an Integral

As you might have guessed, the integral test makes a connection between an infinite **series** and an **integral**. To introduce an integral, we must introduce a function to integrate. What function do you think we could use?

Because the general term of our series is $1/k^2$, let's try the function $f(x) = 1/x^2$. It will be helpful to note here that this function is greater than zero for all x.

Let's also recognize that an integral of a function, **that is non-negative**, can be interpreted as an area, so we can look for a geometrical (or graphical) interpretation of the given problem that involves areas.

In the figure we have a graph of $f(x) = 1/x^2$ along with a set of rectangles. Notice that the areas of the rectangles are



The function 1/x and rectangles approximating the area under the curve. In this example, the rectangles touch the curve at the upper **right** corner.

$$\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \frac{1}{5^2}, \dots$$

By construction, the width of each rectangle is 1. Adding the first N areas together we obtain the partial sum,

$$s_N = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots + \frac{1}{N^2}$$

and taking the limit as N goes to infinity we obtain an infinite series,

$$\lim_{N\to\infty} s_N = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

This infinite series is exactly the infinite series that we were given. Also observe that

1. because the areas are all positive, the partial sum is an increasing sequence, and
2. the infinite sequence, equal to sum of the areas of the rectangles, is less than the total area under the curve

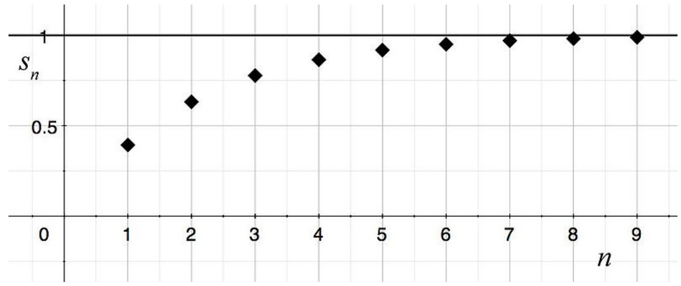
Indeed, the total area under the curve is given by the integral

$$\int_1^{\infty} \frac{1}{x} dx = 2$$

Finally, because we have an increasing sequence, s_n that has a **finite** upper bound, we know that the given sequence must be convergent.

Our final result uses the Monotonic Sequence Theorem for sequences which the reader may not have encountered, but is straightforward. The diagram below demonstrates the concept of the theorem, which is also described in an appendix.

What Assumptions Did We Need?



An arbitrary infinite sequence that is increasing and has an upper bound must converge.

To solve our problem, we needed to make two assumptions. These assumptions are given later in this lesson, but try to go back through our example to uncover what special requirements were needed to get to our result. These assumptions lead us to the necessary conditions to apply the integral test.

2.2.02 A Second Motivating Problem for the Integral Test

Example

Now let's try to determine whether the following series

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

converges. We can use the same approach that we took with

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

to obtain our result, which as you might guess, will show that this series is divergent.

Relating Our Series to an Integral

To introduce an integral, we must introduce a function to work with. Using the same approach with our previous example, because the general term of our series is

$$\frac{1}{\sqrt{k}}$$

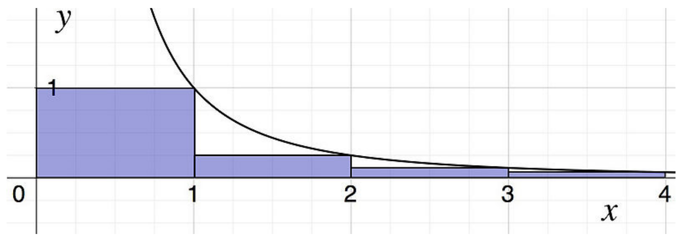
so let's try the function

$$f(x) = \frac{1}{\sqrt{k}}$$

Note that, once again this function meets the following criteria:

- f(x) is greater than zero for all x
- f(x) is continuous
- f(x) is a decreasing function

As in the previous example, we look to a geometrical interpretation of what we are doing. The Figure 1 gives us a graph of $f(x) = 1/x^2$ along with a set of rectangles. Notice that the areas of the rectangles are greater than area under



Starting with the equilateral triangle, this diagram gives the first three iterations of the Koch Snowflake (Creative Commons, Wikimedia Commons, 2007).

$$f(x) = \frac{1}{\sqrt{k}}$$

because of the way we constructed the rectangles.

