ISM Unit 3

Power Series

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3.1 Power Series

What This Lesson Covers

In this lesson we will introduce a few concepts:

- representation of functions with a power series
- conditions for convergence of a power series
- radius of convergence of a power series

Learning Objectives

After reading this lesson and completing a sufficient number of exercices on paper, you should be able to

- expand a function in a power series
- determine the radius of convergence of that series

Topics

1. A Motivating Problem
2. The Power Series
3. A Simple Example
4. Power Series Convergence
5. Power Series Convergence Example
6. Videos on Power Series
7. Final Thoughts on Power Series

3.1.01 A Motivating Problem for Power Series

Earlier in the Infinite Series Module, we introduced the concept of a geometric series,

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \ldots,\ a \neq 0$$

We assumed that *r* was a constant. But what if *r* were not a constant, and instead was a variable?

Let's replace our *r* with *x* in the above summation. We now obtain

$$\sum_{k=1}^{\infty} ax^{k-1} = a + ax + ax^2 + ax^3 + \ldots,\ a \neq 0$$

From this simple substitution, we have created a **function**, and can ask two important questions related to this result.

Key Questions for This Lesson

From our simple construction, we can ask:

1. for what values of *x* does our summation yield a finite result?
2. is there a more general representation of our summation?

The first question will take us to the idea of a **radius of convergence**, and the second question to the concept of a **power series**.

3.1.02 The Power Series

Indeed, a there is a more general representation for the series we introduced on the previous page:

$$\sum_{k=1}^{\infty} ax^{k-1} = a + ax + ax^2 + ax^3 + \ldots,\ a \neq 0$$

The more general representation is given by the following definition.

Definition: Power Series
An infinite series of the form $\sum_{k=0}^{\infty} a_k(x-c)^k$, where *c* is a constant, is a **power series about *c***. The constants, a_k are referred to as the **coefficients** of the series.

What Is A Power Series?

It may help to consider a simple example with partial sums. Let's take the case where a_k = 1, and c = 0, which gives us

$$y(x) = \sum_{k=0}^{N} x^k$$

If we let N = 1, then we have a polynomial of order 1

$$y(x) = \sum_{k=0}^{1} x^k = 1 + x$$

If we take N = 2, then we have a polynomial of order 2

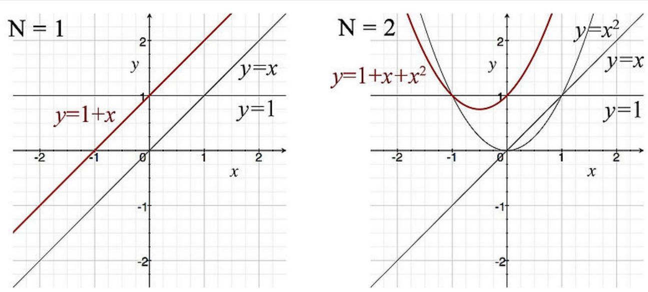
$$y(x) = \sum_{k=0}^{2} x^k = 1 + x + x^2$$

Similarly, with N = 3, then we would obtain a polynomial of order 3

$$y(x) = \sum_{k=0}^{3} x^k = 1 + x + x^2 + x^3$$

In the limit as N goes to infinity, we obtain a **power series**.

The figure below provides a graphical explanation of what a power series is: an infinite polynomial.



This diagram helps to demonstrate the idea of a what a power series is. We consider the simple case when a_k = 1, and c = 0. Using partial sums, the left corresponds to N = 1, and the right to N = 2. As we let N go to infinity, we obtain a power series.

3.1.03 Power Series Example

Example

Consider the infinite series

a) $\sum_{k=0}^{\infty} k!x^k$

b) $\sum_{k=0}^{\infty} \frac{x^k}{k!}$

Determine the domains of these two series

Complete Solution

Part a)

The ratio test gives us:

$$\lim_{k\to\infty}\left|\frac{(k+1)!x^{k+1}}{k!x^k}\right| = \lim_{k\to\infty}|x|(k+1)$$

This limit yields infinity unless $x = 0$. Therefore, the domain is the single point, $x = 0$.

