LINEAR MOTION Q1

## SWING

I WANT TO SWING A SWING AS HIGH AS I CAN WITH A FIXED MAXIMUM FORCE. AT WHAT FREQUENCY SHOULD THE FORCE BE?

A. THE FREQUENCY OF THE SWING IF THERE WERE NO DAMPING.

B TWICE $\omega_A$

C THE FREQUENCY OF THE SWING IF I SET IN MOTION AND LET IT BE

D TWICE $\omega_0$

E. NONE OF THE ABOVE

LINEAR MOTION #7

# HARMONIC OSCILLATORS

- EXPLAIN WHY UNDERSTANDING HARMONIC MOTION IS USEFUL EVEN THOUGH MOST SYSTEMS ARE NOT HARMONIC OSCILLATORS
- SOLVE THE HARMONIC OSCILLATOR WITH DAMPING AND DRIVING FORCES

# LINEAR MOTION

#8

- Near an equilibrium the potential energy function can be written

$$V(x) = V_0 + V'(0)x \quad + \frac{1}{2}V''(0)x^2 + \cdots$$

Since the potential energy is arbitrary up to a constant, we can take $V_0 = 0$.

Because $x=0$ is a point of equilibrium, $V'(0)=0$.

This leaves

$$V(x) \approx \frac{1}{2}V''(0)x^2 .$$

and the force

$$\boxed{F = -V''(0)x = m\ddot{x}}$$

# EQUATION OF MOTION

- Let's look for solutions to the EOM,

$$m\ddot{x} + V'' x = 0$$

This is a <u>linear differential equation</u> **LDEWCC** with <u>constant coefficients</u>.

(1) Only the unknown function and its derivatives appear (no powers etc)

(2) The function and derivatives all are multiplied by constants.

# SOLVING THE EOM

- Such differential equations can be converted to polynomial equations using the guess:

$$x(t) = Ae^{pt}$$

Furthermore because it is linear, solutions can be built by summing other solutions

$$\boxed{\text{SUPERPOSITION}}$$

$$Amp^2e^{pt} + AV''e^{pt} = 0$$

$$\Downarrow$$

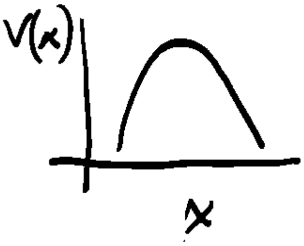
$$mp^2 + V'' = 0$$

$$\Downarrow$$

$$p = \pm\sqrt{-\frac{V''}{m}}$$

# THE SOLUTIONS

- If $V'' < 0$, we have exponentially growing and decaying solutions.

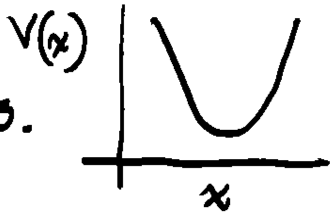


$$x(t) = Ae^{\alpha t} + Be^{-\alpha t}$$

where $\alpha = \sqrt{\frac{-V''}{m}}$

$$\boxed{\text{UNSTABLE}}$$

- If $V'' > 0$, we have oscillatory solutions.



$$x(t) = Ae^{i\omega t} + Be^{-i\omega t}$$

where $\omega = \sqrt{\frac{V''}{m}}$.

To make sense of this use $e^{i\theta} = \cos\theta + i\sin\theta$

$$x(t) = (A+B)\cos\omega t + i(A-B)\sin\omega t$$

For the solution to be real, $A = \bar{B}$

# THE DAMPED OSCILLATOR

- If we have viscous damping the EOM is

$$m\ddot{x} + \lambda\dot{x} + kx = 0$$

where $k = V''$.

Again this is a LDEWCC, so we can try $x(t) = Ae^{pt}$ to get

$$mAp^2e^{pt} + \lambda pAe^{pt} + kAe^{pt} = 0$$

$$mp^2 + \lambda p + k = 0$$

$$\Downarrow$$

$$p = \frac{-\lambda \pm \sqrt{\lambda^2 - 4km}}{2m}$$

$$= -\underset{\substack{\uparrow \\ \gamma \\ \text{damping} \\ \text{constant}}}{\frac{\lambda}{2m}} \pm \sqrt{\left(\frac{\lambda}{2m}\right)^2 - \underset{\substack{\uparrow \\ \omega_0^2 \\ \text{undamped} \\ \text{frequency}}}{\frac{k}{m}}}$$

## LINEAR MOTION

## STRONG DAMPING

- If $\gamma > \omega_0$ then both solutions are real

$$p = -\gamma_\pm \quad \text{with} \quad \gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

$$x = \underset{\substack{\uparrow \\ \text{Dies faster}}}{\tfrac{1}{2} A e^{-\gamma_+ t}} + \underset{\substack{\uparrow \\ \text{Dominates} \\ \text{at late times}}}{\tfrac{1}{2} B e^{-\gamma_- t}}$$