However, it is a known result, often covered in an integral calculus course, that the integral

$$\int_1^\infty f(x)dx$$

is divergent. Because the rectangle area is greater than the area of the curve, the rectangle area must be infinite as well. So the given sequence must be divergent.

2.2.03 The Integral Test

Based on our two previous examples, you may have noticed that we were able to establish that the series converged or diverged because

- our functions were positive on some interval
- our functions were decreasing on some interval

In general, when applying the integral test, we do require these conditions, and also that our function is continuous. The conditions come out of the proof of the integral test.

The Integral Test

The integral test is given by the following theorem.

Theorem: The Integral Test
Given the infinite series $\sum_{k=1}^{\infty} a_k$ if we can find a function *f(x)* such that $a_n = f(n)$ and that is continuous, positive, and decreasing on $[1,\infty)$, then the given series is convergent if and only if $\int_1^\infty f(x)dx$ converges.

Discussion

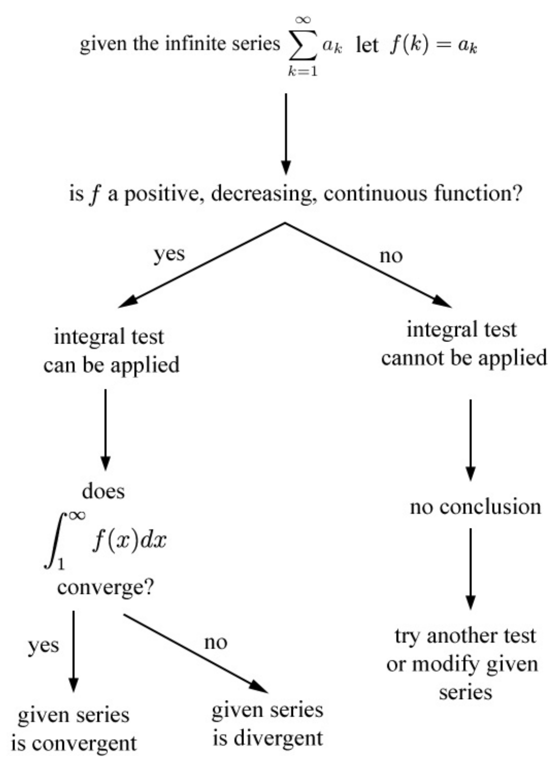
The integral test tells us that if the improper integral

$$\int_1^\infty f(x)dx$$

is convergent (that is, it is equal to a finite number), then the infinite series is convergent. If the improper integral is divergent (equals positive or negative infinity), then the infinite series is divergent. There are of course certain conditions needed to apply the integral test. Our function f must be positive, continuous, and decreasing, and must be related to our infinite series through the relation $f(n) = a_n$. However: when we encounter an infinite series that does **not** meet our criteria to apply the integral test, we can sometimes use some algebraic manipulation. We will explore this concept in a later example.

2.2.04 An Integral Test Flowchart

The steps involved in applying the integral test to an infinite series are given in the flowchart below.



This diagram shows the steps involved in conducting the integral test. Observe that in the case that when the integral test **cannot** be applied, we can try to perform some algebraic manipulation to apply the test. If for example, all terms in the series are negative, we can "pull out the negative sign" to make them positive.

2.2.05 Integral Test Example

Example

Consider the infinite series

$$\sum_{k=0}^{\infty} -ke^{-k}$$

Determine whether it is convergent using the integral test.

Complete Solution

Step 1: Pull Out the Negative Sign

$$\sum_{k=0}^{\infty} -ke^{-k} = -\sum_{k=0}^{\infty} ke^{-k} = -S$$

where

$$S = \sum_{k=0}^{\infty} ke^{-k}$$

If S converges, then the given infinite series converges.

Step 2: Check to see if the integral test can be applied

Let $f(x) = xe^{-x}$. Then

- $f(x)$ is continuous
- $f(x)$ is decreasing
- $f(x)$ is non-negative

Therefore the integral test can be applied.

Step 3: Apply the Integral Test

$$\begin{aligned} \int_1^{\infty} xe^{-x}dx &= -xe^{-x}\Big|_1^{\infty} - \int_1^{\infty} -e^{-x}dx \quad (1) \\ &= \big(-e^{-1} - 0\big) - e^{-x}\Big|_1^{\infty} \quad (2) \\ &= -2e^{-1} \end{aligned}$$

Step 4: Conclusion

The result is finite, so S is convergent by the integral test, so the given series is also convergent.