Part b)

The ratio test gives us:

$$\lim_{k\to\infty}\left|\frac{k!x^{k+1}}{(k+1)!x^k}\right| = \lim_{k\to\infty}\frac{|x|}{(k+1)}$$

This limit is zero for all real values of x. Therefore, the domain is the set of all real numbers.

Explanation of Each Step

Because the procedure was the same for both parts a) and b), we can consider the parts together.

For both parts, we used the ratio test. Recall that the ratio test is given by the following.

The Ratio Test

To apply the ratio test to a given infinite series

$$\sum_{k=1}^{\infty} a_k,$$

we evaluate the limit

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = L.$$

There are three possibilities:

- if $L < 1$, then the series converges
- if $L > 1$, then the series diverges
- if $L = 1$, then the test is inconclusive

In part a) we found that for nonzero x, our limit was infinite, and was equal to zero when x equals zero. In part b), we found that the limit was zero, so the series converged for all x. Theses results give us the required domains of the series.

Possible Challenges

What Convergence Test Should Be Used?

For most problems, the ratio test can be used initially. If the ratio test yields an *interval* for the domain, we need to use other convergence tests to explore what the domain could be at the end points of the interval.

3.1.04 Power Series Convergence

The Sum May Not Converge

Our formula for the power series is

$$\sum_{k=0}^{\infty} a_k (x-c)^k$$

For certain values of x and a_k, a power series can be infinite. Let's go back to the example we introduced earlier in this lesson

$$\sum_{k=1}^{\infty} ax^{k-1} = a + ax + ax^2 + ax^3 + \dots,\ a \neq 0$$

The sum of this series that tells us that the series only converges when $|x| < 1$ (by the divergence test). But in more general power series, there are three distinct possibilities that we can encounter.

Three Possibilities for Convergence

Theorem: Only Three Convergence Results are Possible
x - a | < R, where R is some constant
1. The series converges for any real value of x

The constant R, if it exists, is called the **radius of convergence**. The **interval of convergence** of a power series, is the interval over which the series converges.

3.1.05 Power Series Convergence Example

Example

Determine the radius and interval of convergence of the infinite series

$$\sum_{k=0}^{\infty} \frac{k(x+3)^k}{4^{k+1}}$$

Complete Solution

Step 1: Apply Ratio Test

The ratio test gives us:

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{(k+1)(x+3)^{k+1}}{4^{k+2}}\Bigg/\frac{k(x+3)^k}{4^{k+1}}\right| &= \lim_{k\to\infty}\left|\frac{(k+1)}{k}\cdot\frac{(x+3)^{k+1}}{(x+3)^k}\cdot\frac{4^{k+1}}{4^{k+2}}\right| \\ &= \lim_{k\to\infty}\left|\frac{(k+1)}{k}\cdot(x+3)\cdot\frac{1}{4}\right| \\ &= \left|\frac{x+3}{4}\right|\end{aligned}$$

The ratio test tells us that the power series converges only when

$$\left|\frac{x+3}{4}\right| < 1$$

or $|x+3| < 4$. Therefore, the **radius of convergence** is 4.

Step 2: Test End Points of Interval to Find Interval of Convergence

The inequality $|x+3| < 4$can be written as -7 < x < 1. By the ratio test, we know that the series converges on this interval, but we don't know what happens at the points x = -7 and x = 1.

At x = -7, we have the infinite series

$$\sum_{k=0}^{\infty} \frac{k(-4)^k}{4^{k+1}} = \sum_{k=0}^{\infty} \frac{k(-1)^k}{4}$$

This series diverges by the test for divergence.

At x = 1, we have the infinite series

$$\sum_{k=0}^{\infty} \frac{k4^k}{4^{k+1}} = \sum_{k=0}^{\infty} \frac{k}{4}$$

This series also diverges by the test for divergence.

Therefore, the interval of convergence is -7 < x < 1.

Possible Challenges

What Convergence Test Should Be Used?

For most problems, the ratio test can be used initially. If the ratio test yields an *interval* for the domain, we need to use other convergence tests to explore what the domain could be at the end points of the interval.

3.1.06 Videos

Ratio Test - Radius of Convergence
In this video, a set of examples are explored where the radius of convergence of power series is calculated using the ratio test.
This video can be found on the MIT Opencourseware website [1], and carries a Creative Commons copyright (CC BY-NC-SA).

