JEREMY HEYL

ASTROPHYSICAL PROCESSES

Contents

I Fields 1

1 Radiative Transfer 3

- *Introduction* 3
- *Flux* 3
- *Intensity* 3
  - *Relation to the flux* 4
  - *Spectra* 5
  - *An Astronomical Aside: Magnitudes* 6
- *Energy Density* 7
  - *A Physical Aside: Intensity and Flux* 7
- *Blackbody Radiation* 8
  - *Thermodynamics* 8
  - *Statistical Mechanics* 10
  - *Blackbody Temperatures* 12
- *Radiative Transfer* 13
  - *Emission* 13
  - *Absorption* 14
  - *The Radiative Transfer Equation* 14
- *Thermal Radiation* 16
  - *Einstein Coefficients* 16
  - *A Physical Aside: Einstein coefficients* 18

Calculating the Emission and Absorption Coefficients 18
LTE 18
Non-Thermal Emission 19
Scattering 20
Random Walks 20
Combined Scattering and Absorption 21
Radiative Diffusion 22
Rosseland Approximation 22
Eddington Approximation 24
Problems 25

2 *Basic Theory of Radiation Fields* 29
Maxwell's Equations 29
Waves 31
The Spectrum 32
Polarization 33
Stokes Parameters 35
Poincaré Sphere 36
Electromagnetic Potentials 37
Lorenz Gauge 39
Green's Function 39
Further Reading 41
Problems 41

3 *Radiation from Moving Charges* 45
Retarded Potentials 45
The Fields 46
Distribution in Frequency and Angle 47
Non-relativistic particles 48
How about relativistic particles? 49
Radiation from Systems of Particles 51
A Physical Aside: Multipole Radiation 52

Cherenkov Radiation 52
Thomson Scattering 54
Radiation Reaction 55
Radiation from Harmonically Bound Particles 55
Driven Harmonic Oscillator 56
Further Reading 57
Problems 58

4 *Special Relativity* 61
Back to Maxwell's Equations 61
Lorentz Transformations 61
Length Contraction 62
Adding velocities 62
Doppler Effect 63
Four-Vectors 63
Tensors 66
Transformation of Radiative Transfer 69
Further Reading 72
Problems 72

5 *General Relativity* 77
Back to Newton's Equation 77
A Metric Theory of Gravity 79
Tensors 80
Covariant Derivative 80
Killing Vectors 82
Some Important Tensors 83
Dynamics 84
Static spherically symmetric spacetime 85
Schwarzschild spacetime 88
Periastron Advance 90
Time Evolution 92
Is space curved? 93

The Kerr Metric 94

Epicyclic frequencies 95

Linearized Gravity 98

Gravitational Waves 101

Polarization 102

Production 103

Detection 108

Further Reading 109

Problems 109

II *Radiative Processes* 113

6 *Bremsstrahlung* 115

The Physics of Bremsstrahlung 115

Thermal Bremsstrahlung Emission 118

Thermal Bremsstrahlung Absorption 118

Relativistic Bremsstrahlung 119

Further Reading 123

Problems 123

7 *Synchrotron Radiation* 125

Motion in a magnetic field 125

Spectrum of Synchrotron radiation 126

Qualitative Spectrum 127

Power-Law Distribution of Particle Energies 127

Spectrum — Details 128

Synchrotron Absorption 132

A Complete Synchrotron Spectrum 133

Problems 135

8 *Compton Scattering* 137

The Kinematics of Photon Scattering 137

Inverse Compton Scattering 138

Inverse Compton Power - Single Scattering 139

Inverse Compton Spectra - Single Scattering 140

Repeated Scattering 142

Repeated Scattering with Low Optical Depth 143

Problems 144

III *Quantum Mechanics* 147

9 *Atomic Structure* 149

A single electron in a central field 150

Energies of Electron States 152

Perturbative Splittings 154

Spin-Orbit Coupling 154

Zeeman Effect and Nuclear Spin 155

Thermal Distributions of Atoms 157

Ionization Equilibrium - the Saha Equation 158

A Practical Aside - Orders of Magnitude 159

Problems 161

10 *Radiative Transitions* 163

Perturbation Theory 163

The Perturbation to the Hamiltonian 166

Dipole Approximation 167

Oscillator Strengths 168

Selection Rules 169

Bound-Free Transitions and Milne Relations 170

Line Broadening Mechanisms 173

Problems 174

11 *Molecular Structure* 175

The Born-Oppenheimer Approximation 175

The H_2^+ Molecular Ion 176

Molecular Excitations 179

Problems 182

IV Fluids 183

12 *Fluid Mechanics* 185

Phase-Space Density 185

Particle Current 186

Stress Tensor 187

Ideal Fluids 189

Isentropic flows 190

Hydrostatics 192

Really Little Sound Waves 192

Steady Supersonic Flow 194

Flow through a Channel 195

Real Sound Waves 197

Problems 199

13 *Shock Waves* 201

Non-relativistic Shocks 201

A Spherical Shock - The Sedov Solution 205

Detonation Waves 208

Radiative Shocks 209

Relativistic Shocks 211

Hydraulic Jump 212

Problems 214

14 *Accretion and Winds* 217

Spherical Accretion 217

Accretion Disks 220

Angular Momentum Transport 220

Emission 221

Vertical Structure 221

Modelling the Stress 222

Winds 224

Problems 225

15 *Fluid Instabilities* 229

Gravity Waves and Rayleigh-Taylor Instability 229

Kelvin-Helmholtz or Shearing Instability 232

Gravitational Instability 233

Thermal Instability 234

Problems 235

V *Appendicies* 237

A *Mathematical Appendix* 239

The Integral of $x^3/(e^x-1)$ 239

Parseval's Theorem 241

B *Selected Solutions* 243

Chapter 1 244

Chapter 2 247

Chapter 3 253

Chapter 4 256

Chapter 5 262

Chapter 6 266

Chapter 7 267
Chapter 8 268
Chapter 9 271
Chapter 10 274
Chapter 11 277
Chapter 12 280
Chapter 13 283
Chapter 14 287
Chapter 15 291

Part I

Fields

1

Radiative Transfer

Introduction

We are going to set the stage for a deeper look at astrophysical sources of radiation by defining the important concepts of radiative transfer, thermal radiation and radiative diffusion.

One can make a large amount of progress by realizing that the distances that radiation typically travels between emission and detection or scattering etc. are much longer than the wavelength of the radiation. In this regime we can assume that light travels in straight lines (called rays). Upon these assumptions the field of radiative transfer is built.

Flux

Let's start with something familiar and give it a precise definition. The flux is simply the rate that energy passes through an infinitesimal area (imagine a small window).

$$dE = FdAdt \tag{1.1}$$

For example, if you have an isotropic source, the flux is constant across a spherical surface centered on the source, so you find that

$$E_1 = F_1 4\pi R_1^2 \text{ and } E_2 = F_2 4\pi R_2^2 \tag{1.2}$$

at two radii around the source. Unless there is aborption or scattering between the two radii, $E_1 = E_2$ and we obtain the inverse-square law for flux

$$F_1 R_1^2 = F_2 R_2^2. \tag{1.3}$$

Intensity

Although the flux is a useful quantity, it cannot encapsulate all of our knowledge about a radiation field. For example, one could shine a

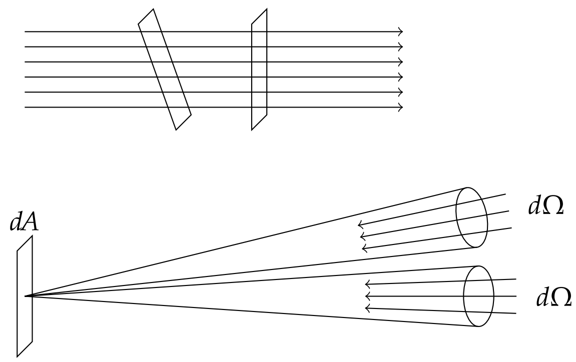


Figure 1.1: Flux and intensity. On the left the power delivered to the two surfaces are equal even though their areas differ. The flux is the power per unit area so the tilted surface gets less flux. On the right intensity is the power delivered to an area from a particular part of the sky (solid angle). Here the two intensities are equal but the upper set of rays delivers less flux.

faint light directly through a window or a bright light through the same surface at an angle. Both of these sitautions are characterized by the same rate of energy flow through the surface, but they are clearly different physical situations.

A more generally useful quantity quantifies the rate that energy flows through a surface in a particular direction (imagine that the window now looks into a long pipe so that only light travelling in a particular direction can pass through,

$$dE = I dA d\Omega dt \tag{1.4}$$

where I is the intensity. Although this quantity seems a bit kludgy, it is actually quite familiar. It is the brightness.

You look at a light bulb. As you move away from the light bulb, your eye receives less flux (F decreases) and the apparent size of the light bulb also decreases ($d\Omega$ decreases). It turns out that these two quantities both decrease as R^{-2}, so the intensity or brightness is conserved along a ray. This result makes the intensity a terrifically useful quantity.

Relation to the flux

From the example at the beginning of this section we can deduce the relationship between the flux and the intensity of the light. Radiation that travels perpendicular to a surface delivers more energy to that surface than radiation travelling at an angle. You can always imagine second surface perpendicular to the light ray through which all of the energy that reaches the first surface travels. We know that intensity is the same along the ray so

$$dE = I dA_1 d\Omega dt = I dA_2 d\Omega dt \tag{1.5}$$

and $dA_2 = \cos\theta dA_1$, so the total flux travelling through the surface is given by a moment of the intensity

$$F = \int I \cos\theta d\Omega \tag{1.6}$$

If I is constant with respect to angle, there is as much energy travelling from left to right as from right to left, so the net flux vanishes, or more mathematically the mean of $\cos\theta$ vanishes over the sphere.

Something to think about The Sun is equally intense in the summer and winter (if you exclude the effects of the atmosphere), then why are winters colder than summers?

A closely related quantity is the pressure that a radiation field exerts on a surface. Pressure is the rate that momentum is delivered to a surface in the direction perpendicular to the surface. The momentum of light is E/c and the rate that energy is delivered to a surface from light travelling around a particular direction is simply $I\cos\theta d\Omega$. The component of the momentum that is directed perpendicular to the surface is $E\cos\theta/c$, so there is a second factor of $\cos\theta$ yielding the following integral.

$$p = \frac{1}{c}\int I\cos^2\theta d\Omega. \quad (1.7)$$

Something to think about Does the radiation pressure from an isotropic radiation field vanish?

Spectra

The quantities that we have defined so far can be examined as a function of the frequency or wavelength of the radiation or the energy of the individual photons, yielding F_ν, F_E, F_λ and also for the intensity, *e.g*

$$dE = F_\nu d\nu dA dt \quad (1.8)$$

and F_ν is called the specific flux. The use of F_ν is so common that astronomers have a special unit to measure F_ν

$$1 \text{ Jansky} = 1 \text{ Jy} = 10^{-26}\text{W m}^{-2}\text{Hz}^{-1} = 10^{-23}\text{erg cm}^{-1}\text{s}^{-1}\text{Hz}^{-1}. \quad (1.9)$$

This unit is most commonly used in the radio and infrared, and sometimes in the x-rays.

A common combination that people use is

$$EF_E = \lambda F_\lambda = \nu F_\nu = \nu\frac{dF}{d\nu} = \frac{dF}{d\ln\nu}. \quad (1.10)$$

This allows you to convert between F_ν and F_λ etc. And it also gives the flux per logarithmic interval in photon energy or frequency. This is really handy since astronomers like to use log-log plots. A spectrum that goes as $F_\nu \propto 1/\nu$ has a constant amount of energy per logarithmic interval.

Something to think about A source emits at 1 Jansky from 100 MHz to 1 GHz and at 1 μJy from 1 to 10 keV. Is it brighter in the radio or x-rays?

An Astronomical Aside: Magnitudes

Astronomers typically speak about the flux of an object in terms of magnitudes. A magnitude is generally defined as

$$m = -2.5 \log_{10} \int_0^\infty g(\nu) F_{\nu,\Omega} d\nu + m_0. \qquad (1.11)$$

What are the different quantities in this expression? Pogson empirically determined the value of "2.5" by comparing the magnitudes of prominent observers of the 1800's. It is remarkably close to $\ln 10 \approx 2.3$, so a change in magnitude of 0.1 is about a ten-percent change in flux.

Another term in the expression is $g(\nu)$, the filter function that determines which part of the electromagnetic spectrum you're are looking at. If $g(\nu) = 1$, the quantity is called a "bolometric magnitude.". It is supposed to quantify the total energy coming from the source. One also hears about a quantity called the "bolometric correction" which is simply the difference between the magnitude of source for a particular filter ($g(\nu)$) and for $g(\nu) = 1$.

$F(\nu, \Omega)$ is the flux coming from the source as a function of frequency integrated over a certain area of sky, Ω. For a star one generally can extrapolate the flux that one observes in the sky to the total flux, but the intensity from a galaxy or other extended source generally falls off gradually so one defines a magnitude within a certain aperture or down to a limiting intensity (surface brightness).

The final term is m_0, the zero point. The value of the zero point is a matter of convention. Two of the standard conventions are the "Vega" convention which states that the magnitude of the star Vega regardless of the function $g(\nu)$ is zero; all of Vega's colours are zero. This works nicely because you could always in principle observe Vega with your equipment.

There are two problems however. One is that the flux from Vega like that of most stars varies a bit. The second is that an object with a flat spectrum (equal energies in each frequency interval) will have an awkward set of colours (the difference in magnitudes for two different $g(\nu)$ functions). This leads to the second convention, the AB system.

$$m(AB) = -2.5 \log_{10} f_\nu - 48.60 = -2.5 \log_{10} \left(\frac{f_\nu}{1 \text{ Jy}} \right) + 8.90 \qquad (1.12)$$

where f_ν is the flux in c.g.s. units. The constants "8.90" and "-48.60" mean that $m(AB) = V$ for a flat spectrum source.

Something to think about How do you define an AB magnitude using a filter?

A final quantity that astronomers talk about is the surface brightness. This is just the intensity that we have been speaking about all along. However, the conventional nomenclature is rather strange, magnitudes per square arcsecond.

Something to think about What is the magnitude of a source that subtends 10 square arcseconds with a surface brightness of 19 magnitudes per square arcsecond?

Energy Density

Let's imagine that light is travelling through a small box. How much energy is in the box at any time? First it is easiest to think about how much energy in the box is travelling in a particular direction through the box during a small time interval such that cdt is the length of the box,

$$dE = u(\Omega) dA c dt d\Omega \tag{1.13}$$

This energy equals the energy that enters the box travelling in the right direction during the time interval dt,

$$dE = I dA dt d\Omega \tag{1.14}$$

so

$$cu(\Omega) = I. \tag{1.15}$$

To get the total energy density you have to integrate over all of the ray directions

$$u = \frac{1}{c} \int I d\Omega = 4\pi \frac{J}{c} \tag{1.16}$$

where J is the mean intensity. Notice how it differs from the flux defined earlier.

Let's revisit the radiation pressure formula. But let's assume that the radiation field is isotropic, so $I = J$ for all directions, we get

$$\begin{aligned} p &= \frac{1}{c} \int J \cos^2 \theta d\Omega & (1.17) \\ &= \frac{1}{c} \int_0^\pi \int_0^{2\pi} J \cos^2 \theta \sin \theta d\theta d\phi & (1.18) \\ &= \frac{1}{c} J \left(2\pi \frac{1}{3} \cos^3 \theta \Big|_0^\pi \right) = \frac{4}{3} \pi \frac{J}{c} = \frac{1}{3} u. & (1.19) \end{aligned}$$

A Physical Aside: What are the Intensity and Flux?

How do the intensity and flux fit in with more familiar concepts like the flux of a vector field? They really don't.

One can define the flux in three perpendicular directions by asking how much energy flows through three mutually perpendicular planes. This flux vector transforms like a vector under rotations, but it doesn't transform like a four-vector under boosts. The flux vector fills in the time-space components of the stress-energy tensor of the electromagnetic field. We have also calculated the time-time component which is the energy density and the space-space components, the pressure. To calculate how the flux transforms with respect to a boost (or Lorentz transformation) by transforming the entire tensor.

The intensity (I_ν) as we shall soon see is simply related to the phase-space density of the ensemble of photons.

Blackbody Radiation

Blackbody radiation is a radiation field that is in thermal equilibrium with itself. In general we will find it convenient to think about radiation that is in equilibrium with some material or its enclosure. Using detailed balance between two enclosures in equilibrium with each other and the enclosed radiation we can quickly derive several important properties of blackbody radiation.

- The intensity (I_ν) of blackbody radiation does not depend on the shape, size or contents of the enclosure.
- Blackbody radiation is isotropic.

What remains is the temperature and the frequency. Because the intensity is a universal function of T and ν, we have

$$\text{Kirchhoff's Law:} \quad I_\nu = B_\nu(T) \text{ for a blackbody at temperature } T. \tag{1.20}$$

Because heat flows from a hotter system to a cooler system we know that if $T_1 > T_2$, $B_\nu(T_1) > B_\nu(T_2)$ for all values of ν. To see this, imagine a filter that only lets light pass over a narrow range of frequencies in the hole between the enclosures. If this condition did not hold, one could have energy flowing from the cooler to the hotter enclosure.

Thermodynamics

The blackbody radiation in its enclosure is a system in equilibrium so we can use the equations of thermodynamics to glean some more of its properties. If we deliver some heat dQ to the blackbody, it can change the internal energy of the blackbody dU or do work pdV. The heat delivered also equals the change in entropy of the system times the temperature of the system.

$$dQ = TdS = dU + pdV. \tag{1.21}$$

Now U is simply the energy density times the volume of the enclosure so $dU = udV + Vdu/dTdT$ and we showed the $p = u/3$. Let's put this together

$$TdS = udV + V\frac{du}{dT}dT + \frac{1}{3}udV. \tag{1.22}$$

If we rearrange and solve for the derivatives we get

$$\left(\frac{\partial S}{\partial T}\right)_V = \frac{V}{T}\frac{du}{dT} \qquad \left(\frac{\partial S}{\partial V}\right)_T = \frac{4}{3}\frac{u}{T} \tag{1.23}$$

Let's take the partial derivative of the first expression with respect to V and the second expression with respect to T and set them equal

$$\frac{1}{T}\frac{du}{dT} = \frac{4}{3}\frac{1}{T}\frac{du}{dT} - \frac{4}{3}\frac{u}{T^2} \tag{1.24}$$

Let's solve for du/dT to get

$$\frac{du}{dT} = \frac{4u}{T} \tag{1.25}$$

so

$$\text{Stefan-Boltzmann Law:} \quad u = aT^4 \tag{1.26}$$

where a is a constant of integration. The value of a is 7.56×10^{15} erg cm^{-3}K^4.

Something to think about Why does du/dV vanish?

We found that for an isotropic radiation field the energy density is simply related to the intensity, $u = 4\pi J/c$. For a blackbody $J = B(T)$ so we have

$$B(T) = \frac{ac}{4\pi}T^4. \tag{1.27}$$

Let's imagine that our blackbody enclosure has a small hole of area dA in it. How much energy emerges through this hole

$$FdA = \int_{\text{out}} \cos\theta B(T) d\Omega = B(T) \int_0^{\pi/2} \int_0^{2\pi} \cos\theta \sin\theta d\theta d\phi = \pi B(T). \tag{1.28}$$

We write this more compactly by defining $\sigma = ac/4$ so we have

$$\text{Another Stefan-Boltzmann Law:} \quad F = \sigma T^4 \tag{1.29}$$

where $\sigma = 5.67 \times 10^{-5}$ erg cm^{-2}s^{-1}K^{-4}.

Using the earlier results we can also derive the entropy of the radiation field

$$S = \frac{4}{3}aT^3V. \tag{1.30}$$

Statistical Mechanics

We have managed to derive several interesting properties of blackbody radiation but we still have no idea what its spectrum is. To figure this out we have to think about the microscopic properties of the radiation field. Let's imagine that we have blackbody radiation in a box whose wavenumber k_x ranges from k_x to $k_x + dk$. How many different types of waves lie in this interval?

You may be tempted to say as many as you want, but the waves are trapped in a box. Let's say that the box has length l_x in the x-direction; therefore, $k_x l_x = 2\pi n_x$ where n_x is an integer so that the radiation field has a node at the edges of the box, so between k_x and $k_x + dk$ there are only $2dkl_x/(2\pi)$ different states. The factor of two arises because the waves have two independent polarization states. If we imagine a small cube in phase space of size $dk_x dk_y dk_z$ we get

$$dN = 2\frac{l_x l_y l_z}{(2\pi)^3} d^3k = 2\frac{dV}{(2\pi)^3} d^3k. \tag{1.31}$$

Now we have $d^3k = k^2 dk d\Omega$ and $k = 2\pi\nu/c$ so

$$d^3k = d\nu d\Omega \frac{(2\pi)^3}{c^3}\nu^2 \tag{1.32}$$

We find that the density of states is given by

$$\rho_s \equiv \frac{dN}{dV d\Omega d\nu} = \frac{2\nu^2}{c^3} \tag{1.33}$$

The energy density ($u_\nu(\Omega)$)of the radiation field is simply the density of states times the mean energy per state and $cu_\nu(\Omega) = I_\nu$.

Classically we find that the mean energy per state is simply given by kT. Let's try this out for size,

$$u_\nu^{\text{classical}}(\Omega) = \frac{2\nu^2}{c^3} kT. \tag{1.34}$$

This is the great Rayleigh-Jeans law and it actually works pretty well, unless you look at large frequencies and find that the total energy $B(T)$ is infinite. This is called the Rayleigh-Jeans (or ultraviolet) catastrophe.

The solution to this problem ushered in the era of modern physics. Planck argued that if light comes in discrete packages (photons) whose energy is proportional to the frequency we can solve this problem ($E = h\nu$). Let's try a really simple minded approach to assume that only photons with $E < kT$ are in the radiation field then we have

$$u_\nu^{\text{classical fixed?}}(\Omega) = \begin{cases} \frac{2\nu^2}{c^3} kT & h\nu < kT \\ 0 & h\nu \geq kT \end{cases} \tag{1.35}$$

Let's integrate u_ν over the frequency range and solid angle to get

$$u^{\text{classical fixed?}} = \frac{8\pi kT}{3c^3}\left(\frac{kT}{h}\right)^3 = \frac{8\pi k^4}{3c^3h^3}T^4. \tag{1.36}$$

This has the right behaviour and the numerical constant differs from the actual value by a factor of about 20.

