600D: ASSIGNMENT 1
SEP 15, 2011

Due date: Sep 27, 2011

(1) Let R be a ring (not necessarily commutative) and $I \subset R$ an ideal. The ideal I is said to be *idempotent lifting* if every idempotent of R/I has the form $x + I$ for some idempotent x in R.

Show that if J is a two sided ideal in R such that J is a nil ideal (i.e. has the property that every x in J satisfies $x^n = 0$ for some n) or that R is J-adically complete, then J is idempotent lifting.

Step 1: For a in R and any $n > 0$, note

$$1 = (a + (1-a))^{2n} = \sum_{j=0}^{2n} \binom{2n}{j} a^{2n-j}(1-a)^j.$$

Define

$$f_n(a) = \sum_{j=0}^{n} \binom{2n}{j} a^{2n-j}(1-a)^j = 1 - \sum_{j=n+1}^{2} n \binom{2n}{j} a^{2n-j}(1-a)^j.$$

Show that $f_n(a) \equiv 0 \bmod a^n R$ and $f_n(a) \equiv 1 \bmod (1-a)^n R$, and deduce that $f_n(a)^2 \equiv f_n(a) \bmod (a(1-a))^n R$.

Step 2: Show that $f_n(a) \equiv f_{n-1}(a) \bmod (a(1-a))^{n-1} R$ and $f_n(a) \equiv a \bmod (1-a)R$.

Step 3: If $a^2 - a$ is nilpotent, use the congruences above to show that for large n, we have $f_n(a) \equiv a \bmod (a^2 - a)R$ and $f_n(a)^2 = f_n(a)$, showing that one can lift idempotents modulo a nil ideal J.

Step 4: If R is J-adically complete, then inductively construct $e_n \in R$ such that $e_1 = a, \cdots, e_n^2 \equiv e_n \bmod J^n$, and $e_{n+1} \equiv e_n \bmod J^n$. (Hint: J/J^n is nilpotent). Using completeness, we get an idempotent e in R such that $e \equiv a \bmod J$.

(2) Let F be a number field, i.e. a finite algebraic extension of $\mathbb{Q}$ and let R be the ring of algebraic integers in F. Show that the class group of R is finite.

(3) If R is a ring and p is an idempotent of $M_n(R)$, show that p defines a projective R-module. Let e and e_1 be two idempotents in $M_n(R)$. Show that the corresponding projective modules P and P_1 are isomorphic if $e_1 = geg^{-1}$ for some $g \in GL_n(R)$. Conversely, if $P \simeq P_1$, show that for some $g \in GL_{2n}(R)$, we have

$$\begin{pmatrix} e_1 & 0 \\ 0 & e_1 \end{pmatrix} = g \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} g^{-1}.$$