

Sample Midterm 1

Name:

October 9, 2012

Mathematics 100-180

Page 2 of 7 pages

Marks:

Short-Answer Questions. Questions 1 and 2 are short-answer questions. Put your answer in the box provided. Full marks will be given for a correct answer placed in the box. Show your work also, for part marks. Each part is worth 3 marks, but not all parts are of equal difficulty. Simplify your answers as much as possible in Questions 1 and 2.

- [9] 1. Determine whether each of the following limits exists, and find the value if they do. If a limit below does not exist, determine whether it "equals" ∞ , $-\infty$, or neither.

(a) [3] $\lim_{x \rightarrow -1} \frac{x^2 - x - 2}{x^2 - 1}$

$$\frac{x^2/x^2 - x/x^2 - 2/x^2}{x^2/x^2 - 1/x^2} = \frac{1 - 1/x - 2/x^2}{1 - 1/x^2} = \frac{1 - 1/1 - 2/1}{1 - 1/1} = \frac{1 - 1 - 2}{1 - 1} = \frac{-2}{0} = \infty$$

Answer ∞ .

(b) [3] $\lim_{t \rightarrow 1} \frac{\sqrt{t^2 + 8} - 3}{t - 1}$

$$\frac{\sqrt{t^2 + 8} - 3}{t - 1} = \frac{t + 2 - 3}{t - 1}$$

$$= \frac{t + 2 - 3}{t - 1} = \frac{0}{0} \text{ L.H.}$$

$$\frac{1}{2} \frac{2t}{\sqrt{t^2 + 8}} = \frac{1}{\sqrt{t^2 + 8}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$$

Answer $1/3$.

(c) [3] $\lim_{x \rightarrow 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

$x \rightarrow 0^-$
 $x < 0$
 $|x| > 0$

$$\frac{1}{x} - \frac{1}{|x|}$$

$$\frac{1}{-0.01} - \frac{1}{|0.01|} = \frac{1}{-0.01} - \frac{1}{0.01} = \frac{-2}{0.01} = \text{big}$$

Answer ∞ .

Continued on page 3

- [12] 2. (a) [3] Find $\frac{d}{dx} \left(\frac{x^4 + x^{7/2}}{x^2} \right)$. Remember (see the instructions above Question 1) that your answers must be completely simplified in Questions 1 and 2.

Answer $\frac{4x^5 + \frac{7}{2}x^3 - 2x^5 + 2x^{9/2}}{x^4}$

$$\frac{(4x^3 + \frac{7}{2}x)x^2 - (x^4 + x^{7/2}) \cdot 2x}{(x^2)^2} = \frac{4x^5 + \frac{7}{2}x^3 - 2x^5 + 2x^{9/2}}{x^4}$$

- (b) [3] If $y = x^2 \cos x$, find the second derivative y'' . Express your answer in the form $p(x) \sin x + q(x) \cos x$ where $p(x)$ and $q(x)$ are polynomials.

Answer $q(x) = 2 - x^2$ $p(x) = 4x^2$

$$y = x^2 \cos x$$

$$y' = 2x \cos x - x^2 \sin x$$

$$y'' = 2 \cos x - 2x \sin x - 2x \sin x - x^2 \cos x$$

$$= \cos x (2 - x^2) - \sin x (2x + 2x)$$

- (c) [3] If $y = x \sin(\sqrt{x} + x)$, find y' .

Answer $4x^{1/2}$

$$y' = \sin(\sqrt{x} + x) + x \cos(\sqrt{x} + x) \cdot \frac{1}{2} x^{-1/2} + 1$$

- (d) [3] f is a function that satisfies $f'(e) = e$. Let $g(x) = f(e^{x^2})$. Find $g'(1)$.

Answer e^2

$$f'(e) = e$$

$$g(x) = f(e^{x^2})$$

$$g'(x) = f'(e^{x^2}) \cdot e^{x^2}$$

$$g'(1) = f'(e) \cdot e$$

$$= e \cdot e = e^2$$

Full-Solution Problems. In questions 3-7, justify your answers and show all your work. If a box is provided, write your final answer there. Unless otherwise indicated, simplification of answers is not required in these questions.

[7] 3. Let $f(x) = \frac{\sqrt{x^2+1}}{3x+5}$.

- (a) [4] Determine the horizontal asymptotes of the graph $y = f(x)$.

Answer

$1/3$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2+1}}{3x+5} = \frac{x+1}{3x+5} = \frac{x}{3x} = \frac{1}{3}$$

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2+1}}{3x+5} = \frac{\sqrt{x^2+1}}{3x+5} = \frac{x^2+1}{(3x+5)\sqrt{x^2+1}}$$

Same for $-\infty$.

- (b) [3] Determine the vertical asymptote(s) of the graph $y = f(x)$. For each vertical asymptote $x = a$, determine whether each of the one-sided limits "equals" ∞ or $-\infty$ as x approaches a .

$$\lim_{x \rightarrow a^+} \frac{\sqrt{x^2+1}}{3x+5} = \frac{\sqrt{a^2+1}}{3a+5}$$

$a = 3x+5=0$ ← asymptote.
 $x = -5/3$

$$\lim_{x \rightarrow -5/3^+} \frac{\sqrt{x^2+1}}{3x+5} \approx \frac{\frac{1}{2}(x^2+1)^{-1/2} \cdot 2x}{3} = \frac{5(5/3^2+1)^{1/2}}{3}$$

$$\lim_{x \rightarrow -5/3^-} \frac{\sqrt{x^2+1}}{3x+5} = \frac{\sqrt{(-5/3)^2+1}}{-3 \cdot 5/3 + 5} = \frac{\sqrt{25/9+1}}{0} = \infty$$

Continued on page 5

- [6] 4. Let a and b be constants, and define

$$f(x) = \begin{cases} (x^2 + b) & \text{if } x < 1 \\ ax + b & \text{if } 1 \leq x \leq 2 \\ 5x - 3 & \text{if } x > 2 \end{cases}$$

Find the values of a and b for which f is continuous at $x = 1$ and $x = 2$. Fully justify your answer.