It turns out that we can do a whole lot better. According to statistical mechanics the probability of a state of energy E is proportional to $e^{-\beta E}$ where $\beta = 1/(kT)$. The energy in a particular state is proportional to the number of photons in the state $E_n = nh\nu$. The mean energy in a state is given by

$$\bar{E} = \frac{\sum_{n=0}^{\infty} E_n e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}} \tag{1.37}$$

Notice that the expression on the top is the derivative of the expression on the bottom with respect to β, so we find

$$\bar{E} = -\frac{\partial}{\partial \beta}\ln\left(\sum_{n=0}^{\infty} e^{-\beta E_n}\right). \tag{1.38}$$

We haven't assumed anything about the states themselves yet, so this result would apply for any system. Here we know that $E_n = nh\nu$ so

$$\sum_{n=0}^{\infty} e^{-\beta E_n} = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} = \frac{1}{1-e^{-\beta h\nu}} \tag{1.39}$$

so

$$\bar{E} = \frac{h\nu}{e^{\beta h\nu}-1} \tag{1.40}$$

For $h\nu \ll kT$ $\beta h\nu \ll 1$ so we have

$$\bar{E} \approx \frac{h\nu}{1+\beta h\nu - 1} = \frac{h\nu}{\beta h\nu} = kT, \tag{1.41}$$

the classical result. But for $h\nu \gg kT$ we have $\beta h\nu \gg 1$

$$\bar{E} \approx h\nu \exp(-\beta h\nu). \tag{1.42}$$

If we use this value with the density of states we get the Wien law.

Let's derive the expression for the spectrum for all frequencies. We have the value of energy density

$$u_\nu(\Omega) = \frac{2h}{c^3}\frac{\nu^3}{\exp(h\nu/kT)-1} \tag{1.43}$$

so that

$$\text{Planck's Law:} \quad B_\nu(T) = \frac{2h}{c^2}\frac{\nu^3}{\exp(h\nu/kT)-1}. \tag{1.44}$$

Wien's displacement law gives the frequency of the peak of the blackbody curve B_ν

$$h\nu_{\text{max}} \approx 2.821439kT \tag{1.45}$$

or $\nu_{\text{max}} = 58.79T$ GHz/K.

Something to think about At what energy does the flux per logarithmic energy interval reach its peak? How about flux per unit wavelength?

Let's try to find the value of a, the constant in the Stefan-Boltzmann Law. First we have

$$u_\nu = 4\pi u_\nu(\Omega) = \frac{8\pi h}{c^3} \frac{\nu^3}{\exp(h\nu/kT) - 1}. \quad (1.46)$$

The total energy density is $\int u_\nu d\nu$

$$u = \int_0^\infty \frac{8\pi h}{c^3} \frac{\nu^3}{\exp(h\nu/kT) - 1} d\nu = \frac{8\pi h}{c^3} \left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3}{e^x - 1} dx. \quad (1.47)$$

The integral can be evaluated using a Taylor series

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \int_0^\infty \frac{x^3 e^{-x}}{1 - e^{-x}} dx = \int_0^\infty x^3 \sum_{n=1}^\infty e^{-nx} dx = \sum_{n=1}^\infty \frac{6}{n^4} \quad (1.48)$$

to yield $\pi^4/15$ (see § A for further details),so

$$u = aT^4 \text{ where } a = \frac{8\pi^5}{15} \frac{k^4}{(hc)^3} = \frac{\pi^2}{15} \frac{k^4}{(\hbar c)^3}. \quad (1.49)$$

The number density of photons can be determined in a similar way but the exponent in the integral is "2" instead of "3" yielding

$$n = \frac{16\zeta(3)\pi k^3}{c^3 h^3} T^3 \quad (1.50)$$

so the mean energy per photon is given by

$$\frac{u}{n} = \frac{\pi^4}{30\zeta(3)} kT \approx 2.701 kT. \quad (1.51)$$

Coincidentally the numeric factor differs from e by less than one percent.

Blackbody Temperatures

A blackbody is of course characterized by a single temperature, T. However, it is often convenient to characterize the radiation from astrophysical sources by assuming that it is a blackbody and using some property of the blackbody spectrum to derive a characteristic temperature for the radiation. There are three characteristic temperatures in common usage: *brightness* temperature, *effective* temperature and the *colour* temperature.

The brightness temperature is determined by equating the brightness or intensity of an astrophysical source to the intensity of a blackbody and solving for the temperature of the corresponding blackbody.

$$I_\nu = B_\nu(T_b). \quad (1.52)$$

This expression is most useful in the regime where the intensity of the blackbody is proportional to the temperature *i.e.* the Rayleigh-Jeans limit. Here we have,

$$T_b = \frac{c^2}{2\nu^2 k} I_\nu. \tag{1.53}$$

The brightness temperature has several nice properties. For one thing it has units of Kelvin rather than something clumsy. Second if a material is emitting thermal radiation one can obtain a simple expression of the radiative transfer equation (see the problems).

Something to think about In what regime does the linear relationship between the brightness temperature and the intensity begin to fail? How can you tell?

The colour temperature is defined by looking at the peak of the emission from the source and using Wien's displacement law to define a corresponding temperature. This may be done in a more sophisicated manner by fitting a blackbody spectrum or something like that.

Finally the effective temperature is the temperature of a blackbody that emits the same flux at its surface as the source, *i.e.*

$$F = \sigma T_{\text{eff}}^4 \tag{1.54}$$

Radiative Transfer

As a ray passes through some material its intensity may increase or decrease depending on the properties of the matter. To understand this process it is helpful to make some defintions.

Emission

Generally material has two routes for the emission of radiation: stimulated emission and spontaneous emission. The rate of the former is proportion to the intensity of the beam so it is convenient to lump it with the absorbing properties of the material. The spontaneous emission is independent of the radiation field.

Let's define the *spontaneous emission coefficient*, j. This is the energy emitted per unit time into unit solid angle from a unit volume, so we have

$$dE = j dV d\Omega dt \tag{1.55}$$

and similarly

$$dE = j_\nu dV d\Omega dt d\nu. \tag{1.56}$$

If the emitter is isotropic or the emitters are randomly oriented then the total power emitted per unit volume and unit frequency is

$$P_\nu = 4\pi j_\nu. \tag{1.57}$$

Often the emission is isotropic and it is convenient to define the emissivity of the material per unit mass

$$dE = \epsilon_\nu \rho dV dt d\nu \frac{\Omega}{4\pi} \tag{1.58}$$

where ρ is the density of the emitting medium. ϵ_ν is simply related to j_ν for an isotropic emitter

$$j_\nu = \frac{\epsilon_\nu \rho}{4\pi}. \tag{1.59}$$

As a beam travels through the material, its intensity increases such that

$$\frac{dI_\nu}{ds} = j_\nu. \tag{1.60}$$

Here is the first term in the equation of radiative transfer. We know what I_ν is and we will spend much effort figuring out what j_ν is for different physical systems.

Absorption

The equation for absorption is similar, except that amount of absorption is proportional to the intensity of the radiation. You can't absorb radiation that isn't there.

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu. \tag{1.61}$$

Phenomenologically you can imagine that there are many independent absorbers in the beam, each with a cross section σ_ν and a number density n. This would yield

$$\alpha_\nu = \sigma_\nu n. \tag{1.62}$$

It is often convenient to define $\alpha_\nu = \rho\kappa_\nu$ where κ_ν is the opacity of the material. You can think of this as the cross section per unit mass of the absorbers.

The quantity α_ν has both positive and negative contributions. The positive contributions are *true absorption* and the negative ones correspond to *stimulated emission*.

The Radiative Transfer Equation

Putting these equations together yields the radiative transfer equation,

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu. \tag{1.63}$$

Once we know α_ν and j_ν for the system of interest, it is straightforward to solve the equations of radiative transfer. We shall see a formal solution a bit later. If there is scattering as well as absorption and emission, things get a bit more complicated.

Let's look at a few examples.

1. **Emission only**

$$\frac{dI_\nu}{ds} = j_\nu \tag{1.64}$$

yields the solution

$$I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu(s')ds' \tag{1.65}$$

The increase in brightness is simply the integral of the emission coefficient along the line of sight. This limit is also know as the *optically thin* regime (absorption can be neglected).

2. **Absorption only**

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu \tag{1.66}$$

which yields

$$I_\nu(s) = I_\nu(s_0)\exp\left[-\int_{s_0}^{s} \alpha_\nu(s')ds'\right] \tag{1.67}$$

The result for pure absorption inspires us to look at the radiative transfer equation again. Let's define, the optical depth,

$$\tau_\nu = \int_{s_0}^{s} \alpha_\nu(s')ds' \tag{1.68}$$

such that $d\tau_\nu = \alpha_\nu ds$

3. **Emission and absorption** Using this defintion we get the following equation of radiative trasnfer,

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu \tag{1.69}$$

where

$$S_\nu \equiv \frac{j_\nu}{\alpha_\nu} \tag{1.70}$$

is called the *source function*. It has the same units as the intensity. It allows a formal solution of the transfer equation

$$I_\nu(\tau_\nu) = I_\nu(s_0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu-\tau'_\nu)} S_\nu(\tau'_\nu)d\tau'_\nu. \tag{1.71}$$

This expression makes a lot of sense. The first term is the radiation that we start with and attentuated through the medium. The second term is the sum of all the radiation emitted in the medium and attentuated from where it is emitted until it escapes. If we have a source with a constant value of S_ν, the solution is much simpler

$$I_\nu(\tau_\nu) = I_\nu(s_0)e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right) = S_\nu + e^{-\tau_\nu}\left(I_\nu(0) - S_\nu\right). \quad (1.72)$$

The intensity field approaches the source function as the optical depth increases.

Thermal Radiation

Let's imagine a blackbody enclosure, and we stick some material inside the enclosure and wait until it reaches equilibrium with the radiation field, $I_\nu = B_\nu(T)$. Now let move the material in a position the blocks our window to the enclosure. What can we say about the source function of the material?

We know that as light travels through the material the intensity field should approach the source function but we also know that the light emerging from the window must have $I_\nu = B_\nu(T)$. If it didn't, we could set up an adjacent blackbody enclosure at the same temperature and energy would flow between them. We must conclude that

$$\text{Another Kirchoff's Law:} \quad S_\nu = B_\nu(T) \text{ for a thermal emitter} \quad (1.73)$$

Furthermore, we can look at the transfer equation that yields,

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + B_\nu(T). \quad (1.74)$$

Because $I_\nu = B_\nu(T)$ outside of the thermal emitting material and $S_\nu = B_\nu(T)$ within the material, we find that $I_\nu = B_\nu(T)$ throughout the enclosure.

If we remove the thermal emitter from the blackbody enclosure we can see the difference between thermal radiation and blackbody radiation. A thermal emitter has $S_\nu = B_\nu(T)$, so the radiation field approaches $B_\nu(T)$ (blackbody radiation) only at large optical depth.

Einstein Coefficients

Kirchoff's law yields a relationship between the emission and absorption coefficients for a thermally emitting material, specifically $j_\nu = \alpha_\nu B_\nu$. This relationship suggests some connection between emission and absorption at a microscopic level. It was Einstein that first elucidated this connection.

Let's imagine a two-level atom. The lower level has energy E and statistical weight g_1. You can think of the statistical weight as the number of ways that the atom can be in the particular state, the degeneracy of the state. The second level has an energy $E + h\nu_0$ and a statistical weight of g_2.

There are three possible transitions,

1. Spontaneous Emission with probability per unit time of A_{21}.
2. Absorption with a rate proportional to the angle-averaged intensity of the radiation field ($J_\nu(\nu_0)$) times the coefficient B_{12}.

   Technically the atom does not absorb at precisely one frequency but over a width $\delta\nu$. To simplify matters we will take $\delta\nu \to 0$ and the line profile $\phi(\nu) \to \delta(\nu - \nu_0)$.
3. Stimulated Emission with a transition rate of $B_{21} J_\nu(\nu_0)$.

For the system to be in thermodynamic equilibrium, the number of transitions from level one to two must equal the reverse transitions,

$$n_1 B_{12} J_\nu(\nu_0) = n_2 A_{21} + n_2 B_{21} J_\nu(\nu_0). \tag{1.75}$$

Let's solve for $J_\nu(\nu_0)$,

$$J_\nu(\nu_0) = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}}. \tag{1.76}$$

We know that the system is in thermodynamic equilibrium so

$$\frac{n_1}{n_2} = \frac{g_1 \exp(-E/kT)}{g_2 \exp(-(E + h\nu_0)/kT)} = \frac{g_1}{g_2} \exp(h\nu_0/kT). \tag{1.77}$$

Let's substitute this into the earlier result

$$J_\nu(\nu_0) = \frac{A_{21}}{(g_1/g_2)\exp(h\nu_0/kT) B_{12} - B_{21}} \tag{1.78}$$

We know that since the system is in thermodynamic equilibrium with the radiation field $J_\nu = B_\nu(T)$

$$\frac{2h}{c^2} \frac{\nu^3}{\exp(h\nu_0/kT) - 1}. = \frac{A_{21}}{(g_1/g_2)\exp(h\nu_0/kT) B_{12} - B_{21}} \tag{1.79}$$

$$= \frac{A_{21}}{B_{21}} \frac{1}{(g_1/g_2)\exp(h\nu_0/kT) B_{12}/B_{21} - 1} \tag{1.80}$$

Because the temperature T may be set arbitrarily we must have the following relationships

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21}, \tag{1.81}$$

$$g_1 B_{12} = g_2 B_{21}. \tag{1.82}$$

Because the Einstein coefficients are properties of the atom alone, they do not depend on the assumption of thermodynamic equilibrium. They are quite powerful. If we calculate the probability of absorption of a photon for example, we can use the Einstein relations to find the rate of stimulated and spontaneous emission. This proof is an example of the principle of *detailed balance* of a microscopic process.

Something to think about Can you use the principle of detailed balance to say anything about the relationship between the stimulating and the stimulated photon?

A Physical Aside: What is deep about the Einstein coefficients?

The Einstein coefficients seem to say something magical about the properties of atoms, electrons and photons. Somehow atoms are forced to behave according to these equations. It turns out that the relationships between Einstein coefficients (1917) are an example of Fermi's Golden Rule (late 1920s). Fermi's Golden Rule relates the cross-section for a process to a quantum mechanical matrix element and the phase space available for the products. Because quantum mechanics for the most part is time reversible, the cross-section for the forward and reverse reactions are related.

Calculating the Emission and Absorption Coefficients

We can write the emission and absorption coefficients in terms of the Einstein coefficients that we have just exmained. The emission coefficient j_ν has units of energy per unit time per unit volume per unit frequency per unit solid angle! The Einstein coefficient A_{21} gives spontaneous emission rate per atom, so dimensional analysis quickly gives

$$j_\nu = \frac{h\nu}{4\pi} n_2 A_{21} \phi(\nu) \tag{1.83}$$

The absorption coefficent may be constructed in a similar manner

$$\alpha_\nu = \frac{h\nu}{4\pi} \phi(\nu) \left(n_1 B_{12} - n_2 B_{21} \right) \tag{1.84}$$

We can now write the absorption coefficient and the source function using the relationships between the Einstein coefficients as

$$\alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12} \left(1 - \frac{g_1 n_2}{g_2 n_1} \right) \phi(\nu) \tag{1.85}$$

$$S_\nu = \frac{2h\nu^3}{c^2} \left(\frac{g_2 n_1}{g_1 n_2} - 1 \right)^{-1}. \tag{1.86}$$

LTE

To derive these relations we have not made any assumptions about whether the photons or the matter are in thermal equilibrium with themselves or each other. An extremely useful assumption is that the matter is in thermal equilibrium at least locally (Local Thermodynamic Equilibrium). This assumption forms the basis of the theory of stellar atmospheres.

In this case the ratio of the number of atoms in the various states is determined by the condition of thermodynamic equilibrium

$$\frac{n_1}{n_2} = \frac{g_1}{g_2} \exp\left(\frac{h\nu_0}{kT}\right). \tag{1.87}$$

This ratio yields the following absorption coefficient and source function

$$\alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12} \left[1 - \exp\left(-\frac{h\nu_0}{kT}\right)\right] \tag{1.88}$$

$$S_\nu = B_\nu(T) \tag{1.89}$$

Something to think about Because the source function equals the blackbody function, does this mean that sources in local thermodynamic equilibrium emit blackbody radiation?

Non-Thermal Emission

In any situation where

$$\frac{n_1}{n_2} \neq \frac{g_1}{g_2} \exp\left(\frac{h\nu_0}{kT}\right). \tag{1.90}$$

(*i.e.*if the radiating particles do not have a Maxwellian distribution) one has to use the full expression for the source function; a power-law distibution oftens occurs astrophysically.

An extreme example of non-thermal emission is the maser. For atoms in thermodynamic equilibrium we have

$$\frac{n_2 g_1}{n_1 g_2} = \exp\left(-\frac{h\nu}{kT}\right) < 1 \tag{1.91}$$

so that

$$\frac{n_1}{g_1} > \frac{n_2}{g_2} \tag{1.92}$$

which means that the absorption coefficient is always positive in thermodynamic equilibrium. However, let us imagine a situation in which

$$\frac{n_1}{g_1} < \frac{n_2}{g_2}. \tag{1.93}$$

This yields a negative absorption coefficient, so the optical depth decreases and becomes negative as one passes through a region with inverted populations and the intensity of the radiation actually increases expontentially as the magnitude of the optical depth increases. So we have to thank Einstein for the laser as well.

Scattering

From the preceding discussion one might think that the theory of radiative transfer simply relies on the application of the formal solution using the Einstein coefficients for various processes of interest.

However, there is a big elephant in the middle of the room that we have been ignoring — **scattering**. Why is scattering a problem? Couldn't you think about scattering as the absorption and re-emission of a photon and include the process in the absorption coefficients and source functions? The answer is no.

The formalism that we have developed so far doesn't allow there to be a correlation between the properties of an absorbed photon and the emitted photon. On the other hand, the initial direction and energy of a scattered photon are generally highly correlated with the photon's final momentum.

We can first look at a process in which the photon is scattered into a random direction without a change in energy. This yields an emission coefficient of

$$j_\nu = \sigma_\nu J_\nu. \tag{1.94}$$

Notice that the emission rate depends on the radiation field through J_ν and not solely on the properties of the scatterer through σ_ν. If isotropic scattering is the only process acting we find that the source function

$$S_\nu = J_\nu = \frac{1}{4\pi}\int d\Omega I_\nu. \tag{1.95}$$

The equation describing the evolution of the radiation field is still rather innocuous looking

$$\frac{dI_\nu}{ds} = -\sigma_\nu \left(I_\nu - J_\nu\right). \tag{1.96}$$

However, it is a completely different beast. The evolution of the intensity of a particular ray depends not only the intensity of the ray and the local properties of the material but also on the intensity of all other rays passing through the same point — we have an **integro-differential equation**.

If you think about things more generally, we had this same problem before introducing scattering because the properties of the emitting and absorbing material usually depend on the radiation field. Even if one neglects scattering, one often has to solve an integro-differential equation.

Random Walks

We can get an order of magnitude feeling for how much scattering will affect the radiation field emerging from a source using the concepts of the mean free path and the random walk.

The mean free path for scattering (or similarly for absorption) is simply the reciprocal of the scattering coefficient σ_ν. So if we imagine a single photon travelling through the material, it will go typically a distance $l = \sigma_\nu^{-1}$ then change direction. The net displacement of the photon after N free paths is

$$\mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \cdots + \mathbf{r}_N. \tag{1.97}$$

However, on average it is as likely to go one way as the other so the mean values of all of the $\mathbf{r}_i$ vectors vanishes as does the sum. Let's ask instead how far the photon typically ends up away from the starting point, here we have

$$|\mathbf{R}|^2 = |\mathbf{r}_1|^2 + |\mathbf{r}_2|^2 + |\mathbf{r}_3|^2 + \cdots + |\mathbf{r}_N|^2 \tag{1.98}$$

$$+ 2\mathbf{r_1} \cdot \mathbf{r_2} + 2\mathbf{r_1} \cdot \mathbf{r_3} + \cdots \tag{1.99}$$

All of the cross terms vanish on average if the scattering is isotropic, but $\left\langle bfr_1{}^2 \right\rangle \approx l = \sigma_\nu^{-1}$ so the net distance travelled after N scatterings is

$$l_*^2 = Nl^2, l_* = \sqrt{N}l. \tag{1.100}$$

If some blob of gas has a typically dimension L we can estimate the number of scatterings through the gas to be $N \approx L^2/l^2$. This is why people sometimes say that it takes a million years for a photon to escape the sun.

In general if τ is large the average number of scatterings $N \approx \tau^2$ while for $\tau \ll 1$, $N \approx \tau$

Combined Scattering and Absorption

In general a material can both scatter and absorb photons. In this case the transfer equation has two terms. Let's focus on coherent isotropic scattering and thermal emission and get

$$\frac{dI_\nu}{ds} = -\alpha_\nu \left(I_\nu - B_\nu\right) - \sigma_\nu \left(I_\nu - J_\nu\right) \tag{1.101}$$

$$= -\left(\alpha_\nu + \sigma_\nu\right)\left(I_\nu - S_\nu\right) \tag{1.102}$$

where

$$S_\nu = \frac{\alpha_\nu B_\nu + \sigma_\nu J_\nu}{\alpha_\nu + \sigma_\nu} \tag{1.103}$$

The net absorption coefficient is $\alpha_\nu + \sigma_\nu$. On average a photon will travel a distance

$$l_\nu = \frac{1}{\alpha_\nu + \sigma_\nu} \tag{1.104}$$

before it is absorbed or scattered. The chance that the free path will end in absorption is

$$\epsilon_\nu = \frac{\alpha_\nu}{\alpha_\nu + \sigma_\nu} \tag{1.105}$$

and the chance that it will be scattered is

$$1 - \epsilon_\nu = \frac{\sigma_\nu}{\alpha_\nu + \sigma_\nu}. \tag{1.106}$$

We can rewrite the source function as

$$S_\nu = (1 - \epsilon_\nu) J_\nu + \epsilon_\nu B_\nu. \tag{1.107}$$

After a photon is emitted it may bounce around several times before it is absorbed; the average number of scatterings per absorption is $N = \epsilon_\nu^{-1}$. After these N scatterings it will typically have travelled a distance,

$$l_* = \sqrt{N} l = \frac{1}{\sqrt{\alpha_\nu(\alpha_\nu + \sigma_\nu)}} \tag{1.108}$$

l_* is the typically distance between the points of creation and destruction of a photon – it is called the *diffusion length*, *the thermalization length*, or the *effective mean free path*. If the material has some thickness L, we can define the *effective optical depth* of the material to be $\tau_* = L/l_*$.

