ISM Unit 1

Sequences and Series

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Unit 1

Unit 1 contains five lessons.

1. infinite sequences
2. sigma notation
3. introduction to infinite series
4. properties of infinite series
5. the telescoping and harmonic series

It is recommended that the student who is unfamliar with this material progress through these lessons in the above order, as each lesson builds upon each other.

1.1 Infinite Sequences

What This Lesson Covers

In this lesson we introduce

- the definition of convergence of an infinite sequence
- various theorems on limits of infinite sequences, including
- the theorem relating between the limit of a function and a limit of a sequence
- the limit laws for infinite sequences
- the squeeze theorem for infinite sequences
- the theorem on the limit of the absolute value of a sequence

Learning Outcomes

After reading this lesson and completing a sufficient number of problems, students should be able to

- determine whether an infinite sequence converges using the theorems presented in this lesson

Simply reading the content in this lesson will not be sufficient. Students will need to get out a pencil and paper and complete problems in order to prepare themselves for assessments related to this lesson.

Topics

It is recommended that students progress through this lessons in the given order, as the sections build upon each other.

1. Introduction to Infinite Sequences
2. Convergence of Infinite Sequences
3. Convergence of Infinite Sequences Example
4. Relationship to Limits of Functions
5. Limit Laws for Infinite Sequences
6. Limit Laws Example
7. Relationship to Sequences of Absolute Values
8. Example Relating Sequences of Absolute Values
9. Squeeze Theorem for Sequences

1.1.01 Introduction to Infinite Sequences

An infinite sequence of numbers is an ordered list of numbers. Examples could include

1. the positive integers: 1, 2, 3, 4, ...
2. the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, ...
3. a sequence defined by Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ x_0 \text{ given}$$

For the purposes of the ISM, we will only consider sequences with real numbers.

Notation

We often will use the notation

$$\{a_n\}_{n=1}^{\infty}$$

to denote a general infinite sequence, whose terms are a_1, a_2, a_3, ... For example, in the case of the Fibonacci sequence,

$a_1 = 1$

$a_2 = 1$

$a_3 = 2$

$a_4 = 3$

Key Questions for This Lesson

Given a sequence of numbers, $\{a_n\}_{n=1}^{\infty}$, we could ask the following questions:

- How do we know if the infinite sequence converges to a finite number?
- If the given infinite sum does yield a finite number, what is it?

These two questions and their answers are the subject of this lesson.

Examples

If we consider examples 1 and 2 above, then we can see that **by inspection**, the sequences does not converge to a finite number because successive terms in the sequences are increasing.

However in some cases, it can be more difficult to establish whether the sequence converges. Depending on the form of $f(x)$ in Example 3 above (Newton's Method), it can be difficult or impossible for us to determine whether this sequence converges by inspection. Indeed, we need a more rigorous method to establish convergence, which we will explore next.

1.1.02 Convergence of Infinite Sequences

Our next task is to establish, given an infinite sequence, whether or not it converges. Knowing whether or not a given infinite sequence converges requires a definition of convergence.

Definition: Convergence of an Infinite Sequence
Suppose we are given an infinite sequence $\{a_n\}_{n=1}^{\infty}$. This sequence has a limit *L*, if a_n approaches *L* as *n* approaches infinity. We write this as $\lim_{n \to \infty} a_n = L, \quad \text{or } a_n \to \infty \text{ as } n \to \infty$ Moreover, if the number *L* exists, it is referred to as the **limit** of the sequence and the sequence is **convergent**. A sequence that is not convergent is **divergent**.

The above definition could be made more precise with a more careful definition of a limit, but this would go beyond the scope of what we need. But our definition provides us with a method for testing whether a given infinite sequence converges: if the limit

$$\lim_{n \to \infty} a_n$$

tends to a finite number, the sequence converges. Otherwise, it diverges.

1.1.03 Convergence of Infinite Sequences Example

Example

Determine whether the sequences

$$a_n = \frac{1}{2 + \frac{1}{n}}$$

and $b_n = (-1)^n$ converge.

Complete Solution

The Sequence a_n

Using the definition of convergence of an infinite sequence, we would evaluate the following limit:

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{2 + \frac{1}{n}} = \frac{1}{2}$$

Because this limit evaluates to a single finite number, the sequence **converges**.

The Sequence b_n

While the sequence a_n converges to 1/2, b_n does not converge because its terms do not approach any number. Instead, the terms in the sequence oscillate between -1 and +1.

The sequence b_n does not converge to a unique number, and so b_n does not converge.

1.1.04 Relationship to Limits of Functions

Our definition of convergence for an infinite sequence may look like the definition of a limit for functions.

Theorem: Relation to Limits of Functions
If we have that $\lim_{n\to\infty} f(x) = L$ and $f(n) = a_n$, where x is a real number and n is an integer, then $\lim_{n\to\infty} a_n = L.$

The above theorem makes it easier for us to evaluate limits of sequences.

Simple Example

Suppose we wish to determine whether the sequence

$$a_n = \frac{1}{n^p}$$

converges, where p is any positive integer.

Since we know that

$$\lim_{x\to\infty} \frac{1}{x^p} = 0, \quad x \in \mathbb{R}$$

for any integer p greater than zero, our theorem yields

$$\lim_{n\to\infty} \frac{1}{n^p} = 0, \quad n \in \mathbb{Z}^+.$$

Therefore, the given sequence

$$a_n = \frac{1}{n^p}$$

converges.

1.1.05 Limit Laws for Infinite Sequences

Convergent sequences have several properties that we can take advantage of. The proofs for the laws below are similar to those for the limit laws for functions, and as such are not provided.

Theorem: Limit Laws of Convergent Infinite Sequences

Suppose we are given two convergent infinite sequences

$$\{a_n\}_{n=1}^{\infty}$$

and

$$\{b_n\}_{n=1}^{\infty}$$

Then

$$1)\ \lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

$$2)\ \lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$$

$$3)\ \lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$4)\ \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \quad \lim_{n\to\infty} b_n \neq 0$$

1.1.06 Limit Laws Example

Example

Determine whether the sequence

$$a_n = \frac{n^2}{2n^2+9}$$

converges.