Explanation of Each Step

Step (1)

If we did not pull out the negative sign, we would not be able to apply the integral test, because this test can only be applied when all terms in the series are **positive**. This simple algebraic manipulation allows us to apply the integral test.

Step (2)

There are only three criteria we need to check before applying the integral test. Because all three criteria are met, we can apply this test.

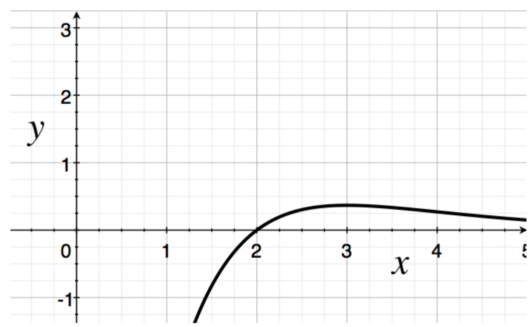
Step (3)

In Step (3) we applied the formula for the integral test, using the method of integration by parts to calculate the integral.

Possible Challenge Areas

Algebraic Manipulation

In the example we were given, we only had to pull out a negative sign, but what if we were asked to determine whether



If we let *a*=2, then we see that the function we use to perform the integral test is strictly positive for *x*>2. Simply pulling out the first few negative terms in the sum that are negative will create a series whose convergence we can test with the integral test.

$$\sum_{k=0}^{\infty} -(k-a)e^{-(k-a)}, \quad a > 0$$

converges? How would we approach this problem?

Observing that the function $f(x) = (x-a)e^{-(x-a)}, \quad a > 0$
is strictly positive for $x>a$, we can pull out the first terms of the sum that are negative, knowing that the remaining terms are positive, and can be used in the integral test.

2.2.06 Integral Test Example with Logarithm

Problem

Using the integral test, determine whether the infinite series

$$\sum_{k=2}^{\infty} \frac{1}{k \ln k}$$

converges or diverges, or if the test cannot be applied.

Complete Solution

First we must establish whether or not the test can be applied. If we let

$$f(k) = a_k = 1/(k \ln k)$$

then we much check if f(x) is continuous, decreasing, and positive.

Step (1): Check Continuity

Both the numerator and denominator are continuous functions on (2,∞), so their ratio will also be continuous. So *f* is continuous.

Step (2): Check Positivity

Both the numerator and denominator are positive functions on (2,∞), so their ratio will also be positive. So *f* is positive.

Step (3): Check to See if *f* is Decreasing

We can apply the usual first derivative test:

$$\frac{d}{dk}\frac{1}{k \ln k} = \frac{0 - 1 \cdot \frac{d}{dk} k \ln k}{k^2 (\ln k)^2} \quad (3.1)$$

$$= \frac{-(\ln k + 1)}{k^2 (\ln k)^2} \quad (3.2)$$

$$= -\frac{\ln k + 1}{k^2 (\ln k)^2} \quad (3.3)$$

The above quantity is negative for

$$k \in (2, \infty)$$

so we have that our function is also decreasing on this interval.

Step (4): Apply Integral Test

Since *f* is positive, continuous, and decreasing, we can apply the integral test.

$$\int_2^{\infty} \frac{1}{k \ln k} = \lim_{b \to \infty} \int_2^{\infty} \frac{1}{k \ln k} \quad (4.1)$$

$$= \lim_{b \to \infty} \ln(k \ln k)|_2^b \quad (4.2)$$

$$= \infty - \ln(\ln(2))$$

Therefore, the infinite series diverges, because the above integral diverges.

Discussion of Each Step

Step (1) and (2)

These checks must be done, but are, in this example, straightforward.

Step (3)

Recall that the first derivative test tells us that a function is decreasing on an interval if the first derivative of that function is negative everywhere on that interval. Using the quotient rule to calculate our derivative, we find that indeed, the function is decreasing on $(2,\infty)$.

Potential Challenges

Getting Started

Given that the question asks us to apply the integral test, we should know immediately how to get started: check for continuity, positivity, and if our function is decreasing. These three checks *must always* be performed before the test can be applied.

2.2.07 The p-series

Problem

For what values of p does the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converge?