Answer

$$x^2 + b = ax + b$$

$$ax + b = 5x - 3$$

$$b = 5x - 3 - ax$$

$$2a + b = 10 - 3$$

$$b = 7 - 2a$$

$$7 - 2a = 5x - 3 - ax$$

$$10 - 3a = 6x$$

$$10 = 6x + 3a$$

$$\lim_{x \rightarrow 1} = x^2 + b = ax + b$$

$$1 + b = a + b$$

$$a = 1$$

$$b = 7 - 2a$$

$$b = 7 - 2 = 5$$

$$\begin{array}{l} a = 1 \\ b = 5 \end{array}$$



Name:

- [4] 5. Prove that the equation $x^3 - 3x + 1 = 0$ has at least two positive real solutions. Carefully cite any theorem you use, and justify why the theorem can be used.

the line will cross since it is neg. and then pos. again. but now pos. & neg.

$$x^3 - 3x + 1 = 0$$

$$x(x^2 - 3) + 1 = 0$$

$$x(x^2 - 3) = -1$$

$$x^2 - 3 = \frac{-1}{x}$$

$$(x + \sqrt{3})(x - \sqrt{3}) = \frac{-1}{x}$$

$$x + \sqrt{3} = \frac{-1}{x}$$

$$x - \sqrt{3} = \frac{-1}{x}$$

$x = 1: 1^3 - 3 \cdot 1 + 1 = 0$	\ominus
$1 - 3 + 1 = 0$	\ominus
neg. $\rightarrow -1 = 0$	
$x = 2: 2^3 - 3 \cdot 2 + 1 = 0$	\oplus
$8 - 6 + 1 = 0$	\oplus
pos. $\rightarrow 3 = 0$	
$x = 0: 0 - 3 \cdot 0 + 1 = 0$	\oplus
pos. $\rightarrow 1 = 0$	

- [6] 6. Find the point(s) on the curve $y = x^2 + 6$ such that the tangent line(s) to the curve at these point(s) pass through the point (2, 1).

Answer $(4, 22)$.

$$y' = 2x$$

$$y' = 2 = 2 \cdot 2 = 4$$

$$y' = 4$$

$$y = 4^2 + 6 = 16 + 6 = 22$$

[6] 7. Use the definition of the derivative in the questions below. No marks will be given for any other method. In this and any other question, you may use the back of a page if necessary, but please indicate so.

(a) [3] Find

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \frac{d}{dx} \left(\frac{1}{1-x} \right)$$

$$\lim_{h \rightarrow 0} = \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \rightarrow 0} = \frac{\frac{1}{1-x+h} - \frac{1}{1-x}}{h}$$

$$\lim_{h \rightarrow 0} = \frac{1-x - (1-x+h)}{(1-x+h)(1-x)h} = \frac{-2x+h}{(1-x+h)(1-x)h}$$

$$= \frac{-2x+h}{h(1-x+h)(1-x)}$$

$$= \frac{-2x}{(1-x)^2}$$

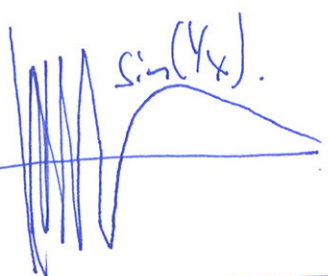
(b) [3] Determine whether the function

$$f(x) = \begin{cases} x^3 \sin \frac{1}{x} & \text{if } x < 0 \\ 4x^2 - x & \text{if } x \geq 0 \end{cases}$$

is differentiable at $x = 0$.

$$\lim_{x \rightarrow 0^-} \frac{3x^2 \sin(1/x) + x^3 \cos(1/x) \cdot (-1/x^2)}{3x^2} = \frac{\cos(1/x) \cdot -x^{-2}}{-3x^{-4}}$$

$$= \frac{\cos(1/x)}{-3} \cdot \frac{x^4}{x^2} \cdot x^2$$



$$\lim_{h \rightarrow 0} \frac{(x+h)^3 \sin(\frac{1}{x+h}) - x^3 \sin(\frac{1}{x})}{h} \quad | \quad f' = 3x^2 \sin(1/x) + x^3 \cos(1/x) \cdot (-1/x^2)$$

The End

$$f' = 3 \cdot 0 \cdot \sin(1/0) + \cos(1/0) \cdot \frac{1}{0}$$

$$f' = 8x - 1 = 8 \cdot 0 - 1 = -1$$

turn over.

$$f' = 3x^2 \sin(1/x) - x^3 \cos(1/x) \frac{1}{x^2}$$

$$= 3x^2 \sin(1/x) - x \cos(1/x).$$

$$\lim_{x \rightarrow 0} = 3x^2 \sin(1/x) - x \cos(1/x).$$



$$\lim_{x \rightarrow 0} = 0.$$

$0 \neq -1$ since the derivative D.N.E.

No!