Complete Solution

Using the limit laws for infinite sequence, we would evaluate

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{2n^2+9} \quad (1)$$

$$= \lim_{n\to\infty} \frac{1}{2+\frac{9}{n^2}} \quad (2)$$

$$= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 2 + 9 \cdot \lim_{n\to\infty} \frac{1}{n^2}} \quad (3)$$

$$= \frac{1}{2+0} \quad (4)$$

$$= \frac{1}{2} \quad (5)$$

Because our limit evaluates to a finite number, the sequence **converges** (and it converges to 1/2).

The above example is trivial, but demonstrates why we need the limit laws. They allow us to evaluate limits of more complicated sequences.

Explanation of Each Step

Step (1)

We applied the definition of convergence of a sequence. Recall that to determine if a sequence is convergent we evaluate

$$\lim_{n\to\infty} a_n$$

and if this limit exists, the sequence converges. If it doesn't the sequence is divergent.

Step (2)

To make the limit easier to evaluate, we divided both the numerator and denominator by n^2. This is as commonly used trick when evaluating limits.

Step (3)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n^2}} \qquad (2)$$

$$= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty}\left(2+\frac{9}{n^2}\right)} \qquad \text{law 4)}$$

$$= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty}(2)+\lim_{n\to\infty}\left(\frac{9}{n^2}\right)} \qquad \text{law 1)}$$

$$= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty}(2)+9\cdot\lim_{n\to\infty}\left(\frac{1}{n^2}\right)} \qquad \text{law 2)}$$

Steps (4, 5)

These two steps consisted of evaluating our limits and simple algebraic manipulation and should be straightforward to students who are familiar with limits of functions.

Possible Challenges

Memorizing the Limit Laws

Students are expected to memorize the laws, but not the labels we assigned to them. For example, students are expected to have memorized that

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

but do not need to memorize that this is the third law.

1.1.07 Relationship to Sequences of Absolute Values

The following theorem can be used to evaluate limits of sequences when the sign of the terms in the sequence alternate between positive and negative.

Theorem: Relationship to Sequences of Absolute Values
The limit $\lim_{n\to\infty} a_n$ equals zero if and only if $\lim_{n\to\infty} \lvert a_n\rvert = 0.$

In other words, if we are given the sequence

$\{a_n\}_{n=1}^{\infty}$

and

$\lim_{n\to\infty} \lvert a_n\rvert = 0$

then

$\lim_{n\to\infty} a_n = 0$

The converse is also true but we will not use it. For us, this theorem is only useful as a test for convergence.

1.1.08 Example Relating Sequences Absolute Values

Example

Determine whether the sequence

$a_n = \dfrac{\cos(n\pi)}{n}$

converges.

Complete Solution

We can quickly determine that the sequence converges by noting that

$$\begin{aligned} \lim_{n\to\infty} a_n &= \left|\frac{\cos(n\pi)}{n}\right| && (1)\\ &= \lim_{n\to\infty} \left|\cos(n\pi)\right| \cdot \left|\frac{1}{n}\right| && (2)\\ &= \lim_{n\to\infty} \frac{1}{n} && (3)\\ &= 0. \end{aligned}$$

Therefore, by our theorem for the relationship to sequences of absolute values, the given sequence **converges**.

Explanation of Each Step

Step (1)

In Step (1), we asserted our theorem for the relationship between sequences of absolute values.

Step (2)

In this step, we used a property of absolute values:

$\left|a \cdot b\right| = \left|a\right| \cdot \left|b\right|$
for any real numbers a and b.

Step (3)

Our trigonometric function has the property that $\cos(n\pi) = (-1)^n$, and so
$\left|\cos(n\pi)\right| = \left|(-1)^n\right| = 1$.

Possible Challenges

Getting Started

It is not always obvious to know which theorem to apply to test for convergence. In this case, we had a specific trigonometric function, $\cos(n\pi)$.

Recognizing that this function takes on values of ±1 gives us a hint that we could try using the relationship to sequences of absolute values, because the absolute value would eliminate the cosine function.

Alternatively, we could have also used the Squeeze Theorem to solve this example, which we explore next.

1.1.09 Squeeze Theorem For Sequences

The Squeeze Theorem for functions can also be adapted for infinite sequences.

Theorem: Squeeze Theorem for Infinite Sequences
Suppose $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$ then $\lim_{n\to\infty} b_n = L.$

This theorem allows us to evaluate limits that are hard to evaluate, by establishing a relationship to other limits that we can easily evaluate. Let's see this in an example.

1.1.10 Squeeze Theorem Example

Example

Determine whether the sequence

$$a_n = \frac{(-1)^n}{n}$$

converges.

Complete Solution

Since

$$\lim_{n\to\infty} \frac{+1}{n} = 0 \qquad (1)$$

and

$$\lim_{n\to\infty} \frac{-1}{n} = 0 \qquad (2)$$

and

$$\frac{-1}{n} \leq \frac{(-1)^n}{n} \leq \frac{+1}{n} \qquad (3)$$

we obtain

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0 \qquad (4)$$

by the Squeeze Theorem for infinite sequences. Therefore, the given sequence **converges**.

Explanation of Each Step

Steps (1) and (2)

To apply the squeeze theorem, we need two functions. One function must be **greater than** or equal to

$$a_n = \frac{(-1)^n}{n}$$

for all *n*, so we can use

$$\frac{(+1)^n}{n}$$

This sequences has the property that its limit is zero.

The other function that we must choose must be **less than** to or equal to a_n for all *n*, so we can use

$$-\frac{(+1)^n}{n}$$

This sequence also has the property that its limit is zero.

There is nothing special about the number zero. We need two sequences that have the same limit at infinity, which in this case happens to be zero.

Step (3)

Our two sequences have the necessary property that one is greater than or equal to a_n for all *n*, and the other is less than or equal to a_n for all *n*.

Step (4)

Our chosen sequences meet the requirements for the Squeeze Theorem to be applied.

Possible Challenge Areas

Knowing When to Use the Squeeze Theorem

We often use the Squeeze Theorem whenever we can easily create two sequences that bound the given sequence and have the same limit. In this example, the functions

$$\frac{(+1)^n}{n}$$

and

$$-\frac{(+1)^n}{n}$$

satisfy these conditions.

Knowing What Sequences to Choose

To apply the squeeze theorem, one needs to create two sequences. Often, one can take the absolute value of the given sequence to create one sequence, and the other will be the negative of the first.