Power Series - Part 1
In this video, the instructor reviews basic definitions of a power series. It also presents two examples of determining the radius and interval of convergence of a series. The ratio, divergence, p-series, and alternating series convergence tests are applied.
More MathIsPower4U videos available here [2].

Power Series - Part 2
This video covers an example of finding the radius and interval of convergence of a power series that is not centered at zero.
More MathIsPower4U videos available here [2].

References

[1] http://ocw.mit.edu/index.htm
[2] http://mathispower4u.yolasite.com

3.2 Taylor Series

What This Lesson Covers

In this lesson we will introduce a few concepts:

- representation of functions with a power series
- conditions for convergence of a power series
- radius of convergence of a power series

Learning Objectives

After reading this lesson and completing a sufficient number of exercices on paper, you should be able to

- expand a given function in a Maclaurin series
- determine the radius of convergence of a Taylor series

Topics

1. A motivating problem
2. The Taylor and Maclaurin expansions
3. The Maclaurin expansion of e^x
4. The Maclaurin expansion of $\sin(x)$
5. The Maclaurin expansion of $\cos(x)$
6. A list of Maclaurin expansions
7. Videos on Taylor Series

3.2.01 A Motivating Problem

A Motivating Problem

In the previous lesson, we found that **if** a power series

$$\sum_{k=0}^{\infty} a_k(x-c)^k$$

has a radius of convergence, *R*, **then** the infinite series represents a function. But, can we instead ask the opposite question? That is, if we were given a function, *f*(*x*), can we find a power series representation for that function? Moreover, what properties does our function have to have in order to have this power series representation? These are questions that we can answer with a short mathematical exploration.

The Taylor Series

Let's assume we are given a function, *f*(*x*), and we will see if there exists a power series representation for that function.

We begin by **assuming** that we can let *f*(*x*) be represented by a power series:

$$f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$$

But, we do not have the coefficients a_0, a_1, a_2, ... and need to find a way to calculate them.

As a first step, substituting $x = c$ into our power series yields $a_0 = f(c)$. To get the next coefficient, a_1, assuming we are able to differentiate our given function, we would obtain

$$f'(x) = 0 + a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \dots$$

Substituting $x = c$ into this expression gives us an expression for a_1,

$$a_1 = \frac{df}{dx}(c)$$

So far, we have two of the coefficients, a_0 and a_1. Continuing with this process, provided that we are able to differentiate our given function again, we would obtain:

$$f''(x) = 0 + 2a_2 + +6a_3(x-c) + \dots$$

Again, substituting $x = c$ into this expression yields

$$a_2 = \frac{1}{2}\frac{d^2 f}{dx^2}(c)$$

Continuing with this process, we can find as many coefficients as we need, provided that we can differentiate our given function as many times as needed. In other words, if we can differentiate our function *N* times, we can calculate *N* power series coefficients, and calculate the series expansion coefficients up to *N* terms. If we can differentiate our function an **infinite** number of times, we can find a power series expansion with an infinite number of terms.

The specific power series representation we introduced here derived what is known as the **Taylor Series** expansion. We will later discuss the conditions that are necessary for a function to have a Taylor series.

Key Questions for This Lesson

We put these conclusions together into a definition on the next page. But from our simple investigation, we can now ask:

1. are there more efficient ways of calculating the power series expansion of a given function, and
2. does the power series expansion that we find for a given function **equal** the given function?

You should be able to find answers to both of these questions within this lesson.

3.2.02 The Taylor Series

We have now found a method to calculate power series expansions for functions who have a certain property: that they have derivatives at all orders. Functions that have derivatives at all orders, on some open interval, are referred to as **analytic** on that interval. Functions that are analytic on an interval have what is called a **Taylor series expansion**.

Definition: The Taylor Series Expansion
Suppose that a given function, $f(x)$, is analytic on an **open** interval that contains the point $x = c$. The **Taylor series expansion for *f*(*x*) at *c*** is $\sum_{k=0}^{\infty} a_k (x-c)^k,$ and the coefficients of the series, a_k are given by $a_k = \frac{f^{(k)}(c)}{k!}$ Here we are using the notation $f^{(k)}$ to denote the k^{th} derivative of the given function, $f(x)$.