If τ_* is small, then a photon after being created in the medium, bounces around until it emerges (it is usually not absorbed). In this case the power emitted by material will simply be

$$\mathcal{L}_\nu = 4\pi\alpha_\nu B_\nu V, \quad (\tau_* \ll 1) \tag{1.109}$$

where B_ν is the source function (a blackbody for thermal emission) without scattering and V is the volume of the material. If $\tau_* \ll 1$ the material is said to be *effectively thin* or *translucent*.

On the other hand, if $\tau_* \gg 1$, the medium is said to be *effectively thick*. In this case only photons emitted within l_* of the surface typically escape without being absorbed. We can estimate the power emitted by

$$\mathcal{L}_\nu = 4\pi\alpha_\nu B_\nu A l_* = 4\pi\sqrt{\epsilon_\nu} B_\nu A, \quad (\tau_* \gg 1) \tag{1.110}$$

Radiative Diffusion

We have used the random walk arguments to show that the radiation field approaches a blackbody within a few effective mean paths (or thermalization lengths) of the surface. Furthermore, the radiation field becomes isotropic within a mean free path of the surface. We will first look at the first situation in which the radiation field is approximately a blackbody.

Rosseland Approximation

Because stellar atmospheres (i.e. the effective mean path) are generally thin compared to the size of the star, we can assume that this region has plane parallel symmetry; that is, the properties of the material depend only on the depth from the surface z. The intensity will generally depend on the depth and the angle that the ray makes with the normal θ. It is generally convenient to use $\mu = \cos\theta$ instead of θ itself.

$$\frac{\partial I_\nu(z,\mu)}{ds} = \cos\theta \frac{\partial I_\nu(z,\mu)}{\partial z} = \mu \frac{\partial I_\nu(z,\mu)}{ds} = -\left(\alpha_\nu + \sigma_\nu\right)\left(I_\nu - S_\nu\right). \tag{1.111}$$

Let's rearrange this as

$$I_\nu(z,\mu) = S_\nu - \frac{\mu}{\alpha_\nu + \sigma_\nu}\frac{\partial I_\nu}{\partial z}. \tag{1.112}$$

Here comes the assumption. Let us assume that the properties of the radiation field do not change much over a mean free path so the second term is much smaller than the first; therefore;

$$I_\nu^{(0)}(z,\mu) \approx S_\nu^{(0)}(T). \tag{1.113}$$

Because this is independent of the angle, $J_\nu^{(0)} = S_\nu^{(0)}(T)$, so $S_\nu^{(0)} = B_\nu$. Let's get a better approximation to the radiation field

$$I_\nu^{(1)}(z,\mu) \approx B_\nu(T) - \frac{\mu}{\alpha_\nu + \sigma_\nu}\frac{\partial B_\nu}{\partial z}. \tag{1.114}$$

Let's calculate the total flux of the radiation field.

$$F_\nu(z) = \int I_\nu^{(1)} \cos\theta d\Omega = -2\pi \frac{\partial B_\nu}{\partial z}\frac{1}{\alpha_\nu + \sigma_\nu}\int_{-1}^{+1}\mu^2 d\mu \tag{1.115}$$

$$= -\frac{4\pi}{3}\frac{1}{\alpha_\nu + \sigma_\nu}\frac{\partial B_\nu}{\partial z} = -\frac{4\pi}{3}\frac{1}{\alpha_\nu + \sigma_\nu}\frac{\partial B_\nu}{\partial T}\frac{\partial T}{\partial z} \tag{1.116}$$

We can integrate this over all frequencies to find the total flux

$$F(z) = \int_0^\infty F_\nu(z) d\nu = -\frac{4\pi}{3}\frac{\partial T}{\partial z}\int_0^\infty \left(\alpha_\nu + \sigma_\nu\right)^{-1}\frac{\partial B_\nu}{\partial T} d\nu. \tag{1.117}$$

Unfortunately the integral above generally cannot be done analytically. However, we can elucidate some of its properties. First, the absorption and scattering coefficients are summed harmonically so regions of the spectrum that have the least absorption or scattering will dominate the energy flow. Furthermore, the harmonic sum is weighted heavily in the region where $\partial B_\nu / \partial T$ is large, near the peak of the blackbody emission.

One can define a mean absorption coefficient by

$$\frac{1}{\alpha_R} \equiv \frac{\int_0^\infty \left(\alpha_\nu + \sigma_\nu\right)^{-1}\frac{\partial B_\nu}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu} = \frac{\pi}{4\sigma T^3}\int_0^\infty \left(\alpha_\nu + \sigma_\nu\right)^{-1}\frac{\partial B_\nu}{\partial T} d\nu \tag{1.118}$$

If we substitute this into the earlier expression we get

$$F(z) = -\frac{16\sigma T^3}{3\alpha_R}\frac{\partial T}{\partial z}. \tag{1.119}$$

where α_R is the Rosseland mean absorption coefficient. In stellar astrophysics one often uses the column density Σ as the independent variable rather than z, $d\Sigma = \rho dz$. Making the substitution yields

$$F(z) = -\frac{16\sigma T^3}{3\alpha_R}\rho\frac{\partial T}{\partial \Sigma}. = -\frac{16\sigma T^3}{3\kappa_R}\frac{\partial T}{\partial \Sigma}. \qquad (1.120)$$

where $\kappa_R = \alpha_R/\rho$ is the Rosseland mean opacity.

Eddington Approximation

What if you are interested in the translucent upper layers of the atmosphere within a few effective mean paths of the surface but still a few mean free paths (scattering lengths) away from the surface. In this region, the radiation field is nearly isotropic, but it need not be close to a blackbody distribution.

Because the intensity is close to isotropic we can approximate it by

$$I_\nu(z,\mu) = a_\nu(z) + \mu b_\nu(z). \qquad (1.121)$$

Let's use the first three moments of the intensity

$$J_\nu \equiv \frac{1}{2}\int_{-1}^{+1} I d\mu = a_\nu, \qquad (1.122)$$

$$H_\nu \equiv \frac{1}{2}\int_{-1}^{+1} \mu I d\mu = \frac{b_\nu}{3}, \qquad (1.123)$$

$$K_\nu \equiv \frac{1}{2}\int_{-1}^{+1} \mu^2 I d\mu = \frac{a_\nu}{3} \qquad (1.124)$$

J is the mean intensity and H and K are proporptional to the flux and the radiation pressure, respectively. The *Eddington approximation* is the result that

$$K = \frac{1}{3}J \qquad (1.125)$$

which we found earlier to hold for strictly isotropic radiation fields. Here we find that it also holds for anisotropic radiation fields of the form defined earlier.

Let's define the normal optical depth,

$$d\tau_\nu = -\left(\alpha_\nu + \sigma_\nu\right) dz \qquad (1.126)$$

yielding the radiative transfer equaition

$$\mu\frac{\partial I_\nu}{\partial \tau} = I_\nu - S_\nu \qquad (1.127)$$

where

$$S_\nu = \frac{\alpha_\nu B_\nu + \sigma_\nu J_\nu}{\alpha_\nu + \sigma_\nu}. \qquad (1.128)$$

The source function S_ν is isotropic, so let's average the radiative transfer equation over direction to yield

$$\frac{\partial H_\nu}{\partial \tau} = J_\nu - S_\nu \tag{1.129}$$

Let's also average the radiative transfer equation times μ over direction to yield

$$\frac{\partial K_\nu}{\partial \tau} = \frac{1}{3}\frac{\partial J_\nu}{\partial \tau} = H_\nu \tag{1.130}$$

Something to think about What happened to S_ν in the equation above?

We can combine the two equations (Eq. 1.129 and 1.130) to yield

$$\frac{1}{3}\frac{\partial^2 J_\nu}{\partial \tau^2} = J_\nu - S_\nu \tag{1.131}$$

and using the definition of the source function gives

$$\frac{1}{3}\frac{\partial^2 J_\nu}{\partial \tau^2} = J_\nu - \frac{\alpha_\nu B_\nu + \sigma_\nu J_\nu}{\alpha_\nu + \sigma_\nu} = \epsilon_\nu \left(J_\nu - B_\nu \right). \tag{1.132}$$

This is sometimes called the *radiative diffusion equation*. If you know the temperature structure of the material you can solve the equation for the mean intensity J_ν and then you know the source function S_ν explicitly and you can use the formal solution to the radiative transfer equation to get the radiation field.

Problems

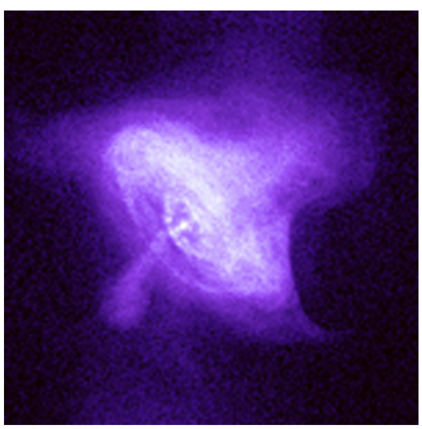


Figure 1.2: The central region of the Crab Nebula as seen by the Chandra X-ray Observatory. Credit: NASA/CXC/SAO)

1. **Hot cloud:**

   X-ray photons are produced in a cloud of radius R at the uniform rate Γ (photons per unit volume per unit time) as in Fig. 1.2. The cloud is a distance d away. Assume that the cloud is optically thin. A detector at Earth has an angular acceptance beam of half-angle $\Delta\theta$ and an effective area ΔA.

(a) If the cloud is fully resolved by the detector what is the observed intensity of the radiation as a function of position?

(b) If the cloud is fully unresolved, what is the average intensity when the source is in the detector?

2. **Brightness Temperature:**

   From the equation of radiative transfer derive an equation describing how the brightness temperature changes as radiation passes through a thermally emitting gas. You may neglect scattering and assume that the emission is in the Rayleigh-Jeans limit. Solve this equation to give the brightness temperature as a function of optical depth, assuming that the gas has a constant temperature.

3. **Neutrino Blackbody:**

   Only one or no neutrinos can occupy a single state. Calculate the spectrum of the neutrino field in thermal equilibrium (neglect the mass of the neutrino). Neutrinos like photons have two polarization states. What is the ratio of the Stefan-Boltzmann constant for neutrinos to that of photons?

4. **Blackbody radiation:**

   (a) Show that if stimulated emission is neglected, leaving only two Einstein coefficients, an appropriate relation between the coefficients will be consistent with thermal equilibrium between an atom and a radiation field with a Wien spectrum, *i.e.* $B_\nu \propto \nu^3 \exp[-h\nu/(kT)]$.

   (b) Derive the relationships between the Einstein coefficients of an atom in equilibrium with a neutrino field.

5. **Surface Emission from the Crab Pulsar:** The neutron star that powers the Crab Pulsar can be assumed to have a mass of $1.4 M_\odot$ and a radius of 10 km with constant internal density and an effective temperature of 10^6 K. The frequency of the Crab Pulsar is 30 Hz and its period increases by 38 ns each day. Compare the power from the surface emission to the power lost as the neutron star spins down. The total power of the Crab Nebulae is about 75,000 times that of the Sun. What is the likely source of this power?

6. **Power-law Atmosphere:** Assume the following

- The Rosseland mean opacity is related to the density and temperature of the gas through a power-law relationship,

$$\kappa_R = \kappa_0 \rho^\alpha T^\beta;$$

- The pressure of the gas is given by the ideal gas law;
- The gas is in hydrostatic equilibrium so $p = g\Sigma$ where g is the surface gravity; and
- The gas is in radiative equilibrium with the radiation field so the flux is constant with respect to z or Σ.

Calculate the temperature of the gas as a function of Σ. Assuming that the Crab pulsar has a surface or effective temperature of 10^6 K, what is the temperature at a density of 10^7 g/cm^{-3} in the interior of the neutron star? You may assume $\alpha = 2$, $g = 10^{14}$ cm/s^2 and

$$\kappa_R = 3.68 \times 10^{22} g_{ff}(1-Z)(1+X)\rho T^{-7/2} \text{cm}^2\text{g}^{-1}.$$

where ρ is given in g/cm^{-3} and T is given in Kelvin.

7. **Goggles:** Calculate from thermodynamic principles how much objects are magnified or demagnified while viewed through goggles underwater.

8. **Intensity and index of refraction:** How does the intensity of light travelling along a ray change when the light enters a material with a different index of refraction?

   (a) Solve this problem using geometry.

   (b) Solve this problem using thermodynamic principles alone. N.B. The wavenumber of a photon of a given frequency is proportional to the index of refraction.

2

Basic Theory of Radiation Fields

Maxwell's Equations

Most of this material is probably familar, so this will only present a quick review. The goals are to understand how charges move under the influence of electromagnetic fields, how when the charges accelerate, they emit electromagnetic radiation and how this radiation transports energy. The electric and magnetic field may be defined operationally by observing the motion of a particle of charge q travelling through the field. The force on the particle is given by the Lorentz force equation,

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right). \tag{2.1}$$

The fields perform work on the particle at a rate

$$\mathbf{v} \cdot \mathbf{F} = q\mathbf{v} \cdot \mathbf{E}. \tag{2.2}$$

One can imagine an ensemble of charged particles of charge density ρ and define the current to be $\mathbf{J} = \rho\mathbf{v}$. In this case we find that power delivered on the charges per unit volume is simply

$$P = \mathbf{J} \cdot \mathbf{E}. \tag{2.3}$$

Note that the magnetic field $\mathbf{B}$ does not perform work on the particle. It changes the direction of the particle's motion but not its speed.

The equations that describe the dynamics of the fields are Maxwell's equations,

$$\begin{array}{cc} \nabla \cdot \mathbf{D} = 4\pi\rho & \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} & \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \end{array} \tag{2.4}$$

in Gaussian units. The fields $\mathbf{D}$ and $\mathbf{H}$ are related to $\mathbf{E}$ and $\mathbf{B}$ through the constitutive relations

$$\mathbf{D} = \epsilon\mathbf{E}, \mathbf{B} = \mu\mathbf{H}, \tag{2.5}$$

where ϵ and μ are in general matrices that depend on the fields applied, but in most situations they are constant scalars and for a vacuum they are simply one.

The equations were discovered by various people. Proceeding left to right then top to bottom we have Gauss's law, a law without a name, Ampere's law and Faraday's law. It might be more appropriate to call the penultimate, Maxwell's equation, because Ampere's law as it was originally formulated was

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} \tag{2.6}$$

Maxwell added the term proportional to the rate of change of the electric field. If one takes the divergence of the complete Ampere's law one obtains,

$$\nabla \cdot (\nabla \times \mathbf{H}) = \frac{4\pi}{c}\nabla \cdot \mathbf{J} + \frac{1}{c}\nabla \cdot \frac{\partial \mathbf{D}}{\partial t} \tag{2.7}$$

The left-hand side vanishes because the divergence of the curl vanishes, on the right hand side one obtains,

$$0 = \frac{4\pi}{c}\nabla \cdot \mathbf{J} + \frac{4\pi}{c}\frac{\partial \rho}{\partial t} \tag{2.8}$$

where we have used the first of Maxwell's equations to simply the result and find that the Maxwell's addition makes the full set of equations consistent with charge conservation.

Let's calculate the work that the field will do on a bunch of charges per unit volume,

$$\mathbf{J} \cdot \mathbf{E} = \frac{1}{4\pi}\left[c\left(\nabla \times \mathbf{H}\right) \cdot \mathbf{E} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}\right] \tag{2.9}$$

We have the following vector identity (the triple product)

$$\mathbf{E} \cdot (\nabla \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{H}) \tag{2.10}$$

Substituting in the earlier result and using Faraday's law yields,

$$\mathbf{J} \cdot \mathbf{E} = \frac{1}{4\pi}\left[-\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - c\nabla \cdot (\mathbf{E} \times \mathbf{H})\right] \tag{2.11}$$

Let's assume that ϵ and μ are constant and get

$$\mathbf{J} \cdot \mathbf{E} + \frac{1}{8\pi}\frac{\partial}{\partial t}\left[\mathbf{H} \cdot \mathbf{B} + \mathbf{E} \cdot \mathbf{D}\right] + \nabla \cdot \left(\frac{c}{4\pi}\mathbf{E} \times \mathbf{H}\right) = 0 \tag{2.12}$$

This is Poynting's theorem. We can identify the work performed on the charges, the change in the field energy per unit volume and the flux of field energy ($\mathbf{S}$) as follows,

$$U_{\text{field}} = \frac{1}{8\pi}\left[\mathbf{H} \cdot \mathbf{B} + \mathbf{E} \cdot \mathbf{D}\right] \text{ and } \mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H} \tag{2.13}$$

Waves

Let's look at Maxwell's equations in a vacuum,

$$\begin{aligned} \nabla \cdot \mathbf{E} = 0 \qquad & \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad & \nabla \times \mathbf{B} = +\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \end{aligned} \tag{2.14}$$

Let's take the curl of the third equation and combine it with the fourth to get

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{2.15}$$

We have the identity that

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}. \tag{2.16}$$

The first term on the right-hand side vanishes so we get the final wave equation,

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{2.17}$$

and a similar equation for the magnetic field.

We write a general solution to the wave equation as a sum of harmonically varying waves such as

$$\mathbf{E} = \Re\left(\hat{\mathbf{a}}_1 E_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right) \text{ and } \mathbf{B} = \Re\left(\hat{\mathbf{a}}_2 B_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right) \tag{2.18}$$

Application of Maxwell's equations to the above solutions shows

$$\nabla \cdot \mathbf{E} = i\mathbf{k}\cdot\mathbf{E} = 0 \text{ so } \hat{\mathbf{a}}_1 \perp \mathbf{k} \tag{2.19}$$

$$\nabla \cdot \mathbf{B} = i\mathbf{k}\cdot\mathbf{B} = 0 \text{ so } \hat{\mathbf{a}}_2 \perp \mathbf{k} \tag{2.20}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = i\mathbf{k}\times\mathbf{E} - i\frac{\omega}{c}\mathbf{B} = 0 \text{ so } \hat{\mathbf{a}}_1 \perp \hat{\mathbf{a}}_2 \tag{2.21}$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = i\mathbf{k}\times\mathbf{B} + i\frac{\omega}{c}\mathbf{E} = 0 \text{ so } \omega = kc \text{ and } E_0 = B_0. \tag{2.22}$$

We would like to calculate the time-averaged energy density and energy flux assoicated with the wave. In general E_0 and B_0 are complex quantities. First let's look at the Poynting vector

$$\begin{aligned} \langle \mathbf{S} \rangle &= \frac{1}{P}\int_0^P \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}dt & (2.23) \\ &= \frac{c\hat{\mathbf{k}}}{4\pi P}\int_0^P dt \left(\Re E_0 \cos\omega t - \Im E_0 \sin\omega t\right) \times \\ &\quad \left(\Re B_0 \cos\omega t - \Im B_0 \sin\omega t\right) & (2.24) \\ &= \frac{c\hat{\mathbf{k}}}{4\pi P}\int_0^P dt \left(\Re E_0 \Re B_0 \cos^2\omega t + \Im E_0 \Im B_0 \sin^2\omega t\right) & (2.25) \\ &= \frac{c\hat{\mathbf{k}}}{8\pi}\left(\Re E_0 \Re B_0 + \Im E_0 \Im B_0\right) = \frac{c\hat{\mathbf{k}}}{8\pi}\Re E_0^* B_0 & (2.26) \\ &= \frac{c\hat{\mathbf{k}}}{8\pi}|E_0|^2 = \frac{c\hat{\mathbf{k}}}{8\pi}|B_0|^2 & (2.27) \end{aligned}$$

The time-averaged energy density is

$$\langle U \rangle = \frac{1}{16\pi} \Re \left(E_0 E_0^* + B_0 B_0^* \right) = \frac{1}{8\pi} |E_0|^2 = \frac{1}{8\pi} |B_0|^2 \tag{2.28}$$

Because the electric and magnetic field have the same behaviour we only have to describe one of the fields to determine the properties of the wave. It is customary to focus on the electric field.

The Spectrum

A general electromagnetic wave can be expressed as a sum of the Fourier components described in the previous section. We have been characterizing the energy in the radiation field with the quantity I_ν the intensity per unit frequency interval. It would be nice to find an relationship between the electric field as a function of time and the intensity.

The first step in obtaining the spectrum is to take a Fourier transform of the electric field of the wave

$$\hat{E}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(t) e^{i\omega t} dt. \tag{2.29}$$

The inverse of the this is

$$E(t) = \int_{-\infty}^{\infty} \hat{E}(\omega) e^{-i\omega t} d\omega. \tag{2.30}$$

Because $E(t)$ is real find that $\hat{E}(-\omega) = \hat{E}^*(\omega)$ so we don't have to keep track of the negative frequencies.

