1 Solutions to assignment 6, due June 23rd

Problem 6.5 We first verify the base case. When $n = 1$, the left-hand side is equal to 1, while the right-hand side is equal to $2(1)^2 - 1 = 1$, and so they are equal.

So suppose that we have the equality

$$\sum_{i=1}^{n}(4i-3) = 2n^2 - n$$

and we want to show that it is true when n is replaced by $n+1$. We compute the left-hand side in that case.

$$\begin{aligned}
\sum_{i=1}^{n+1}(4i-3) &= \sum_{i=1}^{n}(4i-3) + \big(4(n+1)-3\big) \\
&= \sum_{i=1}^{n}(4i-3) + 4n + 1 \\
&= (2n^2 - n) + (4n+1) \qquad \text{by the induction hypothesis} \\
&= 2n^2 + 3n + 1.
\end{aligned}$$

If we then compute the right-hand side, we find

$$2(n+1)^2 - (n+1) = 2n^2 + 4n + 2 - n - 1 = 2n^2 + 3n + 1.$$

As this is equal to the result above, it follows that $\sum_{i=1}^{n+1}(4i-3) = 2(n+1)^2 - (n+1)$, and so by induction, the statement is true for all positive integers n.

Problem 6.6 We first need to come up with a conjectured formula. The simplest way to come up with one is to note that

$$\sum_{i=1}^{n}(3i-2) = 3\sum_{i=1}^{n} i - 2\sum_{i=1}^{n} 1 = 3\frac{n(n+1)}{2} - 2n = \frac{3}{2}n^2 - \frac{n}{2}.$$

It should perhaps be noted that since we already know what $\sum i$ is, that this actually constitutes a proof of this equality! That being said, this is meant to be proven by induction, so by induction we will go.

We work similarly to the previous problem. The base case is easy to see, and so we assume that it holds for some fixed integer n. That is,

$$\sum_{i=1}^{n}(3i-2) = \frac{3n^2 - n}{2}$$

So we will compute the left- and right-hand sides for $n = 1$. The left-hand side is

$$\begin{aligned}
\sum_{i=1}^{n+1}(3i-2) &= \sum_{i=1}^{n}(3i-2) + (3n+1) \\
&= \frac{3n^2 - n}{2} + (3n+1) \qquad \text{by induction} \\
&= \frac{3n^2 + 5n + 2}{2}
\end{aligned}$$

while the right-hand side is

$$\frac{3(n+1)^2-(n+1)}{2}=\frac{3n^2+6n+3-n-1}{2}=\frac{3n^2+5n+2}{2}$$

as desired. Thus by the principle of mathematical induction, the proof is complete.

Problem 6.7 Another formula which could be suggested is to study

$$\sum_{i=1}^{n}(6i-5)$$

which we conjecture (completely blindly, of course) would be equal to $3n^2-2n$. The base case is easily verified.

So assume that this is true for some integer n. Then computing the sum $\sum_{i=1}^{n+1}(6i-5)$ we obtain

$$\begin{aligned}\sum_{i=1}^{n+1}(6i-5)&=\sum_{i=1}^{n}(6i-5)+(6n+1)\\&=(3n^2-2n)+(6n+1)\qquad\text{by the induction hypothesis}\\&=3n^2+4n+1.\end{aligned}$$

The other expression, when evaluated at $n+1$, is

$$3(n+1)^2-2(n+1)=3n^2+6n+3-2n-2=3n^2+4n+1$$

as claimed, and thus the result is true for all integers n.

Problem 6.10 As usual, the base case is clearly true. So we proceed onwards, and assume that

$$\sum_{i=1}^{n-1}ar^i=a\frac{1-r^n}{1-r}$$

Like most problems involving only one more sum, we can work out the left-hand sum for $n+1$ as

$$\begin{aligned}\sum_{i=1}^{n}ar^i&=\sum_{i=1}^{n-1}ar^i+ar^n\\&=a\frac{1-r^n}{1-r}+ar^n\\&=a\Big(\frac{1-r^n}{1-r}+r^n\frac{1-r}{1-r}\Big)\\&=a\Big(\frac{1-r^n+r^n(1-r)}{1-r}\Big)\\&=a\frac{1-r^{n+1}}{1-r}\end{aligned}$$

as claimed, and so the result is true for all integers n by induction.