The Taylor series obtained when we let $c = 0$ is referred to a **Maclaurin series**.

When a Function Equals its Taylor Series

It is possible to show that if a given function is analytic on some interval, then it is *equal* to its Taylor series on that interval. That is, on an interval where $f(x)$ is analytic,

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

We will not prove this result here, but the proof can be found in most first year calculus texts. The proof involves

- a derivation for an expression for the difference, or remainder, between f and the N^{th} order partial sum of a power series expansion, and
- shows that if and only if the remainder goes to zero when N goes to infinity, the Taylor series converges to $f(x)$.

There are functions that are not equal to its Taylor series expansion. But **for the purposes of this module, we will assume that all functions can be expanded as a Taylor series**.

3.2.03 The Maclaurin Series Expansion for exp

Example

Find the Taylor series expansion for e^x when x is zero, and determine its radius of convergence.

Complete Solution

Before starting this problem, note that the Taylor series expansion of any function about the point $c = 0$ is the same as finding its Maclaurin series expansion.

Step 1: Find Coefficients

Let $f(x) = e^x$. To find the Maclaurin series coefficients, we must evaluate

$$\left(\frac{d^k}{dx^k}f(x)\right)\bigg|_{x=0}$$

for $k = 0, 1, 2, 3, 4, \ldots.$

Because $f(x) = e^x$, then all derivatives of $f(x)$ at $x = 0$ are equal to 1. Therefore, all coefficients of the series are equal to 1.

Step 2: Substitute Coefficients into Expansion

By substitution, the Maclaurin series for e^x is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Step 3: Radius of Convergence

The ratio test gives us:

$$\lim_{k\to\infty} \left| \frac{x^{k+1}}{(k+1)!} \Big/ \frac{x^k}{k!} \right| = \lim_{k\to\infty} \frac{|x|}{k+1} = 0$$

Because this limit is zero for all real values of x, the radius of convergence of the expansion is the set of all real numbers.

Explanation of Each Step

Step 1

Maclaurin series coefficients, a_k are always calculated using the formula

$$a_k = \frac{f^{(k)}(0)}{k!}$$

where f is the given function, and in this case is $e(x)$. In step 1, we are only using this formula to calculate coefficients. We found that all of them have the same value, and that value is one.

Step 2

Step 2 was a simple substitution of our coefficients into the expression of the Taylor series given on the previous page.

Step 3

This step was nothing more than substitution of our formula into the formula for the ratio test. Because we found that the series converges for all x, we did not need to test the endpoints of our interval. If however we did find that the series only converged on an interval with a finite width, then we may need to take extra steps to determine the convergence at the boundary points of the interval.

Possible Challenges

What are we Doing?

The following video provides a graphical interpretation of the Taylor approximation to e^x about the point $c = 3$. In the derivation above, we considered an expansion at $c = 0$. The instructor uses the term "Taylor approximation" in the same way we use the term "Taylor expansion".

Visualizing Taylor Series for e^x
A graphical description of the Taylor series of e^x.
This video can be found on the Kahn Academy website [1], and carries a Creative Commons copyright (CC BY-NC-SA 3.0).

Summary

In this example, we found the Macluarin expansion of the exponential function.

The Maclaurin Expansion of e^x
The Maclaurin series expansion for e^x is given by $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ This formula is valid for all real values of x.

References

[1] http://www.khanacademy.org/

3.2.04 The Maclaurin Series Expansion for sin

Example

Find the Taylor series expansion for sin(*x*) at *x* = 0, and determine its radius of convergence.

Complete Solution

Again, before starting this problem, we note that the Taylor series expansion at *x* = 0 is **equal** to the Maclaurin series expansion.

Step 1: Find Coefficients

Let *f*(*x*) = sin(*x*). To find the Maclaurin series coefficients, we must evaluate

$$\left(\frac{d^k}{dx^k}\sin(x)\right)\Bigg|_{x=0}$$

for *k* = 0, 1, 2, 3, 4, ...