To study the energy carried by the wave we look at the Poynting vector

$$\frac{dW}{dtdA} = \frac{c}{4\pi} E^2(t) \tag{2.31}$$

The total energy per unit area in the wave is

$$\frac{dW}{dA} = \frac{c}{4\pi} \int_{-\infty}^{\infty} E^2(t) dt \tag{2.32}$$

Parseval's theorem (see § A for a proof) for Fourier transforms states that

$$\int_{-\infty}^{\infty} |E|^2(t) dt = 2\pi \int_{-\infty}^{\infty} |\hat{E}(\omega)|^2 d\omega. \tag{2.33}$$

Additionally, because $\hat{E}(-\omega) = \hat{E}^*(\omega)$ we have

$$\int_{-\infty}^{\infty} E^2(t) dt = 4\pi \int_0^{\infty} |\hat{E}(\omega)|^2 d\omega. \tag{2.34}$$

and we can write

$$\frac{dW}{dA} = c \int_0^{\infty} |\hat{E}(\omega)|^2 d\omega. \tag{2.35}$$

And we obtain that

$$\frac{dW}{dAd\omega} = c |\hat{E}(\omega)|^2 \tag{2.36}$$

The intensity is related to the energy per unit time. If the pulse repeats on a time scale T or the wave changes only on timescales T much longer than $1/\omega$ we may define

$$\frac{dW}{dAd\omega dt} = \frac{c}{T}|\hat{E}(\omega)|^2 \tag{2.37}$$

Table 2.1 gives a few Fourier transforms of common functions. Although the last one seems rather arcane it has an important property,

$$\int \frac{\sin\left[T(\omega-\omega')\right]}{\pi(\omega-\omega')} d\omega = 1 \tag{2.38}$$

that lets us define the pair

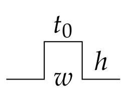
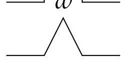
$$E(t) = e^{-i\omega' t} \tag{2.39}$$

$$\hat{E}(\omega) = \delta(\omega-\omega') \tag{2.40}$$

where

$$\int \delta(\omega-\omega') f(\omega) d\omega = f(\omega') \tag{2.41}$$

Table 2.1: Some Useful Fourier transforms

$E(t)$	$\hat{E}(\omega)$
	$\frac{h}{\pi\omega} \sin \frac{\omega w}{2} e^{i\omega t_0}$
	$\frac{2h}{\pi w \omega^2} \left(1 - \cos\frac{\omega w}{2}\right) e^{i\omega t_0}$
$\exp\left(-\frac{(t-t_0)^2}{2\sigma^2}\right)$	$e^{i\omega t_0} \exp\left(-\frac{\omega^2\sigma^2}{2}\right)\sqrt{\frac{\sigma^2}{2\pi}}$
$e^{-a\|t\|}$	$\frac{1}{2\pi}\frac{a}{a^2+\omega^2}$
$\exp\left(-i\omega' t\right)$ for $\|t\| < T$	$\frac{\sin[T(\omega-\omega')]}{\pi(\omega-\omega')}$

Polarization

For a general electromagnetic wave with wavenumber **k** we can define a basis for the polarization of the wave: $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ such that $\boldsymbol{\epsilon}_1 \times \boldsymbol{\epsilon}_2 \parallel \mathbf{k}$. For example a wave can be linearly polarized with its electric field always pointing along $\boldsymbol{\epsilon}_1$ or along $\boldsymbol{\epsilon}_2$. A general solution is a linear combination of these two waves with complex coefficients.

To be more specific we have

$$\begin{aligned}\mathbf{E}_1 &= \boldsymbol{\epsilon}_1 E_1 e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}\\ \mathbf{E}_2 &= \boldsymbol{\epsilon}_2 E_2 e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}\\ \mathbf{B}_j &= \frac{\mathbf{k}\times\mathbf{E}_j}{k}\end{aligned}$$

and a general wave would be $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$. Because the coefficients E_1 and E_2 are complex we can introduce a phase difference between the two perpendicular components. If this phase difference is zero, then the wave is linearly polarized (left panel of Fig. 2.1) with the polarization vector making an angle $\theta = \tan^{-1}(E_2/E_1)$ with $\boldsymbol{\epsilon}_1$ and a magnitude of $E = \sqrt{E_1^2 + E_2^2}$

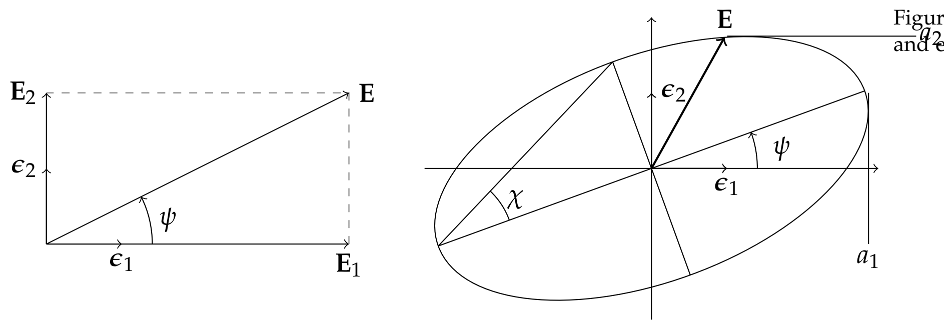


Figure 2.1: Electric field of linearly (left) and elliptically (right) polarized waves

If the phase difference is nonzero, one in genral has an elliptically polarized wave as shown in the right panel of Fig. 2.1. The orientation of the ellipse is characterized by the orientation, tile or azimuth angle ψ which is the angle between the semimajor axis of the ellipse and $\boldsymbol{\epsilon}_1$. The shape of the ellipse is measured by the ellipticity, ϵ, the ratio of the lengths of the major to minor axes. We can also define an ellipticity angle $\chi = \cot^{-1}\epsilon$. The sign of χ is positive for a right-hand circularly polarized wave — in this case the electric field would proceed anti-clockwise in Fig. 2.1.

One could have defined an alternative representation based on the circular polarizations

$$\boldsymbol{\epsilon}_\pm = \frac{1}{\sqrt{2}}\left(\boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2\right). \tag{2.42}$$

Because $\boldsymbol{\epsilon}_\pm$ is now complex one has to be a bit careful about its orthogonality properties. Specifically,

$$\boldsymbol{\epsilon}_\pm^* \cdot \boldsymbol{\epsilon}_\mp = 0, \boldsymbol{\epsilon}_\pm^* \cdot \mathbf{k} = 0, \boldsymbol{\epsilon}_\pm^* \cdot \boldsymbol{\epsilon}_\pm = 1. \tag{2.43}$$

Often it is convenient to use this circular polarization basis rather than the linear polarization basis above (for example, waves travelling through plasma).

Most astronomical detectors blueward of the microwave measure not the electric field directly but rather the energy delivered by the wave. It is possible to recover this polarization information through intensity measurements.

Generally one inserts a filter which collapses the incoming wave onto one of the polarization states and one measures the resulting intensity. For example, the intensity measured through a polarizing filter aligned along the $1-$direction is $|\boldsymbol{\epsilon}_1 \cdot \mathbf{E}|^2$. Let be more explicit and take two examples,

$$\begin{aligned} E_1 &= a_1 e^{i\delta_1}, \quad E_2 = a_2 e^{i\delta_2} \\ E_+ &= a_+ e^{i\delta_+}, \quad E_- = a_- e^{i\delta_-}. \end{aligned}$$

The first wave is given in the linear basis and the second is given in the circular basis. One typically makes a series of intensity measurements through filters and quarter wave plates with different orientations and combines the resulting intensities to form the Stokes parameters, I, Q, U and V or s_0, s_1, s_2 and s_3. The first parameter measures the total intensity of the wave, the sum of the intensities of the two linearly polarized measurements.

Stokes Parameters

In the linear polarization basis we have

$$\begin{aligned} I &= s_0 = |\boldsymbol{\epsilon}_1 \cdot \mathbf{E}|^2 + |\boldsymbol{\epsilon}_2 \cdot \mathbf{E}|^2 = a_1^2 + a_2^2 \\ Q &= s_1 = |\boldsymbol{\epsilon}_1 \cdot \mathbf{E}|^2 - |\boldsymbol{\epsilon}_2 \cdot \mathbf{E}|^2 = a_1^2 - a_2^2 \\ U &= s_2 = 2\Re\left[(\boldsymbol{\epsilon}_1 \cdot \mathbf{E})^*(\boldsymbol{\epsilon}_2 \cdot \mathbf{E})\right] = 2a_1 a_2 \cos\left(\delta_2 - \delta_1\right) \\ V &= s_3 = 2\Im\left[(\boldsymbol{\epsilon}_1 \cdot \mathbf{E})^*(\boldsymbol{\epsilon}_2 \cdot \mathbf{E})\right] = 2a_1 a_2 \sin\left(\delta_2 - \delta_1\right) \end{aligned}$$

and for the circular basis we have

$$\begin{aligned} I &= s_0 = |\boldsymbol{\epsilon}_+ \cdot \mathbf{E}|^2 + |\boldsymbol{\epsilon}_- \cdot \mathbf{E}|^2 = a_+^2 + a_-^2 \\ Q &= s_1 = 2\Re\left[(\boldsymbol{\epsilon}_+ \cdot \mathbf{E})^*(\boldsymbol{\epsilon}_- \cdot \mathbf{E})\right] = 2a_+ a_- \cos\left(\delta_- - \delta_+\right) \\ U &= s_2 = 2\Im\left[(\boldsymbol{\epsilon}_+ \cdot \mathbf{E})^*(\boldsymbol{\epsilon}_- \cdot \mathbf{E})\right] = 2a_+ a_- \sin\left(\delta_- - \delta_+\right) \\ V &= s_3 = |\boldsymbol{\epsilon}_+ \cdot \mathbf{E}|^2 - |\boldsymbol{\epsilon}_- \cdot \mathbf{E}|^2 = a_+^2 - a_-^2 \end{aligned}$$

The fractional polarization is given by

$$\Pi = \frac{\sqrt{Q^2 + U^2 + V^2}}{I}. \tag{2.44}$$

and the fractional linear polarization

$$\Pi_l = \frac{\sqrt{Q^2 + U^2}}{I}. \tag{2.45}$$

The four Stokes parameters satisfy the following relationship for a truly monochromatic wave

$$s_0^2 = s_1^2 + s_2^2 + s_3^2. \quad (2.46)$$

Poincaré Sphere

This result shows that the Stoke's parameters live on a sphere of radius $r \leq s_0$ where the extent of polarization $\Pi = r/s_0$. This sphere of polarization is known as the Poincaré sphere (Fig. 2.2) and the location of the polarization on the sphere is related to the orientation of the polarization ellipse in Fig. 2.1. In particular we have

$$\begin{aligned} s_1 &= Q = \Pi I \cos 2\psi \cos 2\chi \\ s_2 &= U = \Pi I \sin 2\psi \cos 2\chi \\ s_3 &= V = \Pi I \sin 2\chi \end{aligned}$$

which relates Stoke's parameters to the orientation and shape of the polarization ellipse. The two angles defined in Fig. 2.1 related to the latitude (2χ) and longitude (2ψ) of the polarization vector (s_1, s_2, s_3) on the Poincaré sphere.

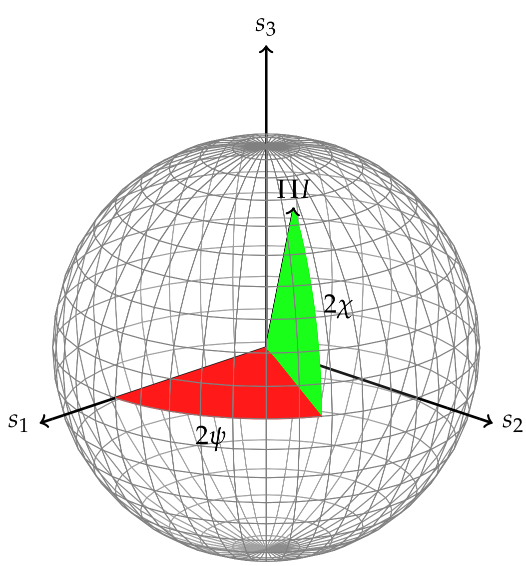


Figure 2.2: The Poincaré sphere

An interesting and useful relationship is that the Stokes parameters are additive for waves whose phases are not correlated. Let's take two waves of frequencies ω_a and ω_b and calculate the value of the first Stokes parameter as an example. We have

$$I = s_0 = |\boldsymbol{\epsilon}_1 \cdot \mathbf{E}|^2 + |\boldsymbol{\epsilon}_2 \cdot \mathbf{E}|^2 \quad (2.47)$$

$$
\begin{aligned}
&= \frac{1}{\Delta t}\int_0^{\Delta t}\left[\left|\boldsymbol{\epsilon}_1\cdot\mathbf{E}_a e^{-i\omega_a t}+\boldsymbol{\epsilon}_1\cdot\mathbf{E}_b e^{-i\omega_b t}\right|^2+\right.\\
&\qquad\left.\left|\boldsymbol{\epsilon}_2\cdot\mathbf{E}_a e^{-i\omega_a t}+\boldsymbol{\epsilon}_2\cdot\mathbf{E}_b e^{-i\omega_b t}\right|^2\right]dt \qquad (2.48)\\
&= \frac{1}{\Delta t}\int_0^{\Delta t}\left\{\,|\boldsymbol{\epsilon}_1\cdot\mathbf{E}_a|^2+|\boldsymbol{\epsilon}_1\cdot\mathbf{E}_b|^2+|\boldsymbol{\epsilon}_1\cdot\mathbf{E}_a|^2+|\boldsymbol{\epsilon}_1\cdot\mathbf{E}_b|^2+\right.\\
&\qquad\left.2\left[(\boldsymbol{\epsilon}_1\cdot\mathbf{E}_a)(\boldsymbol{\epsilon}_1\cdot\mathbf{E}_b)+(\boldsymbol{\epsilon}_2\cdot\mathbf{E}_a)(\boldsymbol{\epsilon}_2\cdot\mathbf{E}_b)\right]\cos\left[(\omega_a-\omega_b)t\right]\right\} \qquad (2.49)\\
&= s_{0,a}+s_{0,b}+\begin{cases}\mathcal{O}\left[\sqrt{s_{0,a}s_{0,b}}\,(\Delta\omega\Delta t)^{-1}\right] & \Delta\omega\Delta t\gg 1\\ \mathcal{O}\left[\sqrt{s_{0,a}s_{0,b}}\,(\Delta\omega\Delta t)\right] & \Delta\omega\Delta t\ll 1\end{cases} \qquad (2.50)
\end{aligned}
$$

where $\Delta\omega = \omega_a - \omega_b$. For example, if we look at a star over a wide range of frequencies (the definition of wide is $\Delta\omega\Delta t \gg 1$), the phase of waves at one end of the frequency range will not correlate with waves at the other end.

When we measure the Stokes parameters in practice we measure for example

$$s_2 = 2\left\langle a_1 a_2 \cos(\delta_2-\delta_1)\right\rangle. \qquad (2.51)$$

Although $\cos^2 x + \sin^2 x = 1$, $0 \le \langle\cos x\rangle^2 + \langle\sin x\rangle^2 \le 1$, so for a quasimonochromatic wave we have

$$s_0^2 \ge s_1^2 + s_2^2 + s_3^2. \qquad (2.52)$$

Because the Stokes parameters are additive and measure the energy content of the wave, they are a natural basis to calculate the radiative transfer of polarized radiation.

Electromagnetic Potentials

Looking at the structure of Maxwell's equations, we can see that we can express the magnetic field as the curl of another field, the vector potential,

$$\nabla\cdot\mathbf{B} = \nabla\cdot(\nabla\times\mathbf{A}) = 0 \qquad (2.53)$$

where the second equality is an identity if $\mathbf{B} = \nabla\times\mathbf{A}$.

Let's substitute this into the other homogenous Maxwell equation,

$$
\begin{aligned}
\nabla\times\mathbf{E}+\frac{1}{c}\frac{\partial}{\partial t}\nabla\times\mathbf{A} &= 0 \qquad (2.54)\\
\nabla\times\left(\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right) &= 0 \qquad (2.55)
\end{aligned}
$$

If

$$-\nabla\phi = \mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} \qquad (2.56)$$

then the second homogeneous Maxwell equation is satisfied as well.

Let's recap

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} && (2.57) \\ \mathbf{E} &= -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} && (2.58) \\ & && (2.59) \end{aligned}$$

The expression of the fields in terms of the vector and scalar potential guarantees that two out of four of Maxwell's equations are satisfied. Let's substitute our results into the remaining Maxwell's equations,

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho && (2.60) \\ \nabla \cdot \left(-\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \right) &= 4\pi\rho && (2.61) \\ \nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \mathbf{A} &= -4\pi\rho && (2.62) \end{aligned}$$

and the second inhomogeneous equation gives

$$\begin{aligned} \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mathbf{J} && (2.63) \\ \nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c}\frac{\partial}{\partial t}\left(\nabla\phi + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \right) &= \frac{4\pi}{c}\mathbf{J} && (2.64) \\ \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} + \frac{1}{c}\frac{\partial}{\partial t}\left(\nabla\phi + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \right) &= \frac{4\pi}{c}\mathbf{J} && (2.65) \end{aligned}$$

Now let's rearrange the last equation a bit more,

$$\begin{aligned} -\nabla(\nabla \cdot \mathbf{A}) + \nabla^2\mathbf{A} - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\phi) - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\frac{4\pi}{c}\mathbf{J} && (2.66) \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} \right) &= -\frac{4\pi}{c}\mathbf{J} && (2.67) \end{aligned}$$

Let's look at the last of the charge density equations equations some more,

$$\begin{aligned} \nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \mathbf{A} &= -4\pi\rho && (2.68) \\ \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{1}{c}\frac{\partial}{\partial t}\left(\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} \right) &= -4\pi\rho && (2.69) \end{aligned}$$

Although it looks like that the equation is a bit more complicated than before, it now has the precise form of the equation with the current.

Wouldn't life be simpler if the quantity in the parenthesis in both equations vanished? Guess what? We can choose for it to vanish by making a good choice of gauge. Only the electric and magnetic fields are measurable so we can make any change to the potentials $\mathbf{A}$ and ϕ that we want as long at $\mathbf{E}$ and $\mathbf{B}$ remain unchanged. Because

$\mathbf{B} = \nabla \times \mathbf{A}$ we can add the gradient of any function ψ to $\mathbf{A}$ without changing $\mathbf{B}$ (the curl of a gradient of a function is zero).

However, if we add a gradient of function to $\mathbf{A}$ the value of $\mathbf{E}$ is affected,

$$\mathbf{E} \rightarrow \mathbf{E} - \frac{1}{c}\frac{\partial}{\partial t}\nabla\psi. \tag{2.70}$$

To fix this we also have to change the scalar potential ϕ at the same time by subtracting $1/c(\partial\psi/\partial t)$ from ϕ. Therefore, we find that the equations of electromagnetism remain unchanged if one replaces

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\psi \;\text{ and }\; \phi \rightarrow \phi - \frac{1}{c}\frac{\partial\psi}{\partial t} \tag{2.71}$$

This is the gauge transformation and it means that we in general have the freedom to set a particular scalar constraint on the potentials.

Lorenz Gauge

In particular we would like to set

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0 \tag{2.72}$$

This is equivalent to finding a function ψ that satisfies

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0 \tag{2.73}$$

and adding it to $\mathbf{A}$ to get $\mathbf{A}'$. It turns out that this is possible so we are free to use the following equations for the potentials,

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho, \;\text{ and }\; \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J}. \tag{2.74}$$

This is the Lorenz gauge (which happens to be *Lorentz* invariant).

Green's Function

Both of the equations have the same form. Let's look at the equation for the electric field

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho \tag{2.75}$$

and write ϕ and ρ in terms of their Fourier transforms in time, e.g.

$$\rho(\mathbf{r}, t) = \int_{-\infty}^{\infty} \hat{\rho}(\mathbf{r}, \omega) e^{-i\omega t} dt \tag{2.76}$$

so the equation for the potential now looks like

$$\nabla^2\hat{\phi}(\mathbf{r}, \omega) + \frac{\omega^2}{c^2}\hat{\phi}(\mathbf{r}, \omega) = -4\pi\hat{\rho}(\mathbf{r}, \omega). \tag{2.77}$$

Now let's look for a particular solution $G(\mathbf{r},\omega)$ for $\hat{\phi}$ where $\hat{\rho}$ vanishes everywhere but the origin

$$\nabla^2 \hat{G}(\mathbf{r},\omega) + \frac{\omega^2}{c^2}\hat{G}(\mathbf{r},\omega) = -4\pi\delta(\mathbf{r}). \tag{2.78}$$

The function $G(\mathbf{r},\omega)$ is the Green's function of the equation (Eq. 2.77). It is a useful concept because Maxwell's equations are linear, so the principle of superposition applies. The electromagnetic fields for two charges is simply the sum of the fields of each charge on its own.

Because the right-hand side only depends on the magnitude of $\mathbf{r}$, the Green's function must be spherically symmetric, so

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}\left[r\hat{G}(r,\omega)\right] + \frac{\omega^2}{c^2}\hat{G}(r,\omega) = -4\pi\delta(r) \tag{2.79}$$

which is nearly the equation for the potential for a point change.