Complete Solution

Step (1): Consider $p > 0$ and $p \neq 1$

When $p > 0$ and $p \neq 1$, the function

$$f(x) = \frac{1}{x^p}$$

is continuous, decreasing, and positive when x is in the interval $[1,\infty)$. Using the integral test,

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p} dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x^p} dx \\ &= \lim_{b\to\infty} \frac{1}{-(p-1)x^{p-1}} \bigg|_1^b \\ &= \begin{cases} \infty & \text{if } p < 1, \\ \frac{1}{p-1} & \text{if } p > 1 \end{cases} \end{aligned}$$

Therefore, the infinite series converges when $p > 1$, and diverges when p is in the interval $(0,1)$.

Step (2): Consider $p \leq 0$ and $p = 1$

If p=1, then we have the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

which we know diverges.

If $p \leq 0$, the infinite series diverges (by the divergence test).

Therefore, the given series only converges for $p > 1$.

The p-Series

The result of this example can be summarized as follows.

The p-Series
The p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Much like a geometric series, we can use this result to determine whether a given infinite series converges by inspection. For example, the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{1+k}}$$

diverges because it is a p-series with p equal to 1/2 (you may want to let u=(1+k) to see this).

2.2.08 Videos on the Integral Test

The Integral Test
An introduction to the integral test and a set of examples on determining whether a series converges using the integral test.
More MathIsPower4U videos available here [1].

The p-Series Test
A set of three examples on determining whether a series converges by using the p-series.
More MathIsPower4U videos available here [1].

Using the Integral Test for Series
An introduction to the integral test and one example on determining whether a series converges using the integral test.
More Patrick JMT videos available here [1].

References

[1] http://mathispower4u.yolasite.com/

2.2.09 Final Thoughts on The Integral Test

The previous lesson on the divergence test gave us a way of determining whether some infinite series diverge. We saw that the divergence test had a limitation: it can tell us if certain infinite series diverges, but it cannot tell us if a given series converges. But there are other convergence tests. The integral test, for example, provides a test for any series

$$\sum_{k=1}^{\infty} a_n$$

whose terms a_n can be related to a continuous, positive, decreasing function. Essentially, we let $f(n) = a_n$, then evaluate the integral

$$\int_1^{\infty} f(x)dx$$

and:

- if the integral **converges**, the infinite series **converges**, and
- if the integral **diverges**, the infinite series **diverges**.

Although this test is limited to functions who are continuous, positive and decreasing, we saw that it led us to a useful convergence theorem for any infinite series of the form

$$\sum_{k=1}^{\infty} \frac{1}{k^p},\ p \in \mathbb{Q}.$$

2.3 The Alternating Series Test

What This Lesson Covers

In this lesson we will only introduce the alternating series test.

Learning Objectives

After reading this lesson and completing a sufficient number of exercices on paper, you should be able to

- recognize whether a given infinite series is an alternating series, and
- use the alternating series test to determine if a given infinite series converges or diverges.

Topics

1. Introduction to the Alternating Series Test
2. The Alternating Series Test
3. A Simple Ratio Test Example
4. Final Thoughts on the Alternating Series Test

2.3.01 A Motivating Problem for The Alternating Series Test

Example

Does the following series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

converge? First let's explore this problem using the tools we already have to see why a new convergence test is needed. This will demonstrate of how, in practice, determining whether an infinite series converges is not a matter of applying a set of algorithms that can be memorized. Rather, we often need to try different approaches and sometimes develop new approaches to solve a given problem.

Why Do We Need Another Convergence Test?

We've covered the divergence dest and the integral test. Does the divergence test tell us if the series converges? Applying the divergence test, we find that the limit

$$\lim_{k\to\infty} \frac{(-1)^k}{k}$$

equals zero. The divergence test yields no information, and **we must use another test**.

Can we apply the integral test? The series have negative and positive values:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots$$

But the integral test can only applied on a sequence if all of its terms are positive. The integral test yields no information, and so **we must use another test**.

In this lesson we introduce a convergence test that can be used to establish that this series converges: the **alternating series test**.