For example, if we were given the sequence

$$b_n = \frac{\cos(n\pi)}{n}$$

we could choose

as one sequence, and choose $c_n = -a_n$ as the other. These sequences would satisfy the conditions we need to apply the theorem (they both tend to the same limit as *n* tends to infinity).

1.1.11 Example rn

Example

Determine the values of *r* so that the sequence

$$\{r^n\}_{n=1}^{\infty}$$

is convergent.

Complete Solution

We can solve this problem by considering cases for the value of *r*.

Case 1

If $r > 1$, then r^n tends to infinity as *n* tends to infinity. The sequence is **divergent** in this case.

Case 2

If $r = 1$, then

$$\lim_{n\to a} r^n = 1$$

so the sequence is **convergent** for this case.

Case 3

If $-1 < r < +1$, then

$$\lim_{n\to\infty} r^n = 0,$$

so the sequence is **convergent** for this case.

Case 4

If $r = -1$, then

$$\lim_{n\to\infty}(-1)^n$$

does not exist, so the sequence is **divergent** for this case.

Case 5

If $r < -1$, then r^ntends to negative infinity as n does not tend to a single finite number. The sequence is **divergent** in this case.

Summary

Therefore, the sequence $\{r^n\}_{n=1}^{\infty}$is convergent when $-1 < r \leq +1$.

Explanation of Each Step

Case 1

Consider the case when $r = 2$. Then our sequence becomes

$$\{2, 4, 8, 16, 32, 64, \ldots\}$$

which tends to infinity.

Case 2

Here we are using a fundamental property of limits, that the limit of a constant equals that constant:

$$\lim_{x\to\infty} c = c$$

for any constant $c$and $x \in \mathbb{R}$.

Case 3

Consider the case when $r = 1/2$. Then our sequence becomes

$$\{1/2, 1/4, 1/8, 1/16, 1/32, \ldots\}$$

which tends to zero.

Similarly, if $r = -1/2$. Then our sequence becomes

$$\{-1/2, +1/4, -1/8, +1/16, -1/32, \ldots\}$$

which also tends to zero.

Case 4

In this case, we have the sequence

$-1, +1, -1, +1, \ldots$

As n approaches infinity the sequence does not approach a unique value, so the limit does not exist.

Case 5

This case is similar to Case 1. Consider the case when $r = -2$. Then our sequence becomes

$\{-2, +4, -8, +16, -32, +64, \ldots\}$

The terms alternate between positive and negative numbers, and do not tend to a single finite number.

Possible Challenge Areas

Connecting Results to Definition of Convergence

In each of the cases, we used a limit to determine whether the sequence is convergent. According to our definition of convergence of a sequence, as long as our respective limits exist, then the sequence converges.

1.1.12 Videos on Infinite Sequences

Patrick JMT: What is a Sequence? Basic Sequence Info
In this video, Patrick JMT introduces the concept of a sequence of numbers and their convergence. At the end of the video, there is a discussion of the sequence r^n.
More Patrick JMT videos available here [1].

Patrick JMT: Sequences – Examples Showing Convergence or Divergence
In this video, Patrick JMT solves two examples involving infinite sequences and convergence.
More Patrick JMT videos available here [1].

Math Is Power 4U: Limits of a Sequence
Definition of convergence of a sequence, and examples.
More MathIsPower4U videos available here [2].

Math Is Power 4U: Limits of a Sequence: The Squeeze Theorem
A different but helpful description of how the squeeze theorem works, and an additional example.
More MathIsPower4U videos available here [2].

References

[1] http://patrickjmt.com/
[2] http://mathispower4u.yolasite.com/

1.1.13 Final Thoughts on Infinite Sequences

Recall our Learning Objectives

In this lesson we covered

- the definition of convergence of an infinite sequence
- various theorems on limits of infinite sequences, including
- the theorem relating between the limit of a function and a limit of a sequence
- the limit laws for infinite sequences
- the squeeze theorem for infinite sequences
- the theorem on the limit of the absolute value of a sequence

After reading this lesson and completing a suitable number of exercises, you should be able to, given an infinite series

- determine whether an infinite sequence converges using the theorems presented in this lesson

What We Didn't Cover

In this lesson, we did not explore the concepts of

- increasing and decreasing sequences
- the Monotonic Sequence Theorem
- bounded sequences

Students are not expected to be familiar with these concepts, but they are are needed to prove theorems that appear later in the ISM. Some of these concepts appear in the ISM Appendices, and we will refer to them when necessary.

1.2.01 Introduction to Sigma Notation

In many areas of mathematics, we are given an infinite sequence

$\{a_n\}_{n=1}^{\infty}$

and we need to add its elements together

$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$

which would produce what we will refer to as an **infinite sum** or **infinite series**. In producing this infinite sum, we are led to the following initial questions:

- Is there a convenient notation that we can use to represent an infinite sum?
- What properties would the sum have?
- Can we find equivalent representations of the sum?

The last question is of particular importance when

- applying formulas to infinite sums, and
- performing algebraic manipulations upon equations with multiple sums,

as we will see later in the module.

1.2.02 Sigma Notation Terminology

Terminology

The infinite sum

$a_1 + a_2 + a_3 + \ldots$

can be written in a more compact way. Using the capital greek letter sigma, Σ, we may write this sum as

$a_1 + a_2 + a_3 + \ldots = \sum_{k=1}^{\infty} a_k$

We often use the following terminology when using this notation

- the **general term** of the sum is a_k
- the **index of summation** is k
- the **limits of summation** are 1 and ∞, although in general the limits can be any integer, $-\infty$, or $+\infty$.

Simple Example

Consider the infinite series

$$\frac{1}{2+3^2}+\frac{2}{2+4^2}+\frac{3}{2+5^2}+\ldots=\sum_{k=3}^{\infty}\frac{k-2}{2+k^2}$$

In this example, the general term is

$$\frac{k-2}{2+k^2},$$

the index of summation is k, and the limits of summation are 3 and infinity.

1.2.03 Sigma Notation Properties

The Properties

As one might expect, sigma notation follows the following properties.