We can start with the solution $1/r$ and add an exponential term to handle the second term in the equation; let's substitute the following ansatz

$$\hat{G}(r,\omega) = \frac{e^{ikr}}{r} \tag{2.80}$$

which yields

$$-k^2\frac{e^{ikr}}{r} + \frac{\omega^2}{c^2}\frac{e^{ikr}}{r} = -4\pi\delta(r) \tag{2.81}$$

which trivially solves the equation everywhere but the origin if $k = \pm\omega/c$. Because Eq. 2.79 is a second-order differential equation we get two solutions,

$$\hat{G}^{\pm}(r,\omega) = \frac{e^{i\pm r\omega/c}}{r} \tag{2.82}$$

and the complete solution is a linear combination of the two. Now because we know the solution for source at a point we can write the solution for a distribution of charge

$$\hat{\phi}^{\pm}(\mathbf{r},\omega) = \int d^3r' \frac{\exp\left(\pm i\omega/c|\mathbf{r}-\mathbf{r}'|/c\right)}{|\mathbf{r}-\mathbf{r}'|}\hat{\rho}(\mathbf{r}',\omega) \tag{2.83}$$

and write out the potential as a function of time

$$\phi^{\pm}(\mathbf{r},t) = \int d^3r' d\omega \frac{\exp\left[-i\omega\left(t \mp |\mathbf{r}-\mathbf{r}'|/c\right)\right]}{|\mathbf{r}-\mathbf{r}'|}\hat{\rho}(\mathbf{r}',\omega). \tag{2.84}$$

We can perform the integral over frequency because it is the same as Eq. (2.76) but with t replaced by $t \mp |\mathbf{r}-\mathbf{r}'|/c$, so we have

$$\phi^{\pm}(\mathbf{r},t) = \int d^3r' \rho\left(\mathbf{r}, t \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\frac{1}{|\mathbf{r}-\mathbf{r}'|}. \tag{2.85}$$

The solution for the vector potential is similar. We have two choices (or a linear combination). The potential here can depend on the locations of charges along the past and the future light cones. Although the latter choice appears to violate causality, it all really depends on what questions that you would like to ask. For example, if you wanted to know given the distribution of fields here and now, what would be the distribution of charges to absorbed the radiation in the future, then the **advanced** potential (with the plus sign after the time coordinate). Here we are generally interested in the radiation that a configuration of charges emit, so we shall use the **retarded** potentials,

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{c} \int d^3 r' \frac{\mathbf{J}\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r} - \mathbf{r}'|} \tag{2.86}$$

$$\phi(\mathbf{r}, t) = \int d^3 r' \frac{\rho\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r} - \mathbf{r}'|}. \tag{2.87}$$

Further Reading

To learn more about faint x-ray structure in the Crab nebula, consult

- Seward, F. D., Tucker, W. H. & Fesen, R. A. 2006, "Faint X-Ray Structure in the Crab Pulsar Wind Nebula," *ApJ*, 652, 1277

and

- Heyl, J. S. & Shaviv, N. J. 2000, "Polarization evolution in strong magnetic fields," *MNRAS*, 311, 555

use the Poincaré sphere extensively to study how the polarization of emission from the surface of the Crab pulsar changes as it travels through the star's magnetic field.

The general development of Maxwell's equations and the polarization of radiation are examined in Chapter 6 and §§ 7.1-7.2 of

- Jackson, J. D., *Classical Electrodynamics*.

Problems

1. **Coulomb's Law**

   Derive Coulomb's law from Maxwell's Equations.

2. **Ohm's Law and Absoption:**

   In certain cases the process of aborption of radiation can be treated by means of the macroscopic Maxwell equations. For example, suppose we have aconducting medium so that the current density $\mathbf{j}$ is related to the electric field $\mathbf{E}$ by Ohm's law: $\mathbf{J} = \sigma \mathbf{E}$ where σ is the conductivity (cgs unit = sec^{-1}). Investigate the propagation of electromagnetic waves in such a medium and show that:

(a) The wave vector $\mathbf{k}$ is complex

$$\mathbf{k}^2 = \frac{\omega^2 m^2}{c^2}$$

where m is the complex index of refraction with

$$m^2 = \mu\epsilon\left(1 + \frac{4\pi i\sigma}{\omega\epsilon}\right)$$

(b) The waves are attenuated as they propagate, corresponding to an absorption coefficient.

$$\alpha = \frac{2\omega}{c}\Im(m)$$

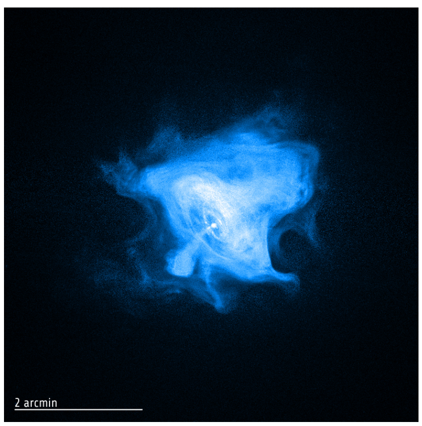


Figure 2.3: A Chandra image of the outskirts of the Crab pulsar wind nebula. Credit: NASA/CXC/SAO/F.Seward et al

3. **The Edge of the Crab**

Fig. 2.3 shows the x-ray emission of the Crab pulsar wind nebula at a distance of 2 kpc. The x-ray emitting gas is contained by magnetic fields causing the x-ray emission regions to end sharply. We can relate the frequency of the emission to the energy of the electrons and the strength of the magnetic field by

$$\omega = \left(\frac{E}{m_e c^2}\right)^2 \frac{eB}{m_e c} \tag{2.88}$$

and assume that the electrons are relativistic so their inertial mass is E/c^2. Use the sharpness of the emission regions to determine the energy of the electrons and the strength of the magnetic field.

4. **Momentum and Angular Momentum:**

This problem is meant to deduce the momentum and angular momentum properties of radiation and does not recesarily represent any real physical system of interest. Consider a charge Q in a viscous medium where the viscous force is proportional to velocity:

$$F_{\text{visc}} = -\beta v$$

Suppose a circular polarized wave passes through the medium. The equation of motion of the charge is

$$m\frac{dv}{dt} = F_{\text{visc}} + F_{\text{Lorentz}}$$

We assume that the terms on the right dominate the inertial term on the left, so that approximately

$$0 = F_{\text{visc}} + F_{\text{Lorentz}}$$

Let the frequency of the wave be ω and the strength of the electric field be E.

(a) Show that to lowest order (neglecting the magnetic force) the crage moves on a circle in a plane normal to the direction of propagation of the wave with speed QE/β and with radius $QE/(\beta\omega)$.

(b) Show that the power transmitted to the fliud by the wave is Q^2E^2/β

(c) By considering the small magnetic force acting on the particle show that the momentum per unit time (force) given to the fluid by the wave is in the direction of propagation and has the magnitude $Q^2E^2/(\beta c)$

(d) Show that the angular momentum per unit time (torque) given to the fliud by the wave is in the direction of propagation and has magnitude $\pm Q^2E^2/(\beta\omega 0$ where the + is for left and - is for right circular polarization.

(e) Show that the absorption cross section of the charge is $4\pi Q^2/(\beta c)$.

(f) If we regard the radiation to be composed of circular polarized photons of energy $E_\gamma = h\nu$, show that these results imply that the photon has momemtum $p = h/\lambda = E_\gamma/c$ and has angular momemtum $J = \pm\hbar$ along the direction of propagation.

(g) Repeat this problem for a linearly polarized wave

5. **Maxwell's equations before Maxwell:**

Show that Maxwell's equations before Maxwell, that is, without the "displacement current" term, $c^{-1}\frac{\partial D}{\partial t}$, unacceptably constrained the sources of the field and also did not permit the existence of waves.

6. **Coulomb gauge** Derive the equations describing the dynamics of the electric and vector potentials in the Coulomb gauge

$$\nabla \cdot \mathbf{A} = 0$$

Look at the equation for the electric potential. What is the solution to the electric potential given the charge density ρ? Why is this called the Coulomb gauge?

How does the expression for the scalar potential in the Coulomb gauge differ from that in the Lorenz gauge? What is strange about it? Is it physical?

Now look at the equation for the vector potential. Show that the LHS can be arranged to be the same as in the Lorenz gauge but the RHS is not just the current but the current plus something else.

Show that the RHS can be expressed as

$$\frac{4\pi}{c}\left(\mathbf{J} - \mathbf{J}_{\text{long}}\right)$$

where

$$\mathbf{J}_{\text{long}} = -\frac{1}{4\pi}\nabla \int \frac{\nabla' \cdot \mathbf{J}}{|\mathbf{x} - \mathbf{x}'|} d^3x$$

3
Radiation from Moving Charges

We will start to look at how radiation gets produced, scattered and absorbed at a microscopic level to derive quantities like j_ν, α_ν and σ_ν.

Retarded Potentials

We saw in the last section that in the Lorenz gauge the equations for the vector and scalar potential are

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho, \text{ and } \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J}. \quad (3.1)$$

which have the following solution

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int d^3r' \frac{\mathbf{J}\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|} \quad (3.2)$$

$$\phi(\mathbf{r},t) = \int d^3r' \frac{\rho\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}. \quad (3.3)$$

An equivalent way of writing $\phi(\mathbf{r},t)$ is

$$\phi(\mathbf{r},t) = \int d^3r' \int dt' \frac{\rho\left(\mathbf{r}', t'\right)}{|\mathbf{r}-\mathbf{r}'|}\delta(t' - t + |\mathbf{r}-\mathbf{r}'|/c) \quad (3.4)$$

and similarly for $\mathbf{A}$

Let's think about a single charge with charge q and position $\mathbf{r}_0(t)$. It can be characterized by

$$\rho(\mathbf{r},t) = q\delta(\mathbf{r}-\mathbf{r}_0(t)) \text{ and } \mathbf{j}(r,t) = q\mathbf{u}\delta(\mathbf{r}-\mathbf{r}_0(t)) \quad (3.5)$$

Let's substitute this expression for ρ,

$$\phi(\mathbf{r},t) = \int d^3r' \int dt' \frac{q\delta(\mathbf{r}'-\mathbf{r}_0(t'))}{|\mathbf{r}-\mathbf{r}'|}\delta(t' - t + |\mathbf{r}-\mathbf{r}'|/c) \quad (3.6)$$

$$= q\int dt' \frac{1}{|\mathbf{r}-\mathbf{r}_0(t')|}\delta(t' - t + |\mathbf{r}-\mathbf{r}_0(t')|/c) \quad (3.7)$$

It is easy to perform integrals over a δ-function if the integral is over the argument of the δ-function, so we would like to perform the following change of variables

$$t'' = t' - t + \frac{|\mathbf{r} - \mathbf{r}_0(t')|}{c} \tag{3.8}$$

yielding the Liénard-Wiechert potentials

$$\begin{aligned} \phi(\mathbf{r}, t) &= q \int dt'' \frac{1}{|\mathbf{r} - \mathbf{r}_0(t')|} \delta(t'') \frac{\partial t'}{\partial t''} && (3.9) \\ &= \frac{q}{R(t_{\rm ret})\kappa(t_{\rm ret})} && (3.10) \\ \mathbf{A} &= \frac{q\mathbf{u}(t_{\rm ret})}{cR(t_{\rm ret})\kappa(t_{\rm ret})} && (3.11) \end{aligned}$$

where

$$\begin{aligned} R(t) &= |\mathbf{r} - \mathbf{r}_0(t)| && (3.12) \\ t_{\rm ret} &= t - \frac{R(t_{\rm ret})}{c} && (3.13) \\ \kappa(t) &= \frac{\partial t''}{\partial t'}. && (3.14) \end{aligned}$$

Let's look at the partial derivative now,

$$\kappa(t) = \frac{\partial t''}{\partial t'} = 1 + \frac{1}{c}\frac{\partial R(t)}{\partial t} \tag{3.15}$$

and looking at $R(t)$

$$R(t)^2 = \mathbf{R}(t) \cdot \mathbf{R}(t) \tag{3.16}$$

so

$$2R(t)\dot{R}(t) = -2\mathbf{R}(t) \cdot \mathbf{u}(t) \quad \text{NB: } \mathbf{R} = \mathbf{r} - \mathbf{r}_0(t) \tag{3.17}$$

so

$$\kappa(t) = 1 - \frac{1}{c}\frac{\mathbf{R}(t) \cdot \mathbf{u}(t)}{R(t)} = 1 - \frac{1}{c}\mathbf{n}(t) \cdot \mathbf{u}(t). \tag{3.18}$$

The Fields

We can use the potentials to determine the electric and magnetic fields produced by the moving particle. Let us define

$$\boldsymbol{\beta} \equiv \frac{\mathbf{u}}{c}, \text{ so } \kappa = 1 - \mathbf{n} \cdot \boldsymbol{\beta} \tag{3.19}$$

which yield the fields (see § 14.1 of Jackson)

$$\begin{aligned} \mathbf{E}(r, t) &= q \left[\frac{(\mathbf{n} - \boldsymbol{\beta})(1 - \beta^2)}{\kappa^3 R^2} \right]_{\rm ret} + \frac{q}{c} \left[\frac{\mathbf{n}}{\kappa^3 R} \times \left[(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right]_{\rm ret} && (3.20) \\ \mathbf{B}(r, t) &= [\mathbf{n} \times \mathbf{E}(r, t)]_{\rm ret}. && (3.21) \end{aligned}$$

It is important to remember that all of the properties of the particle are evaluated at the retarded time.

A few things to notice are that if the particle is not accelerating the electric field points to the current not the retarded position of the particle. This allows us to graphically depict the field for a particle that is stopped suddenly.

The fields have two parts. The first part is proportional to $1/R^2$ and it is simply a generalization of the field for a stationary charge. The second terms are proportional to $1/R$. They are

$$\mathbf{E}_{\text{rad}}(r,t) = +\frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R} \times \left[(\mathbf{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]\right] \quad (3.22)$$

$$\mathbf{B}_{\text{rad}}(r,t) = \left[\mathbf{n} \times \mathbf{E}_{\text{rad}}(r,t)\right]. \quad (3.23)$$

We can calculate the Poynting vector of the radiation fields

$$\mathbf{S} = \mathbf{n}\frac{q^2}{4\pi c \kappa^6 R^2}\left|\mathbf{n} \times \left\{(\mathbf{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right\}\right|^2 \quad (3.24)$$

Distribution in Frequency and Angle

To examine the spectrum of the radiation let us define examine the Fourier transform of the electric field. We know (Eq. 2.36)

$$\frac{dW}{dAd\omega} = c|\hat{E}(\omega)|^2 \quad (3.25)$$

where

$$\hat{E}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(t) e^{i\omega t} dt. \quad (3.26)$$

in our case we are interested in the energy emitted per solid angle so

$$\frac{dW}{d\Omega d\omega} = R^2 \frac{dW}{dAd\omega} = c|R\hat{E}(\omega)|^2 \quad (3.27)$$

and

$$R\hat{E}(\omega) = \frac{q}{2\pi c}\int_{-\infty}^{\infty}\left[\frac{\mathbf{n}}{\kappa^3} \times \left[(\mathbf{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]\right]_{\text{ret}} e^{i\omega t} dt. \quad (3.28)$$

The intergrand is evaluated at the retarded time, $t' + R(t')/c = t$, so we can change the variable of integration from t to t' to yield

$$R\hat{E}(\omega) = \frac{q}{2\pi c}\int_{-\infty}^{\infty}\left[\frac{\mathbf{n}}{\kappa^2} \times \left[(\mathbf{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]\right] e^{i\omega(t'+R(t')/c)} dt'. \quad (3.29)$$

If we assume that the observer is far away from where the acceleration occurs, the unit vector $\mathbf{n}$ can be taken to be constant and $R(t') \approx R_0 - \mathbf{n} \cdot \mathbf{r}(t')$, yielding

$$R\hat{E}(\omega) = \frac{q}{2\pi c}\int_{-\infty}^{\infty}\left[\frac{\mathbf{n}}{\kappa^2} \times \left[(\mathbf{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]\right] e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}(t')/c)} dt'. \quad (3.30)$$

As we did earlier (§ 2), the total energy radiated per unit angle is

$$\frac{dW}{d\Omega d\omega} = \frac{q^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \frac{\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^2} e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}(t')/c)}dt'\right|. \quad (3.31)$$

The expression can be simplied further by noticing that

$$\frac{\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^2} = \frac{d}{dt'}\left[\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})}{1-\boldsymbol{\beta}\cdot\mathbf{n}}\right]. \quad (3.32)$$

and integrating Eq. 3.30 by parts to yield

$$\frac{dW}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c^3}\left|\int_{-\infty}^{\infty} \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)}dt'\right|^2. \quad (3.33)$$

Non-relativistic particles

Let's assume that $|\boldsymbol{\beta}| \ll 1$ and focus on a particular frequency ν so that $\dot{u} \sim u\nu$. We can compare the "acceleration" fields to the "velocity" field

$$\mathbf{E}_{\rm acc} = \frac{q}{c^2 R}\left[\mathbf{n}\times\{\mathbf{n}\times\dot{\mathbf{u}}\}\right] \quad (3.34)$$

$$\mathbf{E}_{\rm vel} = \frac{q\mathbf{n}}{R^2} \quad (3.35)$$

so

$$\frac{E_{\rm acc}}{E_{\rm vel}} \sim \frac{R\dot{u}}{c^2} \sim \frac{Ru\nu}{c^2} = \frac{u}{c}\frac{R}{\lambda} \quad (3.36)$$

so for points in the "near zone", $R \le \lambda$, the velocity field is stronger than the acceleration field by a factor $\ge c/u$; but for points sufficiently far into the "far zone", $R \gg \lambda(c/u)$, the acceleration field dominates.

Let's derive **Larmor's formula** for the radiated energy. Let use the angle Θ to denote the angle between the vectors $\mathbf{n}$ and $\dot{\mathbf{u}}$, so we have

$$|\mathbf{E}_{\rm acc}| = |\mathbf{B}_{\rm acc}| = \frac{q\dot{u}}{Rc^2}\sin\Theta \quad (3.37)$$

Using the formula for the Poynting vector we have

$$\mathbf{S} = \mathbf{n}\frac{c}{4\pi}E_{\rm acc}^2 = \frac{c}{4\pi}\frac{q^2\dot{u}^2}{R^2c^3}\sin^2\Theta \quad (3.38)$$

The power radiated per unit solid angle is simply $(\mathbf{S}\cdot\mathbf{n})R^2$ or

$$\frac{dW}{dtd\Omega} = \frac{q^2\dot{u}^2}{4\pi c^3}\sin^2\Theta \quad (3.39)$$

Let's integrate over all angles to get the power

$$P = \frac{dW}{dt} = \frac{q^2\dot{u}^2}{4\pi c^3}\int \sin^2\Theta d\Omega = \frac{q^2\dot{u}^2}{2c^3}\int_{-1}^{1}\left(1-\mu^2\right)d\mu \quad (3.40)$$

$$= \frac{2q^2\dot{u}^2}{3c^3} \quad (3.41)$$

How about relativistic particles?

Let's look at the Poynting vector again.

$$\mathbf{S} = \mathbf{n}\frac{q^2}{4\pi c\kappa^6 R^2}\left|\mathbf{n}\times\left\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}\right|^2 \quad (3.42)$$

where $\mathbf{S}\cdot\mathbf{n}$ is the energy per unit area per unit time detected at an observation poiint at time t of radiation emitted by the charge at time $t' = 1 - R(t')/c$. Let's calculate the energy radiated away from $t' = T_1$ to $t' = T_2$ we would have

$$E = \int_{t=T_1+[R(T_1)/c]}^{t=T_2+[R(T_2)/c]} [\mathbf{S}\cdot\mathbf{n}]\, dt = \int_{T_1}^{T_2} [\mathbf{S}\cdot\mathbf{n}]\frac{dt}{dt'}dt' \quad (3.43)$$

so we have

$$\frac{dP(t')}{d\Omega} = R^2\left(\mathbf{S}\cdot\mathbf{n}\right)\frac{dt}{dt'} = R^2\left(\mathbf{S}\cdot\mathbf{n}\right)(1-\boldsymbol{\beta}\cdot n) \quad (3.44)$$

When we include the Poynting vector expression we have

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi c}\frac{\left|\mathbf{n}\times\left\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}\right|^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^5} \quad (3.45)$$

Let's start by assuming the $\boldsymbol{\beta}$ is parallel to $\dot{\boldsymbol{\beta}}$, so $\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}} = 0$. We get

$$\frac{dP(t')}{d\Omega} = \frac{q^2\dot{u}^2}{4\pi c^3}\frac{\sin^2\Theta}{(1-\beta\cos\Theta)^5} \quad (3.46)$$

Let's integrate the power over all angles,

$$P = 2\pi\frac{q^2\dot{u}^2}{4\pi c^2}\int_{-1}^{1}\frac{1-\mu^2}{(1-\beta\mu)^5}d\mu = \frac{2}{3}\frac{q^2\dot{u}^2}{c^2}\gamma^6 \quad (3.47)$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{(1+\beta)(1-\beta)}} \quad (3.48)$$

Let's take a second look at the angular distribution of power for small angles and $\beta \approx 1$.

$$\begin{aligned}\frac{dP(t')}{d\Omega} &\approx \frac{q^2\dot{u}^2}{4\pi c^3}\frac{\theta^2}{[1-(1-\gamma^{-2}/2)(1-\theta^2/2)]^5} && (3.49)\\ &\approx \frac{q^2\dot{u}^2}{4\pi c^3}\frac{\theta^2}{[\gamma^{-2}/2+\theta^2/2]^5} && (3.50)\\ &\approx \frac{8}{\pi}\frac{q^2\dot{u}^2}{c^3}\gamma^8\frac{(\gamma\theta)^2}{[1+(\gamma\theta)^2]^5} && (3.51)\end{aligned}$$

Let's repeat the calculation for circular motion, in which $\boldsymbol{\beta}\perp\dot{\boldsymbol{\beta}}$. To be definitive we have to specify two angles for our observer $\mathbf{n}$. Let's

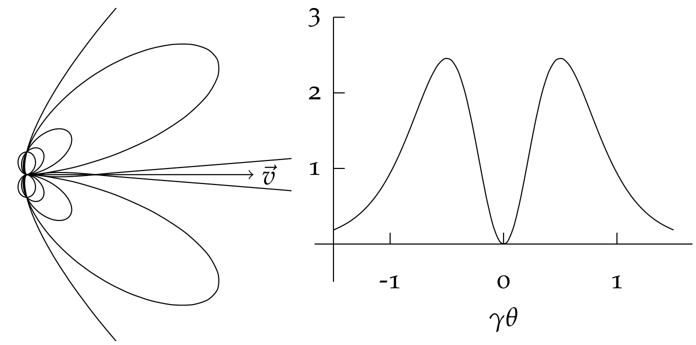


Figure 3.1: Radiated power as a function of angle and $\gamma\theta$ for longitudinal acceleration. The left-hand panel shows from inside out $\beta = 0, 0.2, 0.4, 0.6, 0.8$.

take the velocity to be along the z-axis and the accelation along the z axis, so Θ is the angle between $\mathbf{n}$ and the velocity and ϕ is the angle between the projection of the vector $\mathbf{n}$ into the $x - y$-plane and the acceleration (*i.e.* these are just ordinary spherical coordinates). We obtain in general,

$$\frac{dP(t')}{d\Omega} = \frac{q^2 \dot{u}^2}{4\pi c^3} \frac{1}{(1-\beta\cos\Theta)^3} \left[1 - \frac{\sin^2\Theta\cos^2\phi}{\gamma^2 (1-\beta\cos\Theta)^2} \right] \tag{3.52}$$

If we integrate this over all angles we get

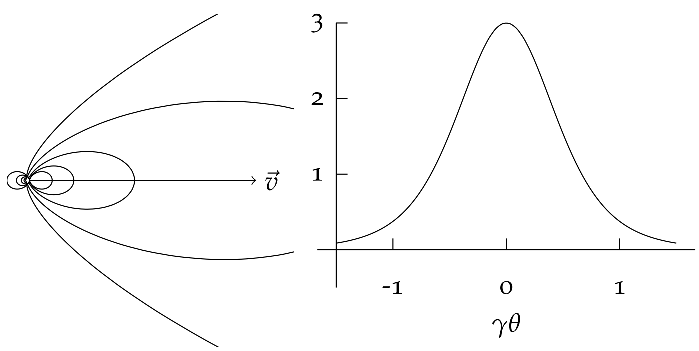


Figure 3.2: Radiated power as a function of angle and $\gamma\theta$ for circular motion in the plane of the circle. The left-hand panel shows from inside out $\beta = 0, 0.2, 0.4, 0.6, 0.8$.

$$P(t') = \frac{2}{3} \frac{q^2 \dot{u}^2}{c^3} \gamma^4 \tag{3.53}$$

At first glance, the power from longitudinal motion seems much larger that the circular motion, but it is important to compare the power emitted for the same applied force ($d\mathbf{p}/dt$).