Investigating the Partial Sums

Remember that for a series to converge, we must have that its partial sums tend to zero. In our example, this means that

$$s_N = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

tends to zero as *N* tends to infinity. Calculating the first 10 partial sums of our series, for *N* from 1 to 10, we obtain the values presented in the table below (numbers rounded to the first two decimal places).

k	$a_k = 1/k$	s_N
1	-1.00	-1.00
2	0.50	-0.50
3	-0.33	-0.83
4	0.25	-0.58
5	-0.20	-0.78
6	0.17	-0.62
7	-0.14	-0.76
8	0.13	-0.63
9	-0.11	-0.75
10	0.10	-0.65
11	-0.09	-0.74
12	0.08	-0.65
13	-0.08	-0.73
14	0.07	-0.66

It appears that the terms of the sequence are tending towards zero, but does it appear that the partial sums converging? They *could* be, but our table doesn't *prove* that the sequence converges to zero. We need a better way to determine whether the series converges. We need a new convergence test.

The alternating series test, applicable to series whose terms alternate between positive and negative, can be applied here. We will next give the alternating series test, and then apply it to show that this series does converge.

2.3.02 The Alternating Series Test

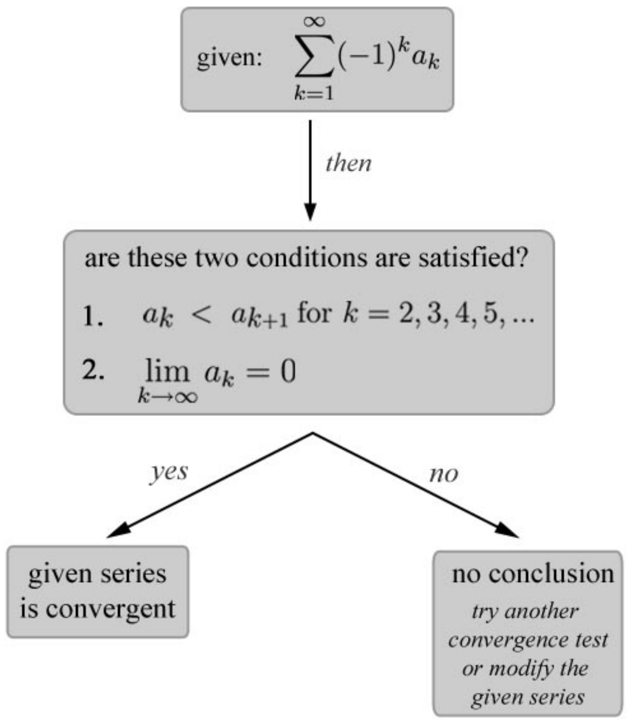
The alternating series test is given by the following.

The Alternating Series Test
If the alternating series $\sum_{k=1}^{\infty}(-1)^k a_k$ satisfies $a_k < a_{k-1}$ for k = 2, 3, 4, 5, ... , and $\lim_{k\to\infty} a_k = 0$ then the series is convergent.

Take care to note that this test, as its name suggests, can only be applied for the special case when successive terms alternate between positive and negative values.

2.3.03 An Alternating Series Test Concept Map

The steps involved in applying the alternating series test are given in the concept map below.



This diagram shows the steps involved in conducting the alternating series test. Observe that in the case that when the test cannot be applied, the test does not tell us if the series is convergent or divergent (other convergence tests can be employed in this case).

2.3.04 An Alternating Series Example

Example

We now return to the example we presented at the beginning of the lesson:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

We wish to determine whether it is convergent using the alternating series test.

Complete Solution

Step 1: Check to see if the alternating series test can be applied

We see that the terms $a_k = 1/k$ satisfy: $a_{k+1} < a_k$. Moreover, the terms in the sequence alternate between positive and negative.

Therefore the alternating series test can be applied.

Step 2: Apply the Alternating Series Test

$$\lim_{k \to \infty} \frac{1}{k} = 0$$

The series is therefore convergent.

Explanation of Each Step

Step (1)

Given a general alternating series,

$$\sum_{k=1}^{\infty} (-1)^k a_k$$

we need only two criteria to apply the alternating series test for a given infinite series:

1. terms in the series must alternate between positive and negative
2. the terms in the sequence must be decreasing (in other words, $a_k < a_{k-1}$)

Our series meets these two criteria.

Step (2)

The alternating series test on the general alternating series

$$\sum_{k=1}^{\infty} (-1)^k a_k$$

only requires that we evaluate

$$\lim_{k \to \infty} a_k$$

If the limit is zero, the series is convergent. In this case, **the limit is zero, so our sequence is convergent**.