Theorem: Sigma Notation Properties
Suppose a_k and b_k are functions of k, m is any integer, and c is any real number. Then $\sum_{k=m}^{\infty} c a_k = c \sum_{k=m}^{\infty} a_k$ $\sum_{k=m}^{\infty} (a_k \pm b_k) = \sum_{k=m}^{\infty} a_k \pm \sum_{k=m}^{\infty} b_k$

For these properties, we also require the infinite sums to exist. We will discuss what it means for an infinite sum to exist in the next lesson. These properties are easy to prove if we can write out the sums without the sigma notation.

Simple Example

The infinite sum

$$\sum_{k=1}^{\infty}\left(\frac{2}{1+k}+3\frac{\sin(k)}{k}\right) \quad (1)$$

can be written as

$$2\sum_{k=1}^{\infty}\frac{1}{1+k}+3\sum_{k=1}^{\infty}\frac{\sin(k)}{k}. \quad (2)$$

Certainly, decomposing the combined sum in (1) into two sums in (2) does not give us a simpler representation. But this decomposition would allow us to more easily perform the convergence tests that we introduce in later lessons.

1.2.04 Changing Summation Limits

In some cases we need to find an equivalent representation of a given summation, but that has different summation limits. For example, we may need to find an equivalent representation of the following sum

$$\sum_{k=2}^{\infty} k = 2 + 3 + 4 + \dots , \quad (1)$$

where the index of summation start at 1 instead of 2. We will introduce two methods for doing this.

Method 1

Introducing the transformation $j = k - 1$, so that when $k = 2$, $j = 1$, yields

$$\sum_{k=2}^{\infty} k = \sum_{j=1}^{\infty} (j + 1) = 2 + 3 + 4 \dots$$

as desired. The summation index now starts at 1 instead of at 2.

Method 1 Observations

If we like, we can go back to calling our summation index *k*,

$$\sum_{k=1}^{\infty} (k + 1)$$

because it does not matter what we call our index. Also observe that the transformation $j = k - 1$ was chosen so that our new index of summation, k, starts at 1. You will also notice that although we accomplished our task, the general term was transformed from k to $k + 1$. This may not be what we need in certain problems, as we will see in other lessons within this Unit. Method 2 requires more work, but circumvents this problem.

Method 2

Now suppose we would like to

- re-write the sum so that we have the index of summation start at 1, but
- **not** change the general term.

Instead of using a change of variable, we can use another trick to accomplish this task.

Our procedure is to add and subtract terms in the sum to shift our index to 1:

$$\sum_{k=2}^{\infty} k = 2 + 3 + 4 \ldots$$

$$\left(1-1\right) + \sum_{k=2}^{\infty} k = \left(1-1\right) + 2 + 3 + 4 \ldots$$

$$= \left(-1\right) + 1 + 2 + 3 + 4 \ldots$$

$$= -1 + \sum_{k=1}^{\infty} k$$

Therefore,

$$\sum_{k=2}^{\infty} k = -1 + \sum_{k=1}^{\infty} k$$

as desired. Using Method 2, the general term (in this example the general term is k) has not changed.

If you find this process confusing or wonder why we would need to use it, fear not. We will go through another example in more detail, and additional lessons in this unit make use of this method.

1.2.05 Changing Summation Limits Example

Problem

Change the following summation

so that the index of summation start at 1 instead of at 3.

Complete Solution

Although not necessary, we will use two method for solving this problem: Method 1 and Method 2.

Method 1

Introducing the transformation

$j = k - 2,$

so that when $k = 3,\ j = 1$ gives us

$$\sum_{k=3}^{\infty} \frac{k}{2+k} = \sum_{j=1}^{\infty} \frac{j+2}{2+(j+2)} = \frac{3}{2+3} + \frac{4}{2+4} + \frac{5}{2+5} + \ldots$$

as desired.

Method 2

Our procedure is to add and subtract terms in the sum to shift our index to 1:

as desired. Therefore

$$\sum_{k=3}^{\infty} \frac{k}{2+k} = -\frac{5}{6} + \sum_{k=1}^{\infty} \frac{k}{2+k}$$

Discussion of Some Steps

Method 1

The transformation $j = k - 2$, was chosen to that the index $j$would start at 1.

Method 2

Most steps in this approach involved straightforward algebraic manipulation. Steps (3) and (5) involve adding and subtracting terms in a way that will allow us to change the summation limits.

More precicesly, in Step (3) we added and substracted the $k = 2$term, and in Step (5) we added and subtracted the $k = 1$term.

Potential Challenges

Method 2 Requires More Work, So Why Should I Use It?

You may not have a choice. In some circumstances, you need to convert your sum in a way that does not change the general term.

1.2.06 Videos on Sigma Notation

Patrick JMT: What is a Series?
In this video, Patrick JMT introduces the idea of an infinite series, and also goes ahead of the concepts presented in this lesson by discussing geometric series, convergence, and the divergence test. All of these topics are part of the ISM, so if you find them confusing or have a question, you may want to read later lessons in the ISM.
More Patrick JMT videos available here [1].

Math Is Power 4U: Introduction to Infinite Series
In this video, the instructor introduces the idea of an infinite series and also goes ahead of the concepts presented in this lesson by discussing convergence and the divergence test. These topics are part of the ISM, so if you find them confusing you may want to read later lessons in the ISM.
More MathIsPower4U videos available here [2].

1.2.07 Final Thoughts of Sigma Notation

Recall our Learning Objectives

In this lesson we covered

- the sigma notation for representing finite and infinite sums
- properties of the sigma notation
- how to find equivalent forms of an infinite series

Learning Outcomes

After reading this lesson and completing a sufficient number of problems, students should be able to

- convert the first few terms of a series, convert the series into sigma notation
- use the properties of sigma notation to simplify sums
- find equivalent forms of an infinite series

1.3.01 Introduction to Infinite Series

Suppose that we are given an infinite sequence

$\{a_n\}_{n=1}^{\infty}$

and that we would like to add its elements together

$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$

which would produce an infinite series or an infinite sum, as was introduced in the previous lesson. Using sigma notation, we can write this sum as

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots = \sum_{k=1}^{\infty} a_k$$

This simple construction leads us to two immediate questions:

1. How do we know if the infinite sum produces a finite number?
2. If the given infinite sum does yield a finite number, what is it?

These two questions and their answers are the subject of this lesson and a key component of this module on sequences and infinite series.