For circular motions the applied force is $\gamma m \dot{u}$, yielding

$$P_{\text{circ}}(t') = \frac{2}{3} \frac{q^2}{m^2 c^3} \gamma^2 \left(\frac{d\mathbf{p}}{dt} \right)^2 \tag{3.54}$$

For longitudinal acceleration, the applied force is given by $m\gamma^3 \dot{u}$

$$P_{\text{long}}(t') = \frac{2}{3} \frac{q^2}{m^2 c^3} \left(\frac{d\mathbf{p}}{dt} \right)^2 \tag{3.55}$$

For a given applied force, the radiation due to the component perpendicular to the motion is much larger than the parallel component. If the particles are ultrarelativistic it is appropriate to neglect the parallel contribution completely.

Radiation from Systems of Particles

Let's focus back on the radiation from a non-relativistic particle, specifically a bunch of such particles. The electric field is linear so the total electric field of the ensemble is the sum of the particle's individual contributions,

$$\mathbf{E}_{\text{tot}} = \sum_i \frac{q_i}{c^2 R_i} \left[\mathbf{n}_i \times \{\mathbf{n}_i \times \dot{\mathbf{u}}_i\}\right] \tag{3.56}$$

This sum could get really cumbersome, especially if you have $\sim 10^{40}$ particles. You have to calculate the retarded position and keep track of the velocity of each one. There is an easier way.

Let's assume that the particles are confined to a region of size l and we are really far from that region, $R_i \gg l$ so $R_i \approx R$ where R is the distance to the centre of the region and $\mathbf{n}_i \approx \mathbf{n}$, a vector pointing to the centre of the region.

For the above expression for the electric field to be valid, R/c must be greater than any timescale (τ for the particles to change position (*i.e.* we are in the "far" zone). Let's also assume that $c\tau \gg l$ which means that we can neglect the difference in retarded time between particles at one end of the region and the other. Let's make these changes

$$\mathbf{E}_{\text{tot}} = \frac{1}{c^2 R} \left[\mathbf{n} \times \left\{\mathbf{n} \times \sum_i q_i \dot{\mathbf{u}}_i\right\}\right] \tag{3.57}$$

Let's define the dipole moment of the ensemble

$$\mathbf{d} = \sum_i q_i \mathbf{r}_{0,i} \tag{3.58}$$

which yields

$$\mathbf{E}_{\text{tot}} = \frac{1}{c^2 R} \left[\mathbf{n} \times \{\mathbf{n} \times \ddot{\mathbf{d}}\}\right] \tag{3.59}$$

We also get

$$\frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3} \sin^2\Theta \text{ and } P = \frac{2\ddot{\mathbf{d}}^2}{3c^3} \tag{3.60}$$

Let's examine the spectrum of dipole radiation. To make things easier, let us assume that the dipole lies in a single direction and varies in magnitude (imagine a negative charge moving up and down a wire). In this case the electric field is parallel to $\mathbf{d}$ and we have

$$E(t) = \ddot{d}(t) \frac{\sin\Theta}{c^2 R} \tag{3.61}$$

Let's define

$$d(t) = \int_{-\infty}^{\infty} e^{-i\omega t} \hat{d}(\omega) d\omega \tag{3.62}$$

so we have

$$\ddot{d}(t) = -\int_{-\infty}^{\infty} \omega^2 e^{-i\omega t} \hat{d}(\omega) d\omega \tag{3.63}$$

so

$$\hat{E}(\omega) = -\frac{1}{c^2 R_0} \omega^2 \hat{d}(\omega) \sin\Theta \tag{3.64}$$

and the power per unit solid angle and frequency is

$$\frac{dW}{d\omega d\Omega} = \frac{1}{c^3} \omega^4 |\hat{d}(\omega)|^2 \sin^2\Theta \text{ and } \frac{dW}{d\omega} = \frac{8\pi\omega^2}{3c^3} |\hat{d}(\omega)|^2 \tag{3.65}$$

A Physical Aside: Multipole Radiation

It is possible to calculate the radiation field to higher order in $L/(c\tau)$. This is necessary if the dipole moment vanishes, for example. We can expand the exponential to yield

$$A_\omega(\mathbf{r}) = \frac{e^{ikr}}{cr} \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathbf{j}_\omega(\mathbf{r}') \left(-ik\mathbf{n} \cdot \mathbf{r}'\right)^n d^3r' \tag{3.66}$$

where $k \equiv \omega/c$ $n = 0$ gives the dipole radiation, $n = 1$ gives the quadrupole radiation and so on.

Cherenkov Radiation

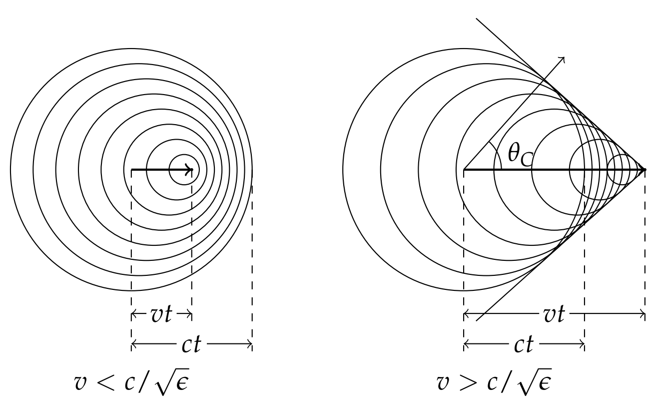


Figure 3.3: The propagation of electromagnetic waves from a source travelling slower and faster than the speed of light in the medium ($c/\sqrt{\epsilon}$ in § 3).

When a charge travels through a medium faster than the speed of light in the medium (taken to be $c/\sqrt{\epsilon}$ in this section), additional complications arise. Fig. 3.3 illustrates how for $v < c/\sqrt{\epsilon}$ each point yields a unique retarded time denoted by the circles. On the other hand if $v > c/\sqrt{\epsilon}$ the space is divided into two regions. In one outside the "Cherenkov cone" one cannot assign a retarded time to a particular point and within one must assign two different times to each point. On the cone one has a range of proper times. We can translate our earlier results, for example Eq. 3.33, by making the following substitutions

$$c \to \frac{c}{\sqrt{\epsilon}} \text{ and } q \to \frac{q}{\sqrt{\epsilon}} \tag{3.67}$$

yielding

$$\frac{dW}{d\omega d\Omega} = \frac{q^2\omega^2\epsilon^{1/2}}{4\pi^2 c^3}\left|\int_{-\infty}^{\infty} \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})e^{i\omega\left(t'-\mathbf{n}\cdot\mathbf{r}_0(t')\epsilon^{1/2}/c\right)}dt'\right|^2. \tag{3.68}$$

Here we have uniform motion in a straight line $\mathbf{r}(t') = \mathbf{v}t'$ so

$$\frac{dW}{d\omega d\Omega} = \frac{q^2\epsilon^{1/2}}{c^3}|\mathbf{n}\times\mathbf{v}|^2\left|\frac{\omega}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t'\left(1-\mathbf{n}\cdot\mathbf{v}\epsilon^{1/2}/c\right)}dt'\right|^2. \tag{3.69}$$

The integral is a Dirac delta function, so we have

$$\frac{dW}{d\omega d\Omega} = \frac{q^2\epsilon^{1/2}\beta^2\sin^2\theta}{c}\left|\delta(1-\epsilon^{1/2}\beta\cos\theta)\right|^2 \tag{3.70}$$

where θ is measured relative to the velocity of the particle. The radiation is only emitted at the angle

$$\cos\theta_C = \frac{1}{\beta\epsilon^{1/2}}. \tag{3.71}$$

In general the dielectric constant is a function of frequency and the frequency dependence does not change the result of Eq. 3.70, so we also find that the radiation is only emitted at frequencies where $\beta\epsilon^{1/2}(\omega) > 1$, or to put it another way at frequencies where the charge exceeds the propagation speed of the radiation.

The total energy radiated according to Eq. 3.70 diverges; this simply results from our assumption that the charge travels through the dielectric material forever and this assumption is easy to relax by replacing the infinite integral with one over a time $2T$ during which the particle travels through the dielectric

$$\frac{\omega}{2\pi}\int_{-T}^{T} e^{i\omega t'\left(1-\mathbf{n}\cdot\mathbf{v}\epsilon^{1/2}/c\right)}dt' = \frac{\omega T}{\pi}\frac{\sin\left[\omega T\left(1-\epsilon^{1/2}\beta\cos\theta\right)\right]}{\left[\omega T\left(1-\epsilon^{1/2}\beta\cos\theta\right)\right]}. \tag{3.72}$$

Again the radiation is sharply peaked at the Cherenkov angle as long as $\omega T \gg 1$ and we can integrate this result over all angles to yield the total energy per frequency emitted as the charge travels through the dielectric

$$\frac{dW}{d\omega} = \frac{q^2\omega}{c^2}\sin^2\theta_c\,(2c\beta T) = \frac{q^2\omega}{c^2}\left[1-\frac{1}{\beta^2\epsilon(\omega)}\right](2c\beta T) \tag{3.73}$$

where $2c\beta T$ is the thickness of the dielectric region.

Thomson Scattering

Let's imagine that an electromagnetic wave hits a charge particle causing it to move according to the Lorentz force equation,

$$\mathbf{F} = q\left(\mathbf{E}_{\rm wave} + \frac{\mathbf{u}}{c} \times \mathbf{B}_{\rm wave}\right). \tag{3.74}$$

To simplify matters let's assume that $u \ll c$ so we can neglect the magnetic term because $E_{\rm wave} = B_{\rm wave}$ so we have.

$$\dot{\mathbf{u}} = \frac{q}{m}\mathbf{E}_{\rm wave} \tag{3.75}$$

Because the wave is accelerated, it radiates electromagnetic radiation

$$\mathbf{E}_{\rm acc} = \frac{q^2}{c^2 m R}\left[\mathbf{n} \times \{\mathbf{n} \times \mathbf{E}_{\rm wave}\}\right] \tag{3.76}$$

The radiated wave has the same frequency content as the incident wave. The electric field of the radiated wave is in the plane containing $\mathbf{E}_{\rm wave}$ and $\mathbf{n}$.

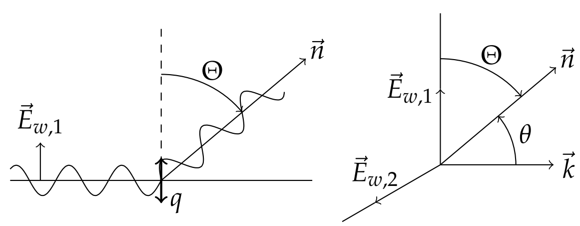


Figure 3.4: Geometry for Thomson scattering

By averaging over a period of the incident radiation we can derive the time-averaged power radiated by the charge

$$\frac{dP}{d\Omega} = \frac{q^4 E_0^2}{8\pi m^2 c^3}\sin^2\Theta \text{ and } P = \frac{q^4 E_0^2}{3m^2c^3} \tag{3.77}$$

where Θ is the angle between the line of sight and the electric field of the incident radiation. The incident radiation carries a flux of $\langle S \rangle = (c/8\pi)E_0^2$, so we can define the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{dP}{d\Omega}\langle S\rangle^{-1} = \frac{q^4}{m^2c^4}\sin^2\Theta = r_0^2\sin^2\Theta \tag{3.78}$$

where $r_0 = 2.82 \times 10^{-13}$ cm for an electron, the classical electron radius.

The total cross section is

$$\sigma = \frac{8\pi}{3}r_0^2 = \sigma_T = 0.665 \times 10^{-24} \text{ cm}^2 \text{ for an electron} \tag{3.79}$$

So far we have examined the scattering of polarized radiation. It is straightforward to think about scattering of unpolarized radiation by taking the incoming beam to be a sum of two beams whose polarization differs by $\pi/2$.

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{unpol}} = \frac{1}{2}\left[\frac{d\sigma(\Theta)}{d\Omega} + \frac{d\sigma(\pi/2)}{d\Omega}\right] \quad (3.80)$$

$$= \frac{1}{2}r_0^2\left(1+\sin^2\Theta\right) = \frac{1}{2}r_0^2\left(1+\cos^2\theta\right). \quad (3.81)$$

The first term in the expression corresponds to light polarized in the plane containing $\mathbf{E}_{w,1}$ and $\mathbf{n}$ and the second term traces light polarized in the plane containing $\mathbf{E}_{w,2}$ and $\mathbf{n}$. They are two orthogonal polarizations. More energy is scattered into the $\mathbf{E}_{w,1} - \mathbf{n}$ plane than in the other in the ratio of $1 : \cos^2\theta$, so the scattered radiation is polarized with

$$\Pi = \frac{1-\cos^2\theta}{1+\cos^2\theta} \quad (3.82)$$

Radiation Reaction

We have found that when a charge is accelerated a certain power is radiated away, so to accelerate the particle we must provide some extra energy to work against a "radiation reaction" force,

$$-\mathbf{F}_{\text{rad}}\cdot\mathbf{u} = \frac{2q^2\dot{u}^2}{3c^3} \quad (3.83)$$

To make sense of this equation, let's consider integrate the power over a period of time

$$-\int_{t_1}^{t_2}\mathbf{F}_{\text{rad}}\cdot\mathbf{u}dt = \frac{2q^2}{3c^3}\int_{t_1}^{t_2}\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}dt = \frac{2q^2}{3c^3}\left[\dot{\mathbf{u}}\cdot\mathbf{u}|_{t_1}^{t_2} - \int_{t_1}^{t_2}\ddot{\mathbf{u}}\cdot\dot{\mathbf{u}}dt\right]. \quad (3.84)$$

We can drop the term from the endpoints if for example the acceleration vanishes at $t = t_1$ and $t = t_2$ or if the acceleration and velocity of the particle are the same at $t = t_1$ and $t = t_2$. We can identify,

$$\mathbf{F}_{\text{rad}} = \frac{2q^2}{3c^3}\ddot{\mathbf{u}} = m\tau\ddot{\mathbf{u}} \quad (3.85)$$

where $\tau = 2r_0/(3c)$.

Radiation from Harmonically Bound Particles

We are going to take the results from the previous section to study particles that are harmonically bound, so their motion satisfies the following equation,

$$-\tau\dddot{x} + \ddot{x} + \omega_0^2 x = 0 \quad (3.86)$$

where the first term contains the radiation reaction. Let's also assume that the radiation reaction is only a small perturbation on the motion so $\dddot{x} \approx -\omega_0\dot{x}$ and we have

$$\ddot{x} + \omega_0^2\tau\dot{x} + \omega_0^2 x = 0. \quad (3.87)$$

Let's solve this by assuming that $x(t) = Ae^{\alpha t}$. Substituting and dividing by the exponential yields the characteristic equation

$$\alpha^2 + \omega_0^2 \tau \alpha + \omega_0^2 = 0 \tag{3.88}$$

and the solutions

$$\alpha = \pm i\omega_0 \sqrt{1 - \omega_0^2 \tau^2} - \frac{1}{2}\omega_0^2 \tau. \tag{3.89}$$

To lowest order we can take the square root to be one and we have the solution

$$x(t) = x_0 e^{-\Gamma t/2} \cos \omega_0 t \text{ where } \Gamma \equiv \omega_0^2 \tau = \frac{2q^2 \omega_0^2}{3mc^3} \tag{3.90}$$

The Fourier transform of this function is

$$\hat{x}(t) = \frac{x_0}{4\pi} \left[\frac{1}{\Gamma/2 - i(\omega + \omega_0)} + \frac{1}{\Gamma/2 - i(\omega - \omega_0)} \right] \tag{3.91}$$

If we focus on positive frequencies, the first term is small so we can approximate the power spectrum of the motion by

$$|\hat{x}|^2 = \left(\frac{x_0}{4\pi} \right)^2 \frac{1}{(\omega - \omega_0)^2 + (\Gamma/2)^2}. \tag{3.92}$$

and the power radiated per unit frequency is

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} |\hat{d}(\omega)|^2 = \frac{8\pi\omega^4}{3c^3} \frac{q^2 x_0^2}{(4\pi)^2} \frac{1}{(\omega - \omega_0)^2 + (\Gamma/2)^2} \tag{3.93}$$

$$= \left(\frac{1}{2} k x_0^2 \right) \frac{\Gamma}{2\pi} \frac{1}{(\omega - \omega_0)^2 + (\Gamma/2)^2} \tag{3.94}$$

The classical line width $\delta\omega = \Gamma$ is a universal constant for electronic oscillators if expressed as a wavelength

$$\frac{\Delta\lambda}{\lambda} = \frac{\Delta\omega}{\omega}, \Delta\lambda = \frac{2e^2\omega_0^2}{3mc^3} \frac{\lambda}{\omega_0} = \frac{4\pi e^2}{3mc^2} = \frac{4\pi}{3} r_0 = 1.2 \times 10^{-12} \text{ cm} \tag{3.95}$$

Driven Harmonic Oscillator

Let's imagine that our harmonic oscillator is driven by incoming electromagnetic wave. Using the assumptions from the section on scattering and the radiation reaction we have

$$m\ddot{x} = -m\omega_0^2 x + m\tau \dddot{x} + qE_0 \cos \omega t. \tag{3.96}$$

Let's divide by the mass and take use the exponential for the cosine

$$\ddot{x} - \tau \dddot{x} + \omega_0^2 x = \frac{qE_0}{m} e^{i\omega t} \tag{3.97}$$

and try a solution of the form

$$x = x_0 e^{i\omega t} \tag{3.98}$$

which gives

$$x_0 \left(-\omega^2 + i\tau\omega^3 + \omega_0^2 \right) = \frac{qE_0}{m} \tag{3.99}$$

so

$$x_0 = -\frac{eE_0}{m} \frac{1}{\omega^2 - \omega_0^2 - i\tau\omega_0^3} \tag{3.100}$$

It is convenient to express

$$x_0 = |x_0| e^{i\delta} \tag{3.101}$$

where

$$\tan\delta = \frac{\omega^3\tau}{\omega^2 - \omega_0^2} \text{ and } |x_0| = \frac{eE_0}{m} \left[\left(\omega^2 - \omega_0^2 \right)^2 + \omega_0^6 \tau^2 \right]^{-1/2}. \tag{3.102}$$

Let's use the dipole formula to calculate the radiated power

$$P = \frac{q^2 |x_0|^2 \omega^4}{3c^3} = \frac{q^4 E_0^2}{3m^2 c^3} \frac{\omega^4}{\left(\omega^2 - \omega_0^2\right)^2 + \left(\omega_0^3 \tau\right)^2} \tag{3.103}$$

Let's divide by the Poynting vector $\langle S \rangle = (c/8\pi) E_0^2$ to get the scattering cross-section

$$\sigma(\omega) = \sigma_T \frac{\omega^4}{\left(\omega^2 - \omega_0^2\right)^2 + \left(\omega_0^3 \tau\right)^2} \tag{3.104}$$

The scattering cross-section has several interesting regimes

- $\omega \gg \omega_0$: $\sigma(\omega) \to \sigma_T$
- $\omega \ll \omega_0$: $\sigma(\omega) \to \sigma_T \left(\frac{\omega}{\omega_0} \right)^4$
- $\omega \approx \omega_0$: In this regime it is convenient to write

$$\omega^2 - \omega_0^2 = (\omega - \omega_0)(\omega + \omega_0) \approx 2\omega_0(\omega - \omega_0) \tag{3.105}$$

  and take $\omega = \omega_0$ elsewhere in the cross-section

$$\sigma(\omega) \approx \frac{\pi\sigma_T}{2\tau} \frac{\Gamma}{2\pi} \frac{1}{(\omega - \omega_0)^2 + (\Gamma/2)^2} \tag{3.106}$$

  Near the resonance the cross-section has the same profile at the spontaneous emission.

Further Reading

To learn more about radiation from moving charges, consult Chapter 14 of

- Jackson, J. D., *Classical Electrodynamics*.

Problems

1. **Constant Velocity Charge**

   Show that if charge is not accelerating, the electric field vector points to the current (not the retarded) position of the charge.

2. **Dipoles:**

   Two oscillating dipole moments (radio antennas) $\mathbf{d}_1$ and $\mathbf{d}_2$ are oriented in the vertical direction and are a horizonal distance L apart. They oscillate in phase at the same frequency ω. Consider radiation at an angle θ with repect to the vertical and in the vertical plane containing the two dipoles.