Calculating the first few coefficients, a pattern emerges:

$$\begin{aligned} f(0) &= \sin(0) = 0 \\ f'(0) &= \cos(0) = 1 \\ f''(0) &= -\sin(0) = 0 \\ f'''(0) &= -\cos(0) = -1 \\ f^{(4)}(0) &= \sin(0) = 0 \\ f^{(5)}(0) &= \cos(0) = 1 \end{aligned}$$

The coefficients alternate between 0, 1, and -1. You should be able to, for the n^{th} derivative, determine whether the n^{th} coefficient is 0, 1, or -1.

Step 2: Substitute Coefficients into Expansion

Thus, the Maclaurin series for sin(*x*) is

$$\begin{aligned} \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}(x-0)^k &= f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\dots \\ &= 0+\left(\frac{1}{1!}x\right)+0+\left(\frac{-1}{3!}x^3\right)+0+\left(\frac{1}{5!}x^5\right)+\dots \\ &= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots \end{aligned}$$

Step 3: Write the Expansion in Sigma Notation

From the first few terms that we have calculated, we can see a pattern that allows us to derive an expansion for the n^{th} term in the series, which is

$$\frac{(-1)^n}{(2n+1)!}x^{2n+1}, \quad n = 0, 1, 2, 3, \ldots$$

Substituting this into the formula for the Taylor series expansion, we obtain

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

Radius of Convergence

The ratio test gives us:

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{(-1)^{k+1}}{(2(k+1)+1)!}x^{2(k+1)+1} \Big/ \frac{(-1)^k}{(2k+1)!}x^{2k+1}\right| &= \lim_{k\to\infty}\frac{(2k+1)!}{(2k+3)!}|x|^2 \\ &= \lim_{k\to\infty}\frac{1}{(2k+3)(2k+2)}|x|^2 \\ &= 0\end{aligned}$$

Because this limit is zero for all real values of x, the radius of convergence of the expansion is the set of all real numbers.

Explanation of Each Step

Step 1

Maclaurin series coefficients, a_k can be calculated using the formula (that comes from the definition of a Taylor series)

$$a_k = \frac{f^{(k)}(0)}{k!}$$

where f is the given function, and in this case is $\sin(x)$. In step 1, we are only using this formula to calculate the first few coefficients. We can calculate as many as we need, and in this case were able to stop calculating coefficients when we found a pattern to write a general formula for the expansion.

Step 2

Step 2 was a simple substitution of our coefficients into the expression of the Taylor series.

Step 3

A helpful step to find a compact expression for the n^{th} term in the series, is to write out more explicitly the terms in the series that we have found:

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}(x-0)^k = (+1)\cdot\frac{x^1}{1!} + (-1)\cdot\frac{x^3}{3!} + (+1)\cdot\frac{x^5}{5!} + \ldots$$

We have discovered the sequence 1, 3, 5, ... in the exponents and in the denominator of each term. We may then find a way to convert this sequence that we have discovered, into the sequence k=0, 1, 2, ... that appears in the final summation. The simple transform $2k+1$ performs this transformation for us.

Step 4

This step was nothing more than substitution of our formula into the formula for the ratio test. Because we found that the series converges for all *x*, we did not need to test the endpoints of our interval. If however we did find that the series only converged on an interval with a finite width, then we may need to take extra steps to determine the convergence at the boundary points of the interval.

An Alternate Explanation

The following Patrick JMT video provides a similar derivation of the Maclaurin expansion for sin(x) that you may find helpful.

Sine Taylor Series at 0
Derivation of the Maclaurin series expansion for sin(*x*).
This video can be found on the Kahn Academy website [1], and carries a Creative Commons copyright (CC BY-NC-SA 3.0).

Possible Challenges

What if we Need the Taylor Series of sin(x) at Some Other Point?

The Maclaurin series of sin(x) is **only** the Taylor series of sin(x) at $x = 0$. If we wish to calculate the Taylor series at any **other** value of x, we can consider a variety of approaches.

Suppose we wish to find the Taylor series of sin(x) at $x = c$, where c is any real number that is not zero. We could find the associated Taylor series by applying the same steps we took here to find the Macluarin series. That is, calculate the series coefficients, substitute the coefficients into the formula for a Taylor series, and if needed, derive a general representation for the infinite sum.