Example

Consider the following example. Suppose we would like to determine the sum of the infinite series

$$2 + \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \ldots$$

Indeed, we are faced with the two questions listed above: does this sequence yield a finite number, and if it does, what is it? Given a general problem, the answers to these two questions are not straightforward. In this lesson, we will introduce one approach to answer these two questions if the given sum has a special form. We will see that because the infinite sequence above has this particular form, we can calculate its sum quite easily.

1.3.02 Convergence of Infinite Series

Our first task is to establish, given an infinite series, whether or not it converges. Knowing whether or not a given infinite series converges requires a definition of convergence.

Definition: Convergence of an Infinite Series

Suppose we are given an infinite series

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots$$

Let s_n denote the n^{th} partial sum of the infinite series:

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots + a_n.$$

If the sequence

$$\{s_n\}_{n=1}^{\infty}$$

is convergent and

$$\lim_{n\to\infty} s_n = s$$

exists, then the infinite series

$$\sum_{k=1}^{\infty} a_k$$

is **convergent** and

Moreover, the number s, if it exists, is referred to as the **sum** of the series. A series that is not convergent is referred to as **divergent**.

The above definition could be made more precise with a more careful definition of a limit, but this would go beyond the scope of what we need. But our definition provides us with a method for testing whether a given infinite series converges: suppose we can construct an expression, s_n, for the partial sum of a given infinite series, and we want to know if the series converges. If the limit

$$\lim_{n\to\infty} s_n$$

tends to a finite number, the series converges. Otherwise, the series diverges.

Example: Partial Sums of $1/k^2$

Unfortunately, it is often difficult to find a formula for s_n that we can use for this test. But to give us a sense of whether a given series converges, we can use a computer to plot s_n for a specified number of terms in the sequence. For example, let's consider

$$a_k = \frac{1}{k^2}$$

and use a computer to try to guess what the partial sum

$$s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{1}{k^2}$$

converges to when we let n tend to infinity. The following widget calculates the partial sums for us, so we can use it do this. The widget plots the partial sums of $1 / k^2$ from 1 to n. Choose a large value of n and try to make a guess as to whether or not the series converges.

You should be able to, using the widget above, guess whether the series converges, and if it does, what the series converges to by looking at what the partial sums converge to when n gets large (try using "infinity": the series converges to π over 6). However, simply plotting the partial sums, although insightful, does not give us an answer to our first key question: given an infinite series, does it converge? To do this, we could find an expression for s_n and take a limit. We will demonstrate this approach in the following section on the geometric series.

1.3.03 The Geometric Series

There are certain forms of infinite series that are frequently encountered in mathematics. The following example

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \ldots,\ a \neq 0$$

for constants a and r is known as the geometric series. The convergence of this series is determined by the constant $r \in \mathbb{R}$, which is the **common ratio**.

Theorem: Convergence of the Geometric Series
Let r and a be real numbers. Then the geometric series $\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$ converges if $\lvert r\rvert < 1$, and otherwise diverges.

Proof

To prove the above theorem and hence develop an understanding the convergence of this infinite series, we will find an expression for the partial sum, s_n, and determine if the limit as n tends to infinity exists. We will further break down our analysis into two cases.

Case 1: $r = 1$

If $r = 1$, then the partial sum s_n becomes

$$s_n = \sum_{k=1}^{n} ar^{k-1} = a + a + a + a + \ldots = na.$$

So as

$$n \to \infty$$

we have that

$$s_n \to \pm\infty.$$

Hence, the geometric series diverges if r = 1.

Case 2: $r \neq 1$

A short derivation for a compact expression for $s_n$will be useful. First note that

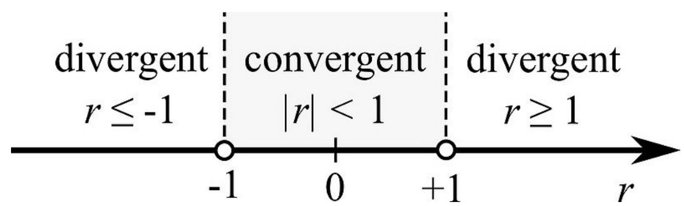
The second equation is the first equation multiplied by r. Subtracting these two equations yields

Using this result, we see that:

- if $|r| < 1$, then as $n \to \infty$, $s_n \to a/(1-r)$
- if $|r| > 1$, then as $n \to \infty$, $s_n \to \infty$
- if $|r| = -1$, then the series is divergent by the Divergence Test (which we cover in a lesson in Unit 2)

Summary

The above results can be summarized in the following figure.



The geometric series only converges when r is between negative one and one.

1.3.04 Geometric Series Example

Example

Determine if the following series is convergent, and if so, find its sum:

$$\sum_{k=0}^{\infty} (-1)^k \frac{2^k}{3^k}.$$

Complete Solution

$$\begin{aligned}\sum_{k=0}^{\infty} (-1)^k \left(\frac{2^k}{3^k}\right) &= \sum_{k=0}^{\infty} \left(\frac{-2}{3}\right)^k && (1)\\ &= \sum_{k=1}^{\infty} \left(\frac{-2}{3}\right)^{k-1} && (2)\\ &= \frac{1}{1-(-2/3)} && (3)\\ &= \frac{3}{5}\end{aligned}$$

The sum of the series is therefore 3/5.

Explanation of Each Step

Step (1)

We first rewrite the problem so that the summation starts at one and is in the familiar form of a geometric series, whose general form is

$$\sum_{k=1}^{\infty} ar^{k-1}.$$

After bringing the negative one and the three fifths together, we see that our given infinite series is geometric with common ratio -3/5.

For a geometric series to be convergent, its common ratio must be between -1 and +1, which it is, and so our infinite series is convergent.

We must now compute its sum.

Step (2)

The given series

$$\sum_{k=0}^{\infty}(-1)^k\Big(\frac{2^k}{3^k}\Big) = \sum_{k=0}^{\infty}(\frac{-2}{3})^k$$

starts the summation at $k = 0$, so we shift the index of summation by one:

$$\sum_{k=0}^{\infty}(\frac{-2}{3})^k = \sum_{k=1}^{\infty}(\frac{-2}{3})^{k-1}.$$

Our sum is now in the form of a geometric series with a = 1, r = -2/3. Since $|r| < 1$, the series converges, and its sum is

$$\frac{a}{1-r} = \frac{1}{1-(-2/3)} = \frac{3}{5}.$$

Step (3)

In Step (3) we applied the formula for the sum of a geometric series:

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

This formula was derived in a previous section of this lesson.