## WEAK DAMPING

- If $\gamma < \omega_0$ then the solutions are complex

$$p = -\gamma \pm i\omega \quad \text{with} \quad \omega = \sqrt{\omega_0^2 - \gamma^2}$$

$$x = \tfrac{1}{2} A e^{i\omega t - \gamma t} + \tfrac{1}{2} B e^{-i\omega t - \gamma t}$$

$$= a e^{-\gamma t} \cos(\omega t - \Theta)$$

$$= A e^{-\gamma t} \cos(\omega t) + B e^{-\gamma t} \sin(\omega t)$$

- If $\gamma = \omega_0$ then $\omega \to 0$ **CRITICAL**

$$x = A e^{-\gamma t} + B\omega t\, e^{-\gamma t} \rightarrow x = (a + bt) e^{-\gamma t}$$

# DRIVEN OSCILLATORS

- What if we apply a force to the oscillator?

$$m\ddot{x} + \lambda\dot{x} + kx = F(t)$$

Because the equation is linear we can add the solution to the unforced oscillation to the particular solution with the forcing.

Let's try $F(t) = F_1 e^{i\omega_1 t}$ because any $F(t)$ can be written as a sum of harmonic oscillations.

$$m\ddot{x} + \lambda\dot{x} + kx = F_1 e^{i\omega_1 t}$$

and a solution $x_p = A_1 e^{i\omega_1 t}$ to get

$$-m\omega_1^2 A_1 + i\lambda A_1\omega_1 + kA_1 = F_1$$

$$A_1 = \frac{F_1}{m} \frac{1}{\omega_0^2 - \omega_1^2 + i(2\gamma\omega_1)}$$

$$A_1 = a_1 e^{-i\Theta_1} \quad a_1 = \frac{F_1}{m} \frac{1}{\sqrt{(\omega_0^2-\omega_1^2)^2 + 4\gamma^2\omega_1^2}}$$

# LINEAR MOTION ✓

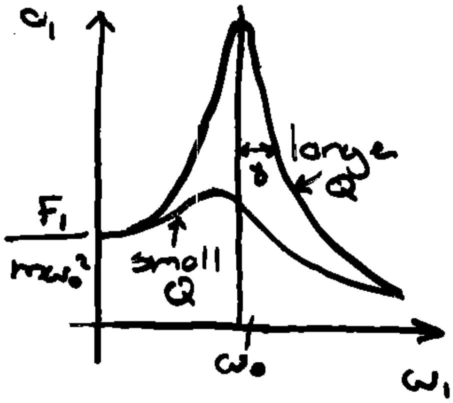
- The complete solution is:

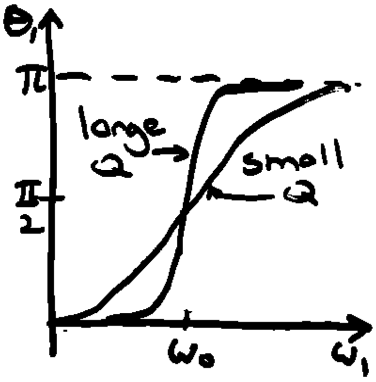
$$x = a_1 \cos(\omega_1 t - \Theta_1) + a e^{-\gamma t} \cos(\omega t - \Theta)$$

with

$$a_1 = \frac{F_1}{m}\left((\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2\right)^{-1/2}$$

$$\tan\Theta_1 = \frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2}$$





$$\boxed{Q = \frac{\omega_0}{2\gamma}}$$

LINEAR MOTION

# GENERAL DRIVING FOR

(1) A general force can be written as a sum of sinusoidal forces:

$$F(t) = \int_{-\infty}^{\infty} F(\omega_1) e^{i\omega_1 t} d\omega_1$$

$$\Downarrow \text{ By superposition}$$

$$x(t) = \int_{-\infty}^{\infty} A(\omega_1) e^{i\omega_1 t} d\omega_1 + \text{transient}$$

where

$$A(\omega_1) = \frac{F(\omega_1)}{m} \frac{1}{\omega_0^2 - \omega_1^2 + i(2\gamma\omega_1)}$$

# GENERAL DRIVING FORCE

(2) A general force can be expressed as a sum of impulses.

$$\boxed{\text{A impulse is a sudden change to the momentum of a system}}$$

$$\Delta p = p(t'+\Delta t) - p(t') = \int_{t'}^{t'+\Delta t} F\,dt$$

Let's take $\Delta t \to 0$, $F \to \infty$ such that $I = \Delta p$ finite.

We get

$$x = \begin{cases} 0, & t < t' \\ \frac{I}{m\omega} e^{-\gamma(t-t')} \sin\omega(t-t') & t > t' \end{cases}$$

Remember that $I = \int_{t'}^{t'+\Delta t} F\,dt$

so we can write $x(t)$ as an integral of

# GREEN'S FUNCTION

If we define

$$G(t-t') = \begin{cases} 0 & t<t' \\ \frac{1}{m\omega} e^{-\gamma(t-t')} \sin\omega(t-t') & t>t' \end{cases}$$

we find that

$$x(t) = \int_{t_0}^{t} G(t-t') F(t') dt' + \text{transient}$$

where $t_0$ is some time when you know the initial condition of the oscillator.

This will work for an system with linear differential equation. Fourier method only works for harmonic oscillator.