2.3.05 Alternating Series Videos

The Alternating Series Test
An introduction to the alternating test and a set of three examples on determining whether a series converges using this test.
More MathIsPower4U videos available here [1].

Alternating Series
An introduction to the alternating series test and a simple example on determining whether a series converges using this test.
More Patrick JMT videos available here [1].

More Alternating Series Examples
As the title suggests, additional examples on determining whether a series converges using this test.
More Patrick JMT videos available here [1].

2.4 The Ratio Test

What This Lesson Covers

In this lesson we will only introduce the ratio test.

Learning Objectives

After reading this lesson and completing a sufficient number of exercices on paper, you should be able to use the ratio test to determine if a given infinite series converges.

Topics

1. Introduction to the Ratio Test
2. The Ratio Test
3. A Ratio Test Flowchart
4. A Simple Ratio Test Example
5. Ratio Test Example with an Exponent
6. Videos on the Ratio Test
7. Final Thoughts on the Ratio Test

2.4.01 Introduction to The Ratio Test

In previous lessons, we explored convergence tests that applied to

- series with positive terms (the integral test)
- series with alternating terms (the alternating series test)

However, there are series for which the above tests cannot be applied. We could ask: **how would we determine if an infinite series converges or diverges if its terms irregularly switch from positive to negative**?

Example

Consider for example the recursive sequence

$$a_1 = 1, \qquad a_{k+1} = \frac{0.1 + \cos(k)}{\sqrt{k}} a_k$$

for k = 1, 2, 3, ...

Approximate values of the first eight terms are given in the table below.

k	a_k
1	1.00000
2	-0.22355
3	+0.11487
4	-0.03180
5	-0.00546
6	-0.00236
7	-0.00076
8	+0.00001

We cannot use the integral test (not all terms are positive).

We cannot use the alternating series test (the signs change irregularly).

We will see that we can use the ratio test to show that this series converges.

2.4.02 The Ratio Test

The Ratio Test

To apply the ratio test to a given infinite series

$$\sum_{k=1}^{\infty} a_k,$$

we evaluate the limit

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L.$$

There are three possibilities:

- if $L < 1$, then the series converges
- if $L > 1$, then the series diverges
- if $L = 1$, then the test is inconclusive

The proof of this test is relatively long, and as such is provided in an appendix on the Proof of the Ratio Test.

Observe Carefully: The Test Can Yield No Information

Before moving on, note that in the case that $L = 1$, the test yields no information. Applying the ratio test to the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

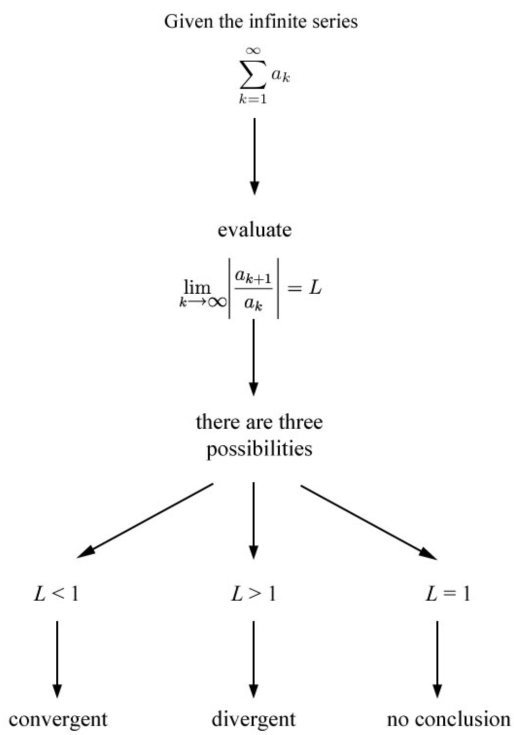
yields

Because the limit equals 1, **the ratio test fails** to give us any information.

But the harmonic series is **not** a convergent series, so in the case where $L = 1$, other convergence tests can be used to try to determine whether or not the series converges.

2.4.03 A Ratio Test Flowchart

The steps involved in applying the ratio test to an infinite series are given in the flowchart below.



This diagram shows the steps involved in conducting the ratio test. Observe that in the case that the limit equals one, the test does not tell us if the series is convergent or divergent.