Possible Challenge Areas

Step (2)

Students who may have been confused by this step may wish to refer to the previous lesson on Sigma Notation, where this process was explained in more detail.

1.3.05 Converting an Infinite Decimal Expansion to a Rational Number

Question

Express

$1.7979797979\ldots = 1.\overline{79}$

as a ratio of integers.

Complete Solution

$$\begin{aligned}
1.\overline{79} &= 1 + 0.79 + 0.0079 + 0.000079 + \ldots && (1)\\
&= 1 + 79 \cdot 10^{-2} + 79 \cdot 10^{-4} + 79 \cdot 10^{-6} + \ldots && (2)\\
&= 1 + \sum_{k=1}^{\infty} (79 \cdot 10^{-2})(10^{-2})^{k-1} && (3)\\
&= 1 + \frac{(79 \cdot 10^{-2})}{1 - 10^{-2}} && (4)\\
&= 1 + \frac{79}{99}\\
&= \frac{178}{99}
\end{aligned}$$

Explanation of Each Step

Step (1)

Although not necessary, writing the repeating decimal expansion into a few terms of an infinite sum allows us to see more clearly what we need to do: relate each term to each other in some way to write the given number using sigma notation.

Step (2)

Each term in the sum is equal to 79 times 10 to a power. Explicitly writing out what these powers are helps us look for a pattern in the individual terms of our sum.

Step (3)

Suppose we allow our infinite series to start with the term $a = 79 \cdot 10^{-2}$. Then each term in the infinite series (after the second term) is related to its previous term by a factor of $r = 10^{-2}$. Using the form of the geometric series:

$$\sum_{k=1}^{\infty} ar^{k-1}$$

and substituting our identified values for r, ainto this formula yields Step (3).

Step (4)

Here, we apply our formula for the sum of an infinite series,

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}.$$

The rest of the problem is algebraic manipulation of fractions to find a simplified ratio of two integers.

Possible Areas of Confusion

Getting Started

Any problem of this type could be started in the same way: by writing out the first few terms of an infinite series. This way, it is easier to see a pattern in the terms of the infinite series.

What About the 1?

Essentially, we solved the given problem by writing $1.\overline{79}$as $1 + 0.\overline{79}$, which isolated the repeating digits, which can be written as a geometric series.

Step (3)

Recall that our general form of the geometric series is

$$\sum_{k=1}^{\infty} ar^{k-1}$$

In Step (2), we can identify $a = 79 \cdot 10^{-2}$, and $r = 10^{-2}$. We may substitute these values into the above general form to obtain Step (3).

1.3.06 Finding the Sum of an Infinite Series

Question

Calculate the sum of the following series:

$$2+\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\dots$$

Solution

Step (1): Express in Sigma Notation

The first four terms in the series are

$$2=2^1$$
$$\frac{1}{2}=2^{-1}$$
$$\frac{1}{8}=2^{-3}$$
$$\frac{1}{32}=2^{-5}$$

Each term in the series is equal to its previous multiplied by 1/4. Hence, the series is a geometric series with common ratio $r=1/4$and first term $a=2$:

$$2+\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\dots=\sum_{k=1}^{\infty}ar^{k-1}=\sum_{k=1}^{\infty}2\left(\frac{1}{4}\right)^{k-1}$$

Step (2): Apply Summation Formula

Explanation of Each Step

Step (1)

In any question where one must find the sum of a series given in the form

$$a_1+a_2+a_3+\dots$$

where each term is positive, we must first convert the sum to sigma notation. Why? Because there are no methods (covered in the ISM) to compute an infinite sum otherwise.

There are no general methods to do this, but by looking for a patterns, one might want to look for a way to relate each term by a common base. In our example here, we found that each term in the series could be related to each other with a common ratio of 1/4.

Step (2)

Applying our summation formula with common ratio $1/4$ and first term $a = 3$ yields Step (2).

Possible Challenges

Getting Started

The question asks us to compute the sum of an infinite series, and there are only two ways we could do this. The only two series that have methods for which we can calculate their sums are geometric and telescoping. Since each term is positive, the sum is not telescoping. So the series must be geometric, which means we can get started by looking for a common ratio, r.

1.3.07 A Geometric Series Problem with Shifting Indicies

Question

Find the sum of the series

$\sum_{k=2}^{\infty}\left(\frac{2}{3}\right)^{k}$.

Complete Solution

$$\begin{aligned}
\sum_{k=2}^{\infty}\left(\frac{2}{3}\right)^{k} &= \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k+1} && (1)\\
&= \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k} - \frac{2}{3} && (2)\\
&= \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k-1} - \frac{2}{3} - 1 && (3)\\
&= \frac{1}{1-2/3} - \frac{2}{3} - 1 && (4)\\
&= 3 - 5/3 = 4/3.
\end{aligned}$$

Discussion of Each Step

Step (1)

Our overall goal is to convert the given series into the form

$$\sum_{k=1}^{\infty} ar^{k-1}$$

so that we can apply our formula for the sum of a convergent geometric series. We can begin by shifting the index of summation from 2 to 1

$$\sum_{k=2}^{\infty}\left(\frac{2}{3}\right)^k = \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k+1}$$

This will allow us to use our formula for the sum of a geometric series, which uses a summation index starting at 1. But we still cannot use our formula, because we need to have our exponent equal to k-1, and it is currently k+1.

Step (2)

Next, we can shift the exponent down by one, to allow us to reduce the exponent by one:

$$\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k+1} = \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k} - \frac{2}{3} \quad (2)$$

This requires that we subtract the term corresponding to the k=1 term.

Step (3)

We repeat the process described in the previous step to obtain

$$\begin{aligned}\sum_{k=2}^{\infty}\left(\frac{2}{3}\right)^k &= \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k+1} && (1)\\ &= \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k} - \frac{2}{3} && (2)\\ &= \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k-1} - \frac{2}{3} - 1 && (3)\end{aligned}$$

Step (4)

Now that our infinite series is in our desired form (k starts at 1 and the exponent is k-1), we apply our summation formula with a = 1 and r = 2/3,

$$\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k-1} = \frac{1}{1-2/3}.$$

The rest of the problem is algebraic manipulation.