Another approach could be to use a trigonometric identity. Consider this approach

$$\begin{aligned}\sin(x) &= \sin(x+c-c)\\ &= \sin(x+c)\cos(c) - \cos(x+c)\sin(c)\\ &= \sin(u)\cos(c) - \cos(u)\sin(c), \quad u = x+c\end{aligned}$$

The functions cos(u) and sin(u) can be expanded in with a Maclaurin series, and cos(c) and sin(c) are constants. We will see the Maclaurin expansion for cosine on the next page.

How Many Terms do I Need to Calculate?

It can be difficult to find an expression for the n^{th} term in the series that allows us to write out a compact expression for an infinite sum. In our example here, we only calculated three terms. It may be helpful in other problems to write out a few more terms to find a useful pattern.

Summary

To summarize, we found the Macluarin expansion of the sine function.

The Maclaurin Expansion of sin(x)

The Maclaurin series expansion for sin(x) is given by

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

This formula is valid for all real values of x.

3.2.05 The Maclaurin Series Expansion for cos

Example

Find the Maclaurin series expansion for cos(x) at $x = 0$, and determine its radius of convergence.

Complete Solution

Step 1: Find the Maclaurin Series

$$\begin{aligned}
\cos(x) &= \frac{d}{dx}\sin(x) \\
&= \frac{d}{dx}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \\
&= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}
\end{aligned}$$

Step 2: Find the Radius of Convergence

The ratio test gives us:

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{(-1)^{k+1}}{(2(k+1))!}x^{2(k+1)} \Bigg/ \frac{(-1)^k}{(2k)!}x^{2k}\right| &= \lim_{k\to\infty}\frac{(2k)!}{(2k+2)!}|x|^2 \\
&= \lim_{k\to\infty}\frac{1}{(2k+1)(2k+2)}|x|^2 \\
&= 0
\end{aligned}$$

Because this limit is zero for all real values of x, the radius of convergence of the expansion is the set of all real numbers.

Explanation of Each Step

Step 1

To find the series expansion, we *could* use the same process here that we used for sin(x) and e^x. But there is an easier method. We can differentiate our known expansion for the sine function.

If you **would** like to see a derivation of the Maclaurin series expansion for cosine, the following video provides this derivation.

Cosine Taylor Series at 0
Derivation of the Maclaurin series expansion for cosine.
This video can be found on the Kahn Academy website [1], and carries a Creative Commons copyright (CC BY-NC-SA 3.0).

Step 2

This step was nothing more than substitution of our formula into the formula for the ratio test.

Possible Challenges

When Can We Differentiate a Power Series?

For the purposes of this module, we will always assume that we can. There is however a theorem on differentiating and integrating power series, which you are not expected to know, that tells us that a power series can only be differentiated if it has a radius of convergence that is greater than zero.

A page in this module on this theorem is coming. In the mean time, this page on Wikipedia may help [1].

Did We Have to Test for Convergence?

The short answer is: no. The theorem mentioned above [1] tells us that, because

- we derived the series for cos(x) from the series for sin(x) through differentiation, and
- we already know the radius of convergence of sin(x),

the radius of convergence of cos(x) will be the same as sin(x). However, we haven't introduced that theorem in this module. You may want to ask your instructor if you are expected to know this theorem.

Summary

The Maclaurin Expansion of cos(x)
The Maclaurin series expansion for cos(x) is given by $\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots$ This formula is valid for all real values of x.

References

[1] http://en.wikipedia.org/wiki/Power_series

3.2.06 Maclaurin Expansions

The following table of Maclaurin expansions summarizes our results so far, and provides expansions for other series that we have not covered.

The Maclaurin Expansions of Elementary Functions
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$
$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$
$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
$\sinh(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$
$\cosh(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$

3.2.07 Videos

Finding Taylor's Series
A set of three examples of calculating Taylor series expansions using known formulas for expansions of elementary functions.
This video can be found on the MIT Opencourseware website [1], and carries a Creative Commons copyright (CC BY-NC-SA).

Taylor's Series of a Polynomial
In this video, the Taylor expansion of a specific polynomial is determined.
This video can be found on the MIT Opencourseware website [1], and carries a Creative Commons copyright (CC BY-NC-SA).

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