2.4.04 A Simple Ratio Test Example

Question

Using only the ratio test, determine whether or not the recursive sequence converges or diverges.

Complete Solution

Applying the ratio test yields

But

$$0 \leq \lim_{k\to\infty} \left|\frac{0.1+\cos(k)}{\sqrt{k}}\right| \leq \lim_{k\to\infty} \frac{1.1}{\sqrt{k}} \quad (3)$$

$$0 \leq \lim_{k\to\infty} \left|\frac{0.1+\cos(k)}{\sqrt{k}}\right| \leq 0 \quad (4)$$

Therefore,

$$\lim_{k\to\infty} \left|\frac{0.1+\cos(k)}{\sqrt{k}}\right| = 0 \quad (5)$$

Since the limit equals 0, the ratio test tells us that the series converges.

Explanation of Each Step

Step (1)

To apply the ratio test, we must evaluate the limit

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$$

In our problem, we can use

$$a_{k+1} = \frac{0.1+\cos(k)}{\sqrt{k}} a_k$$

and substitute this into our limit.

Step (2)

In Step (2), we only cancel the $a_k$in the numerator and denominator.

Step (3)

First observe that

$$0 \leq |0.1+\cos(k)| \leq 1.1$$

Dividing everything by the square root of $k$we obtain

$$0 \leq \left|\frac{0.1+\cos(k)}{\sqrt{k}}\right| \leq \frac{1.1}{\sqrt{k}}$$

Step (4)

In Step (4) we **only** evaluate the limit:

$$\lim_{k\to\infty} \frac{1.1}{\sqrt{k}},$$

which equals zero because the numerator is a constant and the denominator goes to infinity.

Step (5)

In Step (5) we apply the Squeeze Theorem.

Potential Challenge Areas

Getting Started

Because the question asks us to apply the ratio test, we know that we will start our solution by using the formula

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|.$$

Recursive Formula

Most problems involving convergence tests don't involve recursive formulas. But with the ratio test, we apply

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$$

and use the given recursion equation for a_{k+1}. In our case, our recursion equation is

$$a_{k+1} = \frac{0.1+\cos(k)}{\sqrt{k}}a_k$$

which we substitute into the numerator, allowing us to cancel the a_k in the numerator and denominator. This trick is a bit harder to apply for the other convergence tests.

2.4.05 Ratio Test Example with Exponent

Question

Using only the ratio test, determine whether or not the series

converges, diverges, or yields no conclusion.

Complete Solution

Applying the ratio test yields

Since the limit equals $1/5$, the ratio test tells us that the series converges.

Explanation of Each Step

Step (1)

To apply the ratio test, we must evaluate the limit

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$$

In our problem, we have

$$a_k = \frac{k}{5^k}, \quad \text{and} \quad a_{k+1} = \frac{k+1}{5^{k+1}},$$

and we substitute them into our limit.

Step (2)

In Step (2), we use a little algebraic manipulation to make things easier to look at

$$\begin{aligned}\frac{(k+1)/5^{k+1}}{k/5^k} &= \frac{(k+1)/5^{k+1}}{k/5^k}\cdot\frac{5^{k+1}}{5^{k+1}} &(2.1)\\ &= \frac{(k+1)/1}{k/5^k}\cdot\frac{1}{5^{k+1}} &(2.2)\\ &= \frac{k+1}{k/5^k}\cdot\frac{5^k}{5^k}\cdot\frac{1}{5^{k+1}} &(2.3)\\ &= \frac{k+1}{k}\cdot\frac{5^k}{1}\cdot\frac{1}{5^{k+1}} &(2.4)\end{aligned}$$

Step (3)

Step (3) uses a property of absolute values. Recall that for real numbers $a$and b,

$|a \cdot b| = |a| \cdot |b|$.

Step (4)

Step (4) uses a property of limits values. Recall that for functions $f(k)$and $g(k)$, $k \in \mathbb{R}$that

$$\lim_{k \to \infty} \Big(f(k) \cdot g(k) \Big) = \lim_{k \to \infty} f(k) \cdot \lim_{k \to \infty} g(k)$$

Step (5)

Here we evaluate a limit:

$$\lim_{k \to \infty} \left| \frac{k+1}{k} \right| = \lim_{k \to \infty} \left| \frac{1 + k^{-1}}{1} \right| = 1.$$

Step (6)

Some algebraic manipulation helps us see how we can simplify our problem. Recall that, using laws of exponentials, that for real numbers x, a, b, that

$x^{a+b} = x^a \cdot x^b$,

so

$5^{k+1} = 5 \cdot 5^k$.