Possible Mistakes and Challenges

Getting started

Students should immediately recognize that the given infinite series is **geometric** with common ratio 2/3, and that it is not in the form to apply our summation formula,

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}.$$

To convert our series into this form, we can start by changing either the exponent or the index of summation. In the above solution, we started by changing the index of summation (students may want to, as an additional exercise, try to solve this problem by first changing the exponent).

Shifting The Exponent

Our method for shifting the exponent in Steps (3) and (4) may cause some confusion for students. Step (3) is written out in more detail:

$$\begin{aligned}
\sum_{k=2}^{\infty}\left(\frac{2}{3}\right)^{k} &= \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k+1} && (1)\\
&= \left(\frac{2}{3}\right)^{2} + \left(\frac{2}{3}\right)^{3} + \left(\frac{2}{3}\right)^{4} + \dots && (a)\\
&= \left[+\left(\frac{2}{3}\right)^{1} - \left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\mathbf{1}}\right] + \left(\frac{2}{3}\right)^{2} + \left(\frac{2}{3}\right)^{3} + \left(\frac{2}{3}\right)^{4} + \dots && (b)\\
&= -\left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\mathbf{1}} + \left[\left(\frac{2}{3}\right)^{1} + \left(\frac{2}{3}\right)^{2} + \left(\frac{2}{3}\right)^{3} + \left(\frac{2}{3}\right)^{4} + \dots\right] && (c)\\
&= -\frac{\mathbf{2}}{\mathbf{3}} + \left[\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k}\right] && (d)\\
&= \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k} - \frac{\mathbf{2}}{\mathbf{3}} && (2)
\end{aligned}$$

In Step (b) above, we added and subtracted the same term, that term being the common ratio raised to the exponent 1. As mentioned above, the same process for shifting the exponent is used for Step (4).

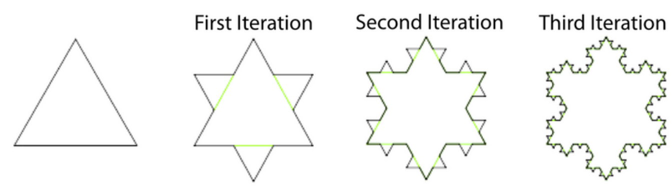
Shifting of exponents was explained in the previous lesson.

1.3.08 Koch Snowflake Example

Problem

Suppose we would like to calculate the area of the "Koch Snowflake". The Koch Snowflake is an object that can be created from the union of infinitely many equilateral triangles (see figure below).

We construct the Koch Snowflake in an iterative process. Starting with an equilateral triangle, each iteration consists of altering each line segment as follows:



Starting with the equilateral triangle, this diagram gives the first three iterations of the Koch Snowflake (Creative Commons, Wikimedia Commons, 2007).

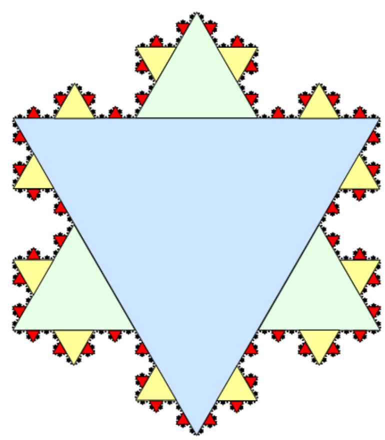
- divide the line segment into three segments of equal length
- draw an equilateral triangle that has the middle segment from step 1 as its base and points outward
- remove the line segment that is the base of the triangle from step 2

The Koch Snowflake is the limit approached as the number of iterations goes to infinity.

Solution

Now, to derive an expression for the area of our construction at the n^{th}iteration, let's start with the fifth iteration. The fifth iteration of the snowflake is shown below, with its iterations in different colours.

Let's assume that the blue triangle as unit area. Each side of the green triangle is exactly 1/3 the length of a side of the blue triangle, and therefore has exactly 1/9 the area of the blue triangle. There are three green triangles, so the green and blue triangles have an area of $1 + 3(1/9) = 4/3$.



The Koch Snowflake after 5 iterations. The first iteration is blue, the second green, the third yellow, the fourth is red, and the fifth is black (Creative Commons, image from Wikimedia Commons).

Similarly, each yellow triangle has 1/9 the area of a green triangle, or 1/27 the area of a blue triangle. The area of the blue, green, and yellow triangles is

$$1 + 3(1/9) + 9(1/27) = 5/3.$$

The total area of the snowflake is given by taking the infinite sequence

$$\left\{3\left(\frac{1}{9}\right)^{n-1}\right\}_{n=1}^{\infty}$$

and adding its terms together to produce the following sum

$$1+3(1/9)+9(1/27)+\ldots=\sum_{n=1}^{\infty}3\left(\frac{1}{9}\right)^{n-1}.$$

Seeing that this is a geometric series with a = 1 and r = 1/9, we immediately conclude that this series converges and is equal to

$$\frac{a}{1-r}=\frac{27}{8}.$$

1.3.09 Videos

Kahn Academy: Sequences and Series Part II
A continuation of the discussion of the geometric series. The formula for the sum of the geometric series is derived.
This video can be found on the Kahn Academy website [1], and carries a Creative Commons copyright (CC BY-NC-SA 3.0).

UBC Mathematics Department: Geometric Interpretation of the Geometric Series
A geometric interpretation of the geometric series, with an algebraic derivation.
Audio transcript of presentation (PDF), PDF version of presentation (106 KB)
This video can be found on the Kahn Academy website [1], and carries a Creative Commons copyright (CC BY-NC-SA 3.0).

Kahn Academy: Series Sum Example
A challenging example involving an infinite geometric series.
This video can be found on the Kahn Academy website [1], and carries a Creative Commons copyright (CC BY-NC-SA 3.0).

Patrick JMT: Geometric Series: Expressing a Decimal as a Rational Number
In this video, the instructor goes through an example involving an infinite geometric series.
More Patrick JMT videos available here [1].