   (a) Show that
   $$\frac{dP}{d\Omega} = \frac{\omega^4 \sin^2\theta}{8\pi c^3}\left(d_1^2 + 2d_1 d_2 \cos\delta + d_2^2\right),$$
   where
   $$\delta \equiv \frac{\omega L \sin\theta}{c}.$$

   (b) Thus show directly that when $L \ll \lambda$, the radiation is the same as from a single oscillating dipole of amplitude $d_1 + d_2$.

3. **Cloud:**

   An optically thin cloud surrounding a luminous object is estimate to be 1 pc in radius and to consist of ionized plasma. Assume that electron scattering is the only important extinction mechanism and that the luminous object emits unpolarized radiation.

   (a) If the cloud is unresolved (angular size smaller than the angular resolution of the detector), what is the net polarization observeed?

   (b) If the cloud is resolved, what is the polarization direction of the observed radiation as a function of position on the sky? Assume only a single scattering occurs.

   (c) If the central object is clear seen, what is an upper bound for the electron density of the cloud, assuming that the cloud is homogeneous?

4. **Synchrotron Cooling:**

   A particle of mass m, charge q, moves in a plane perpendicular to a uniform, static, magnetic field B.

   (a) Calculate the total energy radiated per unit time, expressing it in terms of the constants already defined and the ratio $\gamma = 1/\sqrt{1-\beta^2}$ of the particle's total energy to its rest energy. You can assume that the particle is ultrarelativistic.

(b) If at time $t = 0$ the particle has a total energy $E_0 = \gamma_0 mc^2$, show that it will have energy $E = \gamma mc^2 < E_0$ at a time t, where

$$t \approx \frac{3m^3c^5}{2q^4B^2}\left(\frac{1}{\gamma} - \frac{1}{\gamma_0}\right).$$

5. **Classical HI:**

A particle of mass m and charge q moves in a circle due to a force $\mathbf{F} = -\hat{\mathbf{r}}\frac{q^2}{r^2}$. You may assume that the particle always moves non-relativistically.

(a) What is the acceleration of the particle as a function of r?

(b) What is the total energy of the particle as a function of r? The potential energy is given by $-q^2/r$.

(c) What is the power radiated as a function of r?

(d) Using the fact the $P = -dE/dt$ and the answer to (b), find dr/dt.

(e) Assuming that the particle starts with $r = r_i$ at $t = 0$, find the value of t where $r = 0$.

(f) Let's assume that $q = e$, the charge of the electron, and $m = m_e$, the mass of the electron. Write your answer in (d) in terms of r_i, r_0 (the classical electron radius) and c.

(g) What is the time if $r_i = 0.5$Å (for hydrogen)?

(h) Compare this to the lifetime of a hydrogen atom.

6. **The Eddington Luminosity:**

There is a natural limit to the luminosity a gravitationally bound object can emit. At this limit the inward gravitational force on a piece of material is balanced by the outgoing radiation pressure. Although this limiting luminosity, the Eddington luminosity, can be evaded in various ways, it can provide a useful (if not truly firm) estimate of the minimum mass of a particular source of radiation.

(a) Consider ionized hydrogen gas. Each electron-proton pair has a mass more or less equal to the mass of the proton (m_p) and a cross section to radiation equal to the Thompson cross-section (σ_T).

(b) The radiation pressure is given by outgoing radiation flux over the speed of light.

(c) Equate the outgoing force due to radiation on the pair with the inward force of gravity on the pair.

(d) Solve for the luminosity as a function of mass.

The mass of the sun is 2×10^{33}g. What is the Eddington luminosity of the sun?

7. **The Blue Pool**

   Skectch the spectrum of light emitted by an electron with a total energy of 1 MeV, 3 MeV and 10 MeV travelling through water.

4
Special Relativity

Back to Maxwell's Equations

Earlier we looked at Maxwell's equations in a vacuum,

$$\begin{aligned} \nabla \cdot \mathbf{E} = 0 \qquad & \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \quad & \nabla \times \mathbf{H} = +\tfrac{1}{c}\tfrac{\partial \mathbf{D}}{\partial t} \end{aligned} \tag{4.1}$$

and found that they have wave solutions,

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{4.2}$$

and a similar equation for the magnetic field. The waves travel at a velocity c, which turns out to be the speed of light. The speed of light had been known approximately since the 1600's (does anyone know how?).

Maxwell's and his contemporaries spoke of light travelling through some medium known as the aether. Michelson and Morley attempted to measure the motion of the Earth through the aether, but failed.

Looking at the Michelson-Morley experiment closely shows what is happening. Lorentz proposed that to understand the null result of the experiment objects moving through the aether contract by $\gamma^{-1} = \sqrt{1 - v^2/c^2}$ where γ is the Lorentz factor.

Einstein's insight was that if the speed of light was the same for everyone moving uniformly, one would get the apparent "Lorentz" contraction without needing the aether through which light propagates or for the aether to contract objects. The aether was originally proposed by Aristotle and experiments agreed with it for about 2,200 years, so throwing it away was a big deal.

Lorentz Transformations

Let's imagine two people moving at a velocity v relative to each other in the x-direction. Let's also assume that their coordinate systems coincide at $t = 0$, and that one emits a light pulse at $t = t' = 0$ from $x = x' = 0$. After a time has elapsed the light has reached positions that satisfy,

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \text{ and } x'^2 + y'^2 + z'^2 - c^2t'^2 = 0. \tag{4.3}$$

We can satisfy these equations if

$$x' = \gamma(x - vt) \tag{4.4}$$
$$y' = y \tag{4.5}$$
$$z' = z \tag{4.6}$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \tag{4.7}$$

and the following inverse relations

$$x = \gamma(x' + vt'), t = \gamma\left(t' + \frac{v}{c^2}x'\right) \text{ and the other two equations.} \tag{4.8}$$

Length Contraction

Let's look at the results with the aether again. If we have a rod of length L_0 in the primed frame what it is length in the unprimed frame.

$$L_0 = x_2' - x_1' = \gamma(x_2 - x_1) = \gamma L. \tag{4.9}$$

We have define the length to be the extent of an object measured at a particular time. Notices that someone in the primed frame would claim that the person measured the position of one end of the stick at a different time from the other.

Adding velocities

Let's do a final example. Someone in the primed frame throws a ball in the x'-direction with velocity u_x' from $x' = 0$ at $t' = 0$, what velocity will someone measure in the unprimed frame. After a time t' the ball will be at $x' = u_x't'$. Let's use the inverse transformation to calculate its coordinates in the unprimed frame,

$$x = \gamma(u_x't' + vt'), t = \gamma\left(t' + \frac{v}{c^2}u_x't'\right). \tag{4.10}$$

The velocity u_x in the unprimed frame is

$$u_x = \frac{x}{t} = \frac{\gamma(u_x't' + vt')}{\gamma\left(t' + \frac{v}{c^2}t'u_x'\right)} = \frac{u' + v}{1 + vu_x'/c^2} \tag{4.11}$$

If the particle had velocity components in the y' or z' directions the corresponding components in the unprimed frame are

$$u_y = \frac{y}{t} = \frac{\gamma(u_y't')}{\gamma\left(t' + \frac{v}{c^2}t'u_x'\right)} = \frac{u_y'}{\gamma(1 + vu_x'/c^2)} \tag{4.12}$$

and similarly for the z-direction.

The apparent direction of the particle is different in the two frames,

$$\tan\theta = \frac{u_y}{u_x} = \frac{u'_y}{\gamma(u'_x + v)} = \frac{u' \sin\theta'}{\gamma(u'\cos\theta' + v)}. \tag{4.13}$$

This is the aberration equation. Let's for an example take $u' = c$ and $\theta' = \pi/2$. We could imagine that this is the emission from a dipole moving at a velocity v. We get

$$\tan\theta = \frac{c}{\gamma v} \text{ or } \sin\theta = \frac{1}{\gamma} \tag{4.14}$$

Doppler Effect

We have a radio transmitter in the primed frame radiating at a frequency ω'. According to the time dilation, in the unprimed frame it oscillates more slowly at a time inverval $\Delta t = 2\pi\gamma/\omega$.. The time between the arrival for two crests of the wave in the unprimed frame is given by,

$$\Delta t_A = \Delta t - \frac{d}{c} = \Delta t \left(1 - \frac{v}{c}\cos\theta\right). \tag{4.15}$$

so

$$\omega = \frac{2\pi}{\Delta t_A} = \frac{\omega'}{\gamma\left(1 - \frac{v}{c}\cos\theta\right)} \tag{4.16}$$

Four-Vectors

We have found many strange properties of special relativity in a rather *ad hoc* manner. All of these properties resulted from the fact that

$$s^2 = c^2\tau^2 = c^2t^2 - x^2 - y^2 - z^2 \tag{4.17}$$

is the same for all observers travelling uniformly relative to each other. In three dimensions we can think about vectors whose length $x^2 + y^2 + z^2$ is invariant with respect to rotations. Once we establish that a certain quantity is a vector we can use the transformation properties of the vectors under rotation to determine what its value is in any other frame.

Similarly in relativity, it is convenient to define something called a four-vector whose components transform between rotated frames and frames moving at different velocities such that the equation above holds. A four vector is simply

$$x^\mu = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \tag{4.18}$$

and

$$s^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu} x^\mu x^\nu \equiv \eta_{\mu\nu} x^\mu x^\nu \equiv x^\mu x_\mu \tag{4.19}$$

where

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \text{ and } x_\mu = [ct \ \ -x \ \ -y \ \ -z] \,. \tag{4.20}$$

This tensor $\eta_{\mu\nu}$ defines the metric for flat spacetime. It is called the metric because you need it to convert various four vectors (and other objects tensors) into scalars that we can measure. We have selected the particular convention that time-time component is negative (like in Misner, Thorne and Wheeler). Jackson use the opposite convention. If the index is upstairs the vector is contravariant and if it is downstairs it is convariant.

Now we can write the transformation between two frames very concisely

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{4.21}$$

where

$$\Lambda^\mu{}_\nu = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.22}$$

This matrix looks remarkably similar to a rotation matrix. For example,

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.23}$$

This is no coincidence. A boost (shift between frames with two different velocities) is like a rotation in spacetime. However, we have in the rotation case we have

$$\cos^2\theta + \sin^2\theta = 1 \tag{4.24}$$

while in the boost case we have

$$\gamma^2 - (\beta\gamma)^2 = \gamma^2(1-\beta^2) = 1. \tag{4.25}$$

Sometimes people define the rapidity ζ such that $\gamma = \cosh\zeta$. The nice thing about the rapidity is that like the angle θ it is additive for successive boosts.

What about the transformation of the covariant vector?

$$s^2 = x^\mu x_\mu = x'^\alpha x'_\alpha = \Lambda^\alpha_\mu \tilde{\Lambda}^\nu_\alpha x^\mu x_\nu \tag{4.26}$$

which tells us that

$$\Lambda^{\alpha}_{\mu}\tilde{\Lambda}^{\nu}_{\alpha} = \delta^{\nu}_{\mu} \tag{4.27}$$

so the covariant vector transforms using the inverse matrix.

Let's try to find some physically meaningful four-vectors. We know that a displacement is a four-vector. Let's try to find a four-vector related to the velocity of a particle.

$$U^{\mu} = \frac{dx^{\mu}}{dt?} \tag{4.28}$$

The numerator is a displacement that transforms as a four-vector. For the left-hand side also to be a four-vector the denominator must be the same in all frames (a Lorentz scalar). The only one is $d\tau$ which we defined earlier. This is the time measured by someone moving with the particle. We have

$$U^{\mu} = \gamma_u \begin{bmatrix} c \\ u_x \\ u_y \\ u_z \end{bmatrix} = \gamma_u \begin{bmatrix} c \\ \mathbf{u} \end{bmatrix} \tag{4.29}$$

What this means is that for each second measured by someone moving with the particle, γ times one second elapses for us and the particle travels $\gamma\mathbf{u}$ times one second.

What is the magnitude of U^{μ}?

$$U^{\mu}U_{\mu} = (\gamma_u c)^2 - (\gamma_u \mathbf{u})^2 = c^2\gamma_u^2\left(1-\beta^2\right) = c^2 \tag{4.30}$$

If a particle is at rest its four-velocity is given by $U^0 = c$ and $U^i = 0$.

In non-relativistic mechanics, we define the momentum to be the mass times the velocity, similarly we can define the four-momentum to $p^{\mu} = mU^{\mu}$. Let's look at the properties of this vector in more detail. Its components are

$$p^{\mu} = \gamma_u m \begin{bmatrix} c \\ u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} \gamma_u mc \\ \mathbf{p} \end{bmatrix} \tag{4.31}$$

Let's expand the first component to see what it is

$$p_t = \gamma_u mc = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} mc \approx mc + \frac{1}{2}m\frac{v^2}{c} \tag{4.32}$$

$$= \frac{mc^2 + \mathrm{KE}}{c} = \frac{E}{c} \tag{4.33}$$

(4.34)

If we calculate $p^\mu p_\mu$ we find the relativistic relationship between energy and momentum

$$p^\mu p_\mu = (mc)^2 = \frac{E^2}{c^2} - \mathbf{p}\cdot\mathbf{p} \quad (4.35)$$

$$E = \sqrt{c^2 p^2 + m^2 c^4} \quad (4.36)$$

Another important four-vector shows up in the equation for the propagation of an electromagnetic wave. The electric and magnetic fields of the wave are proportional to $\cos(\mathbf{k}\cdot\mathbf{x} - \omega t)$. If both fields vanish at a point and time in spacetime, all observers should agree on this regardless of their motion so

$$\omega t - \mathbf{k}\cdot\mathbf{x} = k_\mu x^\mu \quad (4.37)$$

is a scalar. Because x^μ is a four-vector,

$$k^\mu = \left[\begin{array}{c} \omega/c \\ \mathbf{k} \end{array}\right] \quad (4.38)$$

must be one as well. This leads to a quick way to derive the redshift formula. The person observing a wave finds

$$\omega' = k^\mu U'_\mu = \gamma\left(\omega - \mathbf{k}\cdot\mathbf{u}\right) = \gamma\omega\left(1 - \frac{v}{c}\cos\theta\right) \quad (4.39)$$

Tensors

We have essentially stumbled upon a few nice four-vectors, but there is a more systematic way of dealing with four-vectors, scalars and other quantities like the transformation matrix $\Lambda^\mu{}_\nu$. All of these objects are examples of tensors.

We can work out how tensors transform by looking at a few examples. The quantity

$$T^{\mu\nu} = A^\mu B^\nu \quad (4.40)$$

is a tensor. Let's use the Lorentz matrix to transform to a new frame

$$T'^{\alpha\beta} = A'^\alpha B'^\beta = \Lambda^\alpha{}_\mu A^\mu \Lambda^\beta{}_\nu B^\nu = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu T^{\mu\nu}. \quad (4.41)$$

We can find similar results for mixed tensors and covariant tensors.

Right now, we can build a contravariant vector by taking a set of coordinates x^i for a event in spacetime and we can construct a covariant vector by applying the metric $\eta_{\mu\nu}$ to lower the index of the vector. How else can we make a covariant vector?

Let's say there is a scalar field defined over all spacetime. This just means a Lorentz invariant number at each point and time. We could ask how much this number changes as one goes from one event in spacetime to another:

$$\Delta f = f\left(x^\mu + \Delta x^\mu\right) - f\left(x^\mu\right) \tag{4.42}$$

The quantity on the left is clearly a scalar because it is the different in the value of a scalar field at two points. Let's imagine that we take Δx^μ to be really smaller so that Δf is proportional to Δx^μ then we have

$$\Delta f = \frac{\partial f}{\partial x^\mu}\Delta x^\mu \equiv f_{,\mu}\Delta x^\mu \tag{4.43}$$

Because Δx^μ transforms as a contravariant vector and Δf doesn't transform, $f_{,\mu}$ must transform as a covariant vector.

We could also imagine taking the derivative of the vector field to create a tensor, for example,

$$A^\mu_{,\nu} = \frac{\partial A^\mu}{\partial x^\nu} \tag{4.44}$$

If we take A^μ to be the vector potential plus the scalar potential,

$$A^\mu = \left[\begin{array}{c} \phi \\ \mathbf{A} \end{array}\right], \tag{4.45}$$

we have

$$\partial_\nu \partial^\nu A^\mu = \frac{4\pi}{c} J^\mu \text{ and } \partial_\alpha A^\alpha = 0 \tag{4.46}$$

gives the equations of electrodynamics in the Lorenz gauge, where

$$J^\mu = \left[\begin{array}{c} c\rho \\ \mathbf{J} \end{array}\right]. \tag{4.47}$$

We have argued that we can only measure the fields themselves, so we would like to figure out how the fields transform. Under rotations the fields act like vectors. Can we generalize the electric and magnetic field to be four-vectors?

The answer is no. Let's take a look at definitions of the fields in terms of the potentials,

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{4.48}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{4.49}$$

Let's look at the $x-$components of the fields

$$E_x = -\frac{\partial \phi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t} = A_{0,1} - A_{1,0} \tag{4.50}$$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = A_{3,2} - A_{2,3} \tag{4.51}$$

so the electric and magnetic fields seem to be the components of the second rank tensor,

$$F_{\alpha\beta} = -\left(A_{\alpha,\beta} - A_{\beta,\alpha}\right) = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix} \tag{4.52}$$

where the index α labels the rows and β labels the columns. Let's look first at the Lorentz force equation,

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \tag{4.53}$$

To generalize this we know that $\mathbf{p}$ transforms as the space-part of the four-vector p^μ. We also need to use the proper time τ instead of the coordinate time t, this gives

$$\frac{d\mathbf{p}}{d\tau} = \frac{q}{c}\left(U_0\mathbf{E} + \mathbf{U} \times \mathbf{B}\right) \tag{4.54}$$

Something to think about. Why did the velocity terms on the right-hand side become four velocities?

We also need an equation for the time-like component of the four-momentum.

$$\frac{dp_t}{d\tau} = \frac{q}{c}\mathbf{U} \cdot \mathbf{E}. \tag{4.55}$$

We can combine these equations into a single equation using the field tensor,

$$\frac{dp^\alpha}{d\tau} = m\frac{dU^\alpha}{d\tau} = \frac{q}{c}F^\alpha{}_\beta U^\beta \tag{4.56}$$

or

$$\frac{d}{d\tau}\begin{bmatrix} E \\ p_x \\ p_y \\ p_z \end{bmatrix} = \frac{\gamma q}{c}\begin{bmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}\begin{bmatrix} c \\ v_x \\ v_y \\ v_z \end{bmatrix} \tag{4.57}$$

which defines the field tensor without reference to the potentials. The timelike components of this mixed tensor are not antisymmetric but it does have the advantage that its components are independent of the signature that you are using.

Similarly Maxwell's equations can be written in the compact form

$$F^{\alpha\beta}{}_{,\beta} = \frac{4\pi}{c}J^\alpha \text{ and } \mathcal{F}^{\alpha\beta}{}_{,\beta} = 0 \tag{4.58}$$

where

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix} \tag{4.59}$$

To construct $\mathcal{F}^{\alpha\beta}$ from $F^{\alpha\beta}$ we put $\mathbf{E} \rightarrow \mathbf{B}$ and $\mathbf{B} \rightarrow -\mathbf{E}$. This is called a duality transformation.

Transformation of Radiative Transfer

The equations of radiative transfer follow the intensity of the radiation field. We would like to understand how this and other radiative transfer quantities transform relativistically.

As I argued earlier, the intensity of a radiation field is related to the phase space density of photons. Phase space density is simply the number of particles in a certain range of momenta in a particular location,

$$f = \frac{dN}{d^3\mathbf{p}' d^3\mathbf{x}'}. \tag{4.60}$$

We would like to see how f transforms relativistically. First the numerator is simply the number of particles in the region of phase space that we can count and all should agree upon. The second term in the denominator is the volume that the particles occupy. Let's assume that the primed frame is moving a velocity βc in the $x-$direction relative to the unprimed frame. For convenience let's assume that the origins of the two coordinate systems coincide at $t = t' = 0$. These assumptions cover all of the possibilities because the volumes $d^3\mathbf{x}$ and $d^3\mathbf{p}$ are invariant under rotations. The following derivation follows one by Jeremy Goodman. We will take $c = 1$ to simply the proof.