2.4.06 Videos on The Ratio Test

Using the Ratio Test to Determine if a Series Converges #1
An introduction to the ratio test and a set of examples on determining whether a series converges using this test.
More Patrick JMT videos available here [1].

Using the Ratio Test to Determine if a Series Converges #2
A set of three examples on determining whether a series converges using the ratio test.
More Patrick JMT videos available here [1].

2.5 Review of Convergence Tests

What This Lesson Covers

In this lesson we will review the convergence tests covered in the ISM and discuss strategies for determining whether an infinite series converges.

Learning Objectives

After reading this lesson and completing a sufficient number of exercices on paper, you should be able to determine whether a given infinite series converges or diverges.

Topics

1. Strategies for Testing for Convergence
2. Example
3. Additional Video Examples

2.5.01 Strategies for Testing for Convergence

It can be hard to determine which convergence test to apply to a given infinite series. While is tempting to search for an optimal order of convergence tests that can be applied to any given infinite series, a more efficient approach is to develop a firm understanding of what requirements each of the tests require to be applied. This way, certain convergence tests can be ruled out or considered by simple inspection of the **form** of the infinite series.

Infinite Series Forms

Knowing what convergence test to apply for a given series can involve classifying the series according to its **form**. For example, if the series has the form $\Sigma\ 1/n^p$, then the series is a **p-series**, which we know is convergent if p is greater than 1, and divergent otherwise.

Form	Convergence Test	Conditions for Convergence
$\Sigma\ 1/n^p$	p-series	$p > 1$
$\Sigma\ ar^{k-1}$	geometric series	$\lvert r\rvert < 1$
$\Sigma\ (-1)^k a_k$	alternating series	$a_k \to 0$ as $k \to \infty$
$\Sigma\ a_k$, $a_k = f(k)$, and f is continuous, positive and decreasing	integral test	integral of $f(x)$ from 1 to infinity exists

In all convergence tests, some preliminary algebraic manipulation may be required to bring a series into one of these forms. Moreover, most tests can also tell you if a series diverges, and all tests yield inconclusive results if they cannot be applied.

The Ratio Test

The ratio test doesn't fit into the above table as well as the other tests, but can be applied to any infinite series. It is often helpful when the general term of the series contains factorials, or constants raised to a power of k.

2.5.02 Example

In the following problem, we don't work out details on applying convergence tests. Rather, we discuss general strategies for convergence testing.

Problem

State what convergence test you would use to determine whether the following series converge, and explain why:

$$a)\ \sum_{k=0}^{\infty}(-1)^k\frac{2^k}{k7^{k+1}}$$

$$b)\ \sum_{k=0}^{\infty}\frac{3^k}{k4^{k+1}}$$

Complete Solutions (and Discussion)

Part a)

Because of the $(-1)^k$ term, the series is alternating, so we can try using the alternating series test (the ratio test could also be applied, but would require slightly more work to compute the ratio).

Part b)

Unlike Part a), we do **not** have an alternating series, but the presence of constants raised to the power k suggests that the ratio test could be applied.

Possible Challenges

Applying the Divergence Test

You may have noticed that in parts a) and b) that the divergence test yields an inconclusive result. Because the divergence test is easy to apply, you may find that it is one of the first tests you tend to think about, but it isn't one that can tell us when a series converges.

Not Recognizing The Form of the Series

Knowing how to classify a given infinite series requires being able to recognize **what components of the series are helpful**. If we consider Part a):

$$\sum_{k=0}^{\infty}\mathbf{(-1)^k}\frac{\mathbf{2^k}}{\mathbf{k7^{k+1}}}$$

The (-1)^{k} term (highlighted in blue) suggests that we **might** try using the alternating series test. Other components (highlighted in purple) tell us what the outcome of the test will be.

2.5.03 Video Example

Strategy for Testing Series
In this example, Patrick JMT reviews a number of convergence tests. Some of the convergence tests covered include the root test and the comparison test, which are not covered on this website. Students may want to ask their instructor which convergence tests they are required to be familiar with.
More Patrick JMT videos available here [1].

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