Math Is Power 4U: Infinite Geometric Series
In this video, the instructor provides a nice connection between the definition of convergence of an infinite series and the geometric series. The previous lesson mentioned at the beginning deals with finite geometric series, which students may not be responsible for.
More MathIsPower4U videos available here [2].

References

[1] http://www.khanacademy.org/

1.3.10 Final Thoughts on Infinite Series

After reading this lesson and completing a sufficient number of related exercises, you should be familiar with these concepts:

- the definition of an infinite series
- the definition of convergence of an infinite series
- a test for determining if a given infinite series converges based on its partial sums
- the definition of the geometric series
- a formula for the sum of a convergent geometric series

Although the method that we introduced in this lesson for testing whether a given series converges is important, it is limited. Our method depends on us being able to find an expression for a partial sum, which is often not possible. As such, further lessons in this course explore other strategies that answer our two key questions: given an infinite series, does the series converge, and if it does, what does it converge to?

1.4 Properties of Convergent Series

What This Lesson Covers

In this short lesson we only provide four properties of convergent series.

Learning Objectives

After reading this lesson and completing a sufficient number of exercices on paper, students should be able to

- use the given properties of convergent series to help them determine whether a given series converges.

Topics

1. The Properties of Convergent Series
2. Example: Properties of Convergent Series

Additional Resources

Simply reading the content in this lesson will not be sufficient. You will need to get out your pencil and paper and complete problems in order to prepare yourself for assessments related to this lesson.

1.4.01 The Properties of Convergent Series

The following properties may not come as a surprise to students, but are useful when determining whether more complicated series are convergent or divergent. Proofs of the theorem below can be found in most introductory Calculus textbooks and are relatively straightforward.

Theorem (Properties of Convergent Series)
If the two infinite series $\sum_{k=1}^{\infty} a_k \ , \ \sum_{k=1}^{\infty} b_k$ are both convergent, and c is a real constant, then 1) $\sum_{k=1}^{\infty} (a_k \pm b_k)$ is convergent 2) $\sum_{k=1}^{\infty} c a_k$ is convergent 3) $\sum_{k=1}^{\infty} c a_k = c \sum_{k=1}^{\infty} a_k$ 4) $\sum_{k=1}^{\infty} (a_k \pm b_k) = \sum_{k=1}^{\infty} a_k \pm \sum_{k=1}^{\infty} b_k$

1.4.02 Properties of Convergent Series Example

Problem

Find the sum of the series

$$\sum_{k=1}^{\infty} \left(\frac{4}{5^{k-1}} - \frac{2^k}{3^k} \right).$$

Complete Solution

We can break this problem down into parts and apply the theorem for convergent series to combine each part together.

Step (1): Find the Sum of the First Term

The first term in the problem is a geometric series that can be simplified:

Step (2): Find the Sum of the Second Term

$$\sum_{k=1}^{\infty} \frac{2^k}{3^k} = \sum_{k=2}^{\infty} \left(\frac{2}{3}\right)^{k-1} \quad (2.1)$$

$$= \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k-1} - 1 \quad (2.2)$$

$$= \frac{a}{1-r} - 1 \quad (2.3)$$

$$= \frac{1}{1-2/3} - 1 \quad (2.4)$$

$$= 3 - 1$$

$$= 2.$$

Step (3): Combining Results

Combining results from Steps (1) and (2) yields

$$\sum_{k=1}^{\infty} \left(\frac{4}{5^{k-1}} - \frac{2^k}{3^k}\right) = 5 + 2 = 7.$$

Discussion of Each Step

Step (1)

The infinite series

is a geometric series with common ratio $r = 1/5$and first term $a = 4$. Therefore, we can apply our formula

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

for computing the sum of a geometric series.

Step (2.1)

The infinite series is geometric, and so we can find its sum by working it into the the form

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

to apply our summation formula. One way of starting this process is to making the exponent equal to $k - 1$, which we carried out by shifting the summation index up by one (other approaches may be used, see discussion below in the Possible Challenges section).

Step (2.2)

The index of summation can be shifted down to 1, requiring that we subtract 1. This process was described in the lecture on sigma notation.

Steps (2.3) and (2.4)

Now that our geometric series in the needed form to apply our summation formula, we may equate our series to

$$\frac{a}{1-r} - 1 \quad (2.3)$$

and substitute our values of r and a to obtain Step (2.4).

The rest of Step (2) is algebraic manipulation (fractions).

Step (3)

Because we have found two convergent infinite series, we can invoke the fourth property of convergent series (the sum of two convergent series is a convergent series) to compute the sum of the given problem:

$$\sum_{k=1}^{\infty}\left(\frac{4}{5^{k-1}} - \frac{2^k}{3^k}\right) = 5 + 2 = 7.$$

For demonstration purposes, more steps were shown than what students may find that are needed to solve problems during assessments.

Possible Challenge Areas

Converting the Series Into the Needed Form

Students completing this problem may have noticed that our geometric series formula

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

can only be applied when the summation starts at 1 and the exponent is $k-1$. In simplifying the second term, we needed to convert our problem into this form, which could be solved in another way. Another way of making the conversion is as follows:

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{2^k}{3^k} &= \sum_{k=1}^{\infty}\frac{2^k}{3^k} + (1-1) \\ &= \left[\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k\right] - 1, \text{ absorb the positive 1 into sum} \\ &= \left[\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k-1}\right] - 1, \text{ shift summation index up by 1} \\ &= \frac{1}{1-2/3} - 1, \text{ using the summation formula for a convergent geometric series} \\ &= 2.\end{aligned}$$

Yet even more methods of finding the sum of this series could be used. While we have applied the formula

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

some calculus textbooks introduced the equivalent formula

$$\sum_{k=0}^{\infty} ar^{k} = \frac{a}{1-r}$$

which leads to other similar approaches to finding the sum of the infinite series.

1.5 The Telescoping and Harmonic Series

What This Lesson Covers

In this lesson we introduce

- the telescopic series
- the harmonic series

Learning Objectives

After reading this lesson and after completing a sufficient number of the problems, students should be able to

- determine if a given series is a telescopic or harmonic series
- calculate the sum of a telescopic series

Topics

- Introduction: The Telescoping and Harmonic Series
- The Harmonic Series
- The Telescoping Series
- Video Examples
- Final Notes on Telescoping and Harmonic Series