First let's write momenta in the primed frame in terms of its values in the unprimed frame, we have

$$p'_t = \gamma\left(p_t - \beta p_x\right) \tag{4.61}$$
$$p'_x = \gamma\left(p_x - \beta p_t\right) \tag{4.62}$$
$$p'_y = p_y \tag{4.63}$$
$$p'_z = p_z. \tag{4.64}$$

Now let's construct the Jacobian of the transformation,

$$\begin{vmatrix} \frac{\partial p'_x}{\partial p_x} & \frac{\partial p'_x}{\partial p_y} & \frac{\partial p'_x}{\partial p_z} \\ \frac{\partial p'_y}{\partial p_x} & \frac{\partial p'_y}{\partial p_y} & \frac{\partial p'_z}{\partial p_z} \\ \frac{\partial p'_z}{\partial p_x} & \frac{\partial p'_z}{\partial p_y} & \frac{\partial p'_z}{\partial p_z} \end{vmatrix} = \begin{vmatrix} \gamma\left(1 - \beta\frac{\partial p_t}{\partial p_x}\right) & -\gamma\beta\frac{\partial p_t}{\partial p_y} & -\gamma\beta\frac{\partial p_t}{\partial p_z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \tag{4.65}$$

yielding the value of the Jacobian,

$$\gamma\left(1 - \beta\frac{\partial p_t}{\partial p_x}\right). \tag{4.66}$$

Now we can calculate $\partial p_t/\partial p_x$ from the relationship between the four momentum and mass

$$p_t^2 - \mathbf{p}^2 = m^2 \tag{4.67}$$

so $\partial p_t / \partial p_x = p_x / p_t$ and the Jacobian is

$$\gamma \left(1 - \beta \frac{p_x}{p_t} \right) = \frac{\gamma (p_t - \beta p_x)}{p_t} = \frac{p'_t}{p_t}. \tag{4.68}$$

so

$$d^3 \mathbf{p}' = \frac{p'_t}{p_t} d^3 \mathbf{p} \text{ so } \frac{d^3 \mathbf{p}}{p_t} \text{ is invariant.} \tag{4.69}$$

Now let's look at the transformation of the length interval dx. Let's take two particles travelling at the same velocity but separated by some distance. First in the primed frame we have

$$x'_A = v' t'_A + x'_A(0), x'_B = v' t'_B + x'_B(0). \tag{4.70}$$

To measure the distance between them in the primed frame we take $t'_A = t'_B$ so $\Delta x' = x'_A(0) - x'_B(0)$. Let's substitute the values of x'_A and t'_A in terms of x_A and t_A to yield

$$\gamma (x_A - \beta t_A) = v' \gamma (t_A - \beta x_A) + x_A(0) \tag{4.71}$$

and solving for x_A yields

$$x_A = \frac{\beta + v'}{1 + \beta v'} t_A + \frac{x'_A(0)}{\gamma (1 + \beta v')}. \tag{4.72}$$

Notice how the particle travels at a different velocity in the new frame and the relativistic addition of velocities. Now we find that

$$\Delta x = \frac{\Delta x'}{\gamma (1 + \beta v')}. \tag{4.73}$$

Looking at the denominator, we have

$$\gamma (1 + \beta v') = \frac{\gamma (p'_t + \beta p'_x)}{p'_t} = \frac{p_t}{p'_t} \tag{4.74}$$

where we have used $v' = p'_1 / p'_t$ and the inverse Lorentz transformation, so we find

$$\Delta x = \Delta x' \frac{p'_t}{p_t} \text{ so } p_t d^3 \mathbf{x} \text{ is invariant.} \tag{4.75}$$

Therefore, $d^3 \mathbf{x} d^3 \mathbf{p}$ is Lorentz invariant and

$$f = \frac{dN}{d^3 \mathbf{p}' d^3 \mathbf{x}'}. = \frac{dN}{d^3 \mathbf{p} d^3 \mathbf{x}}, \tag{4.76}$$

phase-space density is a Lorentz invariant.

Let's calculate the energy density of the photon field,

$$h\nu f d^3 \mathbf{p} = h\nu f p^2 dp d\Omega = u_\nu(\Omega) d\Omega d\nu \tag{4.77}$$

$$h\nu f \left(\frac{h\nu}{c} \right)^2 d \left(\frac{h\nu}{c} \right) d\Omega = \frac{I_\nu}{c} d\Omega d\nu \tag{4.78}$$

$$h^4 \nu^3 c^{-3} f d\nu d\Omega = \frac{I_\nu}{c} d\Omega d\nu \tag{4.79}$$

$$\frac{h^4}{c^2} f d\nu d\Omega = \frac{I_\nu}{\nu^3} d\nu d\Omega \tag{4.80}$$

Because the left-hand side is a bunch of Lorentz invariants we find that

$$\frac{I_\nu}{\nu^3} = \text{Lorentz invariant} \tag{4.81}$$

A second more heuristic way to find this result is to focus on the intensity of blackbody radiation

$$I_\nu = B_\nu(T) = \frac{2h}{c^2} \frac{\nu^3}{\exp(h\nu/kT) - 1}. \tag{4.82}$$

To preserve the shape of the blackbody function, the ratio $h\nu/kT$ should be invariant with boosts. The constants, h and c, must also be invariant, so we have

$$\frac{I_\nu}{\nu^3} = \frac{B_\nu(T)}{\nu^3} = \frac{2h}{c^2} \frac{1}{\exp(h\nu/kT) - 1} = \text{Lorentz invariant.} \tag{4.83}$$

Because the source function S_ν appears in the equations of radiative transfer as $I_\nu - S_\nu$, S_ν must have the same transformation properties as I_ν, *i.e.*

$$\frac{S_\nu}{\nu^3} = \text{Lorentz invariant} \tag{4.84}$$

The optical depth τ is simply the logarithm of the fraction of radiation that remains after passing through a slab of absorbing material we have

$$\tau = \frac{l\alpha_\nu}{\sin\theta} = \frac{l}{\nu \sin\theta} \nu\alpha_\nu = \frac{lc}{k_y} \nu\alpha_\nu = \text{Lorentz invariant} \tag{4.85}$$

If we move relative to the slab in the $x-$direction, the thickness of the slab in the $y-$direction, l, does not change. Although ν and $\sin\theta$ will change, k_y will not change because it is not in the direction of the motion, so we have

$$\nu\alpha_\nu = \text{Lorentz invariant} \tag{4.86}$$

Finally, we have $j_\nu = \alpha_\nu S_\nu$, so

$$\frac{j_\nu}{\nu^2} = \text{Lorentz invariant} \tag{4.87}$$

These relations allow us to calculate the radiative transfer through a medium in whichever frame is convenient. We could calculate the source function and absorption in the rest-frame of the material and the radiative transfer in the "lab" frame. Or we could calculate everything in the rest frame and translate the intensity to the "lab" frame.

Further Reading

To learn more about special relativity, consult Chapter 12 of

- Jackson, J. D., *Classical Electrodynamics*.

Please note that Jackson uses the opposite signature to here, so some formulas may differ.

Problems

1. **The Ladder and the Barn: A Spacetime Diagram:**

   This problem will work best if you have a sheet of graph paper. In a spacetime diagram one draws a particular coordinate (in our case x) along the horizontal direction and the time coordinate vertically. People also generally draw the path of a light ray at 45°. This sets the relative units of the two axes.

   (a) Draw a spacetime diagram and label the axes x and t. The t-axis is the path of Emma through the spacetime.

   (b) Draw the world line of someone travelling at $\frac{3}{5}$ of the speed of light. The world line should intersect with the origin of the spacetime diagram. Label this world line t'. The t'-axis is the path of Kara through the spacetime.

   (c) Draw the x' axis on the graph. Here's a hint about where it should go. The light ray bisects the angle between the x and t axes. Kara who is travelling along t' will find that the speed of light is the same for her, so the light ray must also bisect the angle between x' and t'.

   (d) Parallel to Emma's time axis draw the walls of the barn in pencil. The barn is 4.5 meters wide in Emma's frame.

   (e) Draw Kara's ladder along Kara's x-axis. The ladder is 5 meters long in Kara's frame. How long is it in Emma's frame.

   (f) Draw the world lines of the ends of Kara's ladder. These lines are parallel to Kara's time axis.

   (g) Erase a portion of the barn walls to allow Kara's ladder to fit through.

   (h) Using the diagram, explain how Kara and Emma can understand how the too-long ladder fits in the too-small barn.

2. **The Fermi Process:**

   One model to understand how cosmic rays are accelerated is through shocks. The main idea is that a charge particle can cross a shock and turned around by the tangled mangetic field and recross the shock. Each time the charge does this it gains energy.

To understand this let's use a simplified model in which two mirrors are travelling toward each other at some velocity v. When a particle hits the mirror, its energy in the frame of the mirror remains unchanged but its velocity and therefore the spacelike components of the four-momentum change sign.

(a) Draw a diagram with the two mirrors.

(b) For argument's sake, let's first focus on the mirror on the left and consider that the mirror on the right is moving. What is the four-velocity in this frame of the mirror on the left (U_l^μ)? What is the four-velocity in this frame of the mirror on the right (U_r^μ)?

(c) Now let's focus on the mirror on the right and consider that the mirror on the left is moving. What is the four-velocity in this frame of the mirror on the left ($U_l'^\mu$)? What is the four-velocity in this frame of the mirror on the right ($U_r'^\mu$)?

(d) To start let's assume that the particle of mass m approaches the mirror on the left at the velocity of the mirror on the right. What is the four-momemtum of the particle (p^μ) in the frame of the mirror on the left?

(e) The particle bounces off of the mirror. What is its four-momentum now?

(f) Now the particle is approaching the mirror on the right. What is the zeroth component of the four-momentum of the particle in the frame of the right-hand mirror? One could do a Lorentz transformation but it is easier to use $U_r^\mu p_\mu$ to determine the energy of the particle in the primed frame.

(g) Using the answer to 6, construct the four-momentum of the particle in the frame of the right-hand mirror (p'_μ).

(h) The particle bounces off of the mirror. What is its four-momentum now?

(i) Now the particle is approaching the mirror on the left. What is the zeroth component of the four-momentum of the particle in the frame of the left-hand mirror? Again one could do a Lorentz transformation but it is easier to use $U_l'^\mu p'_\mu$ to determine the energy of the particle in the unprimed frame.

(j) Compare the energy of the particle in step (d) to the energy of the particle in step (i). Has the energy of the particle increased? Let's let the relative velocity of the mirrors approach the speed of light.

$$\beta \approx 1 - \frac{1}{2\gamma^2}$$

By what factor does the energy of the particle increase each time it goes back and forth.

(k) The final element is the fact that only a tiny fraction of the particles bounce back and forth. Let's take that fraction to be 10^{-5} and $\gamma = 100$. What can you say about the final distribution of particle energies?

3. **Boosting** We are going to figure out how times and energies measured by someone in motion differ from what we might measure.

   (a) Use special relativity (the Minkowski metric) to figure this out. I measure a photon to have an energy E. What is the four-momentum of the photon?

   (b) My pal is travelling toward me in the opposite direction of the photon at a velocity βc. What is his four-velocity? Use the definition $\gamma = \left(1-\beta^2\right)^{-1/2}$ to simplify the expression. What energy would he measure for the photon? What does the expression look like as γ gets much larger than one?

   (c) If my pal observes the photon to have an energy of 100 MeV while I say its energy is less than 500 keV, what is the minimal value of γ for my pal (take $\beta \approx 1$ to make life easier)?

   (d) My pal is still coming toward me at a velocity βc. When he is a distance r away from me (at a time t_0) he emits a photon toward me. How long does it take this photon to reach me?

   (e) From his point of view a short time Δt later he emits another photon toward me. How long is Δt in my frame and when do I receive the second photon? What is the difference in time between when I receive the first and second photons? What does the expression look like as γ gets much larger than one? Compare it with you answer to (b).

4. **Precession** We will calculate the transformation that results from a pair of boosts in different directions.

   (a) Write out the Lorentz transformation matrix for a boost in the $x-$direction to a velocity β_x.

   (b) Write out the Lorentz transformation matrix for a boost in the $y-$direction to a velocity β_y.

   (c) Write out the Lorentz transformation matrix for a boost in the $x-$direction to velocity β_x followed by boost to a velocity β_y in the $y-$direction.

   (d) Write out the Lorentz transformation matrix for a boost in the $x-$direction to velocity β_x, followed by boost to a velocity β_y in the $y-$direction, followed a boost in the $x-$direction to velocity $-\beta_x$ followed by boost to a velocity $-\beta_y$ in the $y-$direction.

(e) Now you have undone both boosts and have zero velocity, did you get the identity? Why or why not?

5 General Relativity

Needless to say, the subject of general relativity cannot be adequately treated in a single chapter. This is not the goal here. The goal is to introduce a bit of the kinematics (how matter and light moves in general relativity) under the influence of gravity alone. In high-energy astrophysics, three particular structures are important: the gravitational field outside a spherically symmetric mass (e.g. a star or a non-rotating black hole), the gravitational field of a rotating black hole, and gravitational radiation. We will examine these three situations as well, but first a bit of introduction.

Back to Newton's Equation

At the turn of the 20th century, the laws of electrodynamics and mechanics contradicted each other. Galiean mechanics contained no reference to the speed of light, but Maxwell's equations and experiments said that light goes at the speed of light no matter how fast you are going. To deal with this people argued that there should be new rules to add velocities and that the results of measuring an object's mass or length as it approaches the speed of light would defy one's expectations.

Einstein argued that the constancy of the speed of light was a property of space(time) itself. The Newtonian picture was that everyone shared the same view of space and time marched in lockstep for everybody. So people would agree on the length of objects and the duration of time between events

$$dl^2 = dx^2 + dy^2 + dz^2, dt$$

. Einstein argued that spacetime was the important concept and that the interval between events was what everyone could agree on.

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2.$$

This simple idea explained all of the nuttiness that experiments with light uncovered but it also cast the die for the downfall of Newtonian gravity.

Newtonian gravity was action at a distance (Newton himself wasn't happy about this). This means that if you move a mass, its gravitational field will change everywhere instantaneously. In special relativity this leads to contradictions. In particular effects can precede causes.

Instead of trashing the brand-new special theory of relativity, Einstein decided to rework the venerable theory of gravity. He came upon the general theory of relativity. Newtonian gravity looks a lot like electrostatics:

$$\nabla^2 \phi = 4\pi G \rho.$$

Let's generalize it as a relativistic scalar field:

$$\nabla^2 \phi - c^2 \frac{d^2 \phi}{dt^2} = 4\pi G \rho$$

. What is ρ? The mass (or energy) density that one measures depends on velocity but the L.H.S. does not, so this equation is not Lorentz invariant.

The relativistic generalization of the mass or energy density is the energy-momentum tensor. For a perfect fluid we have

$$T^{\alpha\beta} = \left(\rho + \frac{P}{c^2}\right) u^\alpha u^\beta + P g^{\alpha\beta}$$

where

$$g^{\alpha\beta} = \mathrm{diag}\,(1, -1, -1, -1)$$

is the metric tensor. We can get a scalar by taking

$$T = g_{\alpha\beta} T^{\alpha\beta} = \rho c^2 - 3P$$

so

$$\nabla^2 \phi - c^2 \frac{d^2 \phi}{dt^2} = 4\pi \frac{G}{c^2} T$$

The energy-momentum tensor of an electromagnetic field is traceless so $T = 0$. This means that photons or the energy in a electric field does not generate gravity. This is bad, bad, bad. If photons feel gravity, momentum is not conserved. If photons don't feel gravity, energy is not conserved. The next obvious step would be a vector field like electromagnetism, but it isn't obvious how to make a vector from the energy-momentum tensor. How about a tensor field? So

$$\Box h^{\alpha\beta} = 4\pi \frac{G}{c^2} T^{\alpha\beta}$$

But we would like gravitational energy to gravitate, so h should be on both sides. One can develop a theory equivalent to GR like this.

A Metric Theory of Gravity

Einstein assumed that all objects follow the same paths in a gravitational field regardless of their mass or internal composition (strong equivalence principle). He suggested that gravity is the curvature of spacetime. In the absence of forces other than gravity, objects follow extremal paths in the spacetime (geodesics). Therefore, the metric itself ($g_{\alpha\beta}$) contains the hallmarks of gravity.

Let's make some definitions:

$$u^\alpha = \frac{dx^\alpha}{ds}, u_\alpha = g_{\alpha\beta} u^\beta, g_{\alpha\beta,\gamma} = \frac{dg_{\alpha\beta}}{dx^\gamma}$$

Using the definition of the metric

$$\begin{aligned} \delta\left(ds^2\right) &= 2ds\delta\left(ds\right) = \delta\left(g_{\alpha\beta}dx^\alpha dx^\beta\right) \\ &= dx^\alpha dx^\beta g_{\alpha\beta,\gamma}\delta x^\gamma + 2g_{\alpha\beta}dx^\alpha d\left(\delta x^\beta\right) \end{aligned}$$

Solving for $\delta(ds)$ and integrating by parts yields

$$\begin{aligned} \delta s &= \int \delta(ds) = \int \left[\frac{1}{2}u^\alpha u^\beta g_{\alpha\beta,\gamma}\delta x^\gamma + g_{\alpha\gamma}u^\alpha \frac{d\delta x^\gamma}{ds}\right] ds \\ &= \int \left[\frac{1}{2}u^\alpha u^\beta g_{\alpha\beta,\gamma}\delta x^\gamma - \frac{d}{ds}\left(g_{\alpha\gamma}u^\alpha\right)\delta x^\gamma\right] ds \end{aligned}$$

Because the variation (δx^γ) is arbitrary we can set its coefficient equal to zero

$$\frac{du_\gamma}{ds} - \frac{1}{2}u^\alpha u^\beta g_{\alpha\beta,\gamma} = 0.$$

Although this equation is somewhat awkward, it yields an important result that if $g_{\alpha\beta,\gamma}$ is zero (i.e. none of the metric coefficients depend on the coordinate x^γ), then the component of the covariant four-velocity u_γ is constant along a geodesic. This statement of course depends on the coordinate system that one uses; nevertheless, it is very useful. Furthermore, it can be generalized to yield a coordinate independent result.

We can manipulate this further so that all of the four-velocities are contravariant (with superscripts).

$$\begin{aligned} \frac{1}{2}u^\alpha u^\beta g_{\alpha\beta,\gamma} - \frac{d}{ds}\left(g_{\mu\gamma}u^\mu\right) &= 0 \\ \frac{1}{2}u^\alpha u^\beta g_{\alpha\beta,\gamma} - g_{\mu\gamma}\frac{du^\mu}{ds} - u^\mu u^\beta g_{\mu\gamma,\beta} &= 0 \\ g_{\mu\gamma}\frac{du^\mu}{ds} + \frac{1}{2}u^\alpha u^\beta\left(g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma}\right) &= 0 \end{aligned}$$

After rearranging we can write

$$g_{\mu\gamma}\frac{du^\mu}{ds} + \Gamma_{\gamma,\alpha\beta}u^\alpha u^\beta = 0$$

where the connection coefficient or Christoffel symbol is given by

$$\Gamma_{\gamma,\alpha\beta} = \frac{1}{2}\left(g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma}\right)$$

Tensors

The quantities like $T_{\alpha\beta}$ that we have been manipulating are called tensors, and they have special properties. Specifically they transform simply under coordinate transformations.

$$T_{\alpha\beta} = \frac{\partial x'^{\gamma}}{\partial x^{\alpha}}\frac{\partial x'^{\delta}}{\partial x^{\beta}}T'_{\gamma\delta}, T^{\alpha\beta} = \frac{\partial x^{\alpha}}{\partial x'^{\gamma}}\frac{\partial x^{\beta}}{\partial x'^{\delta}}T'^{\gamma\delta},$$

Also if the metric is not constant you would expect derivatives to depend on how the coordinates change as you move too.

Covariant Derivative

We want a derivative that transforms like a tensor (this is also called the connection), The derivative of a scalar quality should be simple; it does not refer to any directions, so we define the covariant derivative to be $\phi_{;\alpha} = \phi_{,\alpha}$. Let's assume that the chain and product rules work for the covariant derivative like the normal one that we are familiar with (also linearity). Let's prove a result about the metric, the tensor that raises and lowers indices.

$$A_{\beta;\alpha} = \left(g_{\beta\gamma}A^{\gamma}\right)_{;\alpha} = g_{\beta\gamma;\alpha}A^{\gamma} + g_{\beta\gamma}A^{\gamma}_{;\alpha} = g_{\beta\gamma;\alpha}A^{\gamma} + A_{\beta;\alpha}$$

so because the vector field A_{β} is arbitrary, $g_{\beta\gamma;\alpha} = 0$.

Let's define the covariant derivative of a tensor that is compatible with our requirements.

$$A_{\alpha\beta;\gamma} = A_{\alpha\beta,\gamma} - A_{\delta\beta}\Gamma^{\delta}{}_{\alpha\gamma} - A_{\alpha\delta}\Gamma^{\delta}{}_{\beta\gamma}.$$

Let's apply this to the metric itself to get

$$g_{\alpha\beta;\gamma} = g_{\alpha\beta,\gamma} - g_{\delta\beta}\Gamma^{\delta}{}_{\alpha\gamma} - g_{\alpha\delta}\Gamma^{\delta}{}_{\beta\gamma}.$$

The left-hand side is zero. Furthermore, if we also assume that the derivative is symmetric (torsion-free), then $\Gamma^{\delta}{}_{\beta\gamma}$ is symmetric in its lower indices. We can use the symmetry of the metric and the connection to get the following three relations.

$$\begin{aligned}
0 &= g_{\alpha\beta,\gamma} - g_{\delta\beta}\Gamma^{\delta}{}_{\alpha\gamma} - g_{\alpha\delta}\Gamma^{\delta}{}_{\beta\gamma} \\
0 &= g_{\gamma\beta,\alpha} - g_{\delta\beta}\Gamma^{\delta}{}_{\alpha\gamma} - g_{\gamma\delta}\Gamma^{\delta}{}_{\beta\alpha} \\
0 &= g_{\alpha\gamma,\beta} - g_{\delta\gamma}\Gamma^{\delta}{}_{\alpha\beta} - g_{\alpha\delta}\Gamma^{\delta}{}_{\beta\gamma}
\end{aligned}$$

To get the second expression, we swapped α and γ. To get the third expression, we swapped β and γ. Now let's add the first two and subtract the third, cancelling terms that are equal by symmetry.

$$0 = g_{\alpha\beta,\gamma} + g_{\gamma\beta,\alpha} - g_{\alpha\gamma,\beta} - 2g_{\delta\beta}\Gamma^{\delta}{}_{\alpha\gamma}$$

$$\Gamma^{\delta}{}_{\alpha\gamma} = \frac{1}{2}g^{\delta\beta}\left(g_{\alpha\beta,\gamma} + g_{\gamma\beta,\alpha} - g_{\alpha\gamma,\beta}\right)$$

How the components of a tensor vary reflect both the change in the coordinates and the physical variance. We can remove the coordinate part and focus on the physics.

$$\begin{aligned}
A^{\alpha}{}_{;\beta} &= A^{\alpha}{}_{,\beta} + \Gamma^{\alpha}{}_{\gamma\beta}A^{\gamma} \\
A_{\alpha;\beta} &= A_{\alpha,\beta} - \Gamma^{\gamma}{}_{\alpha\beta}A_{\gamma} \\
A^{\alpha\beta}_{;\gamma} &= A^{\alpha\beta}_{,\gamma} + \Gamma^{\alpha}{}_{\delta\gamma}A^{\delta\beta} + \Gamma^{\beta}{}_{\delta\gamma}A^{\alpha\delta}
\end{aligned}$$

where

$$\Gamma^{\delta}{}_{\alpha\beta} = \frac{1}{2}g^{\delta\gamma}\left(g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma}\right)$$

Notice that the geodesic equation that we derived earlier can be written using the covariant derivative

$$g_{\alpha\gamma}\frac{du^{\alpha}}{ds} + \Gamma