Problem 6.14 Our base case is $n = 4$. We compute

$$4! = 24 \qquad\qquad 2^4 = 16$$

and so the base case is true. So let us assume the result for some integer n. Then

$$\begin{aligned}(n+1)! &= n!(n+1) \\ &> 2^n(n+1) \\ &> 2^n \cdot 2 \qquad \text{since } n+1 > 2 \\ &= 2^{n+1}\end{aligned}$$

which proves the result.

Problem 6.17 Once again, the base case is clear, as the two sides are equal when $n = 1$. So we assume that $(1+x)^n \geq 1 + nx$, and we look at

$$\begin{aligned}(1+x)^{n+1} = (1+x)^n(1+x) &\geq (1+nx)(1+x) \qquad \text{by induction} \\ &= 1 + (n+1)x + nx^2 \\ &\geq 1 + (n+1)x \qquad \text{since } x^2 \geq 0\end{aligned}$$

and so Bernoulli's inequality is true for every integer n.

Side note: Can you see where the assumption $x \geq -1$ is used? It is subtle, but it is there!

Problem 6.18 The base case is clear, since for $n = 0$, we have $5^n - 1 = 0$. We move boldly forward by induction. Unfortunately, we cannot say that all of the exercises are solved by induction at this point though!

So assume for some fixed n that $4 \mid (5^n - 1)$. Consider $5^{n+1} - 1$. We can write this as

$$5^{n+1} - 1 = 5 \cdot 5^n - 1 = (4+1)5^n - 1 = 4 \cdot 5^n + (5^n - 1).$$

As both terms are divisible by 4 (the former, because, well... and the latter, by the induction hypothesis), it follows that $5^{n+1} - 1$ must also be divisible by 4 as claimed.

Problem 6.20 Base case, yadda yadda.

So assume that it is true for some integer n i.e. we can write

$$7k = 3^{2n} - 2^n$$

for some integer k. We look at the next case and find

$$\begin{aligned}3^{2n+2} - 2^{n+1} &= 9 \cdot 3^{2n} - 2 \cdot 2^n \\ &= 9 \cdot 3^{2n} - 9 \cdot 2^n + 7 \cdot 2^n \qquad \text{by clever trickery} \\ &= 9(3^{2n} - 2^n) + 7 \cdot 2^n \qquad \text{by induction} \\ &= 9 \cdot 7k + 7 \cdot 2^n = 7(9k + 2^n)\end{aligned}$$

and so we see that $7 \mid 3^{2n+2} - 2^{n+1}$ as claimed. Thus by induction, the statement is true for all integers $n \geq 0$.

Problem 6.22 This one is a little trickier. We use as a base case $n = 2$, in which case the statement is

Prove that if A_1, A_2 are sets, then $(A_1 \cap A_2)^c = A_1^c \cup A_2^c$.

which hopefully one recognizes as a statement of one of De Morgan's laws. Thus the base case has already been proven in the past, and so we will not redo it here.

So let us move on by induction on the number of sets. Suppose that it is true for some integer n, or that

$$(A_1 \cap \cdots \cap A_n)^c = A_1^c \cup \cdots \cup A_n^c$$

and consider $(A_1 \cap \cdots \cap A_n \cap A_{n+1})^c$. We can write this as $\big((A_1 \cap \cdots \cap A_n) \cap A_{n+1}\big)^c$, or to be more precise, we write it as the intersection of two sets: $A_1 \cap \cdots \cap A_n$ and A_{n+1}. Thus we compute

$$\begin{aligned}\big((A_1 \cap \cdots \cap A_n) \cap A_{n+1}\big)^c &= (A_1 \cap \cdots \cap A_n)^c \cup A_{n+1}^c && \text{by De Morgan's law}\\ &= (A_1^c \cup \cdots \cup A_n^c) \cup A_{n+1}^c && \text{by induction}\\ &= A_1^c \cup \cdots \cup A_n^c \cup A_{n+1}^c\end{aligned}$$

as we desired. Thus by induction, the statement is true for any number of sets.

Problem 6.25 (a) We induct on the number of elements in the set $S = \{x_1, \ldots x_n\}$. If $n = 1$, then being alone, it must be the largest element.

So assume it is true for every set of real numbers with n elements, and let S be a set with $n + 1$ elements. If we consider $S - \{x_{n+1}\}$, then this is a set with n elements, and so it has a largest element x_i. We have two cases to consider.

Case 1: $x_i > x_{n+1}$. In this case, then x_i is the largest element in S; it is larger than all elements including x_{n+1}, and so we are done.

Case 2: $x_{n+1} > x_i$. In this case, since x_i is the largest element of $S - \{x_{n+1}\}$, we must have that x_{n+1} is the largest element of S, and so S still has a largest element.

Thus by induction, we are done.

(b) If S is finite, then the set $-S = \{x \mid -x \in S\}$ is also finite, and so by the previous exercise it has a largest element. That is, there is some $-x \in -S$ such that for all $-y \in -S$, $-y \leq -x$. However, this implies that $y \geq x$ for all $y \in S$ i.e. that $x \in S$ is the least element of S.

Problem 6.32 We will compute the first few numbers:

$$\begin{array}{lll} a_1 = 1 & a_2 = 2 & a_3 = 4 \\ a_4 = 8 & a_5 = 16 & a_6 = 31 \end{array}$$

... Ok, I'm just kidding about that last one. $a_6 = 2a_5 = 32$, as I'm sure you know. These are all powers of 2, and it ceratinly looks like $a_n = 2^{n+1}$, a fact which is verified for these first few terms, all of which now form our base case(s).

So assume that our formula is true for all integers up to n. Then we compute that

$$a_n = 2a_{n-1} = 2 \cdot 2^{n-1+1} = 2 \cdot 2^n = 2^{n+1}$$

and so induction (strong induction?) proves it to be true for all n.

Problem 6.34 Without even looking at any recursive terms, I will conjecture that this sequence is given by $a_n = n^2$. Not that many sequences begin with $1, 4, 9, \ldots$

So the base case is given, and we proceed onwards. We assume that it is true for all integers less than or equal to some fixed n, and we compute that

$$\begin{aligned} a_{n+1} &= a_n - a_{n-1} + a_{n-2} + 4n - 2 \\ &= n^2 - (n-1)^2 + (n-2)^2 + 4n - 2 \\ &= n^2 - (n^2 - 2n + 1) + (n^2 - 4n + 4) + 4n - 2 \\ &= n^2 + 2n + 1 = (n+1)^2 \end{aligned}$$

and so by induction, it follows that this is true for every n.

Problem 6.36 We will make the following conjecture: For every integer $n > 0$, we have

$$\sum_{i=(n-1)^2+1}^{n^2} i = (n-1)^3 + n^3.$$

Note that this is verified in the book for the first 4 terms.

To simplify things, we will introduce the notation

$$\ell_n = \sum_{i=(n-1)^2+1}^{n^2} i \qquad \text{and} \qquad r_n = (n-1)^3 + n^3.$$

Now, we note that if we sum together the ℓ_i's, that we obtain

$$\sum_{i=1}^{n} \ell_i = \sum_{i=1}^{n^2} i = \frac{n^2(n^2+1)}{2}.$$

We will use this fact as the basis for our induction.

Suppose that our statement (that $\ell_n = r_n$) is true for all integers $1 \leq k < n$. By the statement above, we find that

$$\ell_n = \frac{n^2(n^2+1)}{2} - (\ell_1 + \cdots + \ell_{n-1}).$$

By the induction hypothesis, each of the ℓ_k on the right-hand side can be written as r_k; that is,

$$\ell_n = \frac{n^2(n^2+1)}{2} - (r_1 + \cdots + r_{n-1}).$$

However, as $r_k = (k-1)^3 + k^3$, we can rewrite this as

$$\begin{aligned}
\ell_n &= \frac{n^2(n^2+1)}{2} - \sum_{i=1}^{n-1} r_i \\
&= \frac{n^2(n^2+1)}{2} - \sum_{i=1}^{n-1} \left((i-1)^3 + i^3\right) \\
&= \frac{n^2(n^2+1)}{2} - \sum_{i=1}^{n-2} i^3 - \sum_{i=1}^{n-1} i^3 \\
&= \frac{n^2(n^2+1)}{2} - \frac{(n-2)^2(n-1)^2}{4} - \frac{(n-1)^2n^2}{4} \\
&= \frac{2n^2(n^2+1) - (n-2)^2(n-1)^2 - (n-1)^2n^2}{4}
\end{aligned}$$

which after a fair amount of simplification is equal to $(n-1)^3 + n^3$ as desired (check this yourself!), and so the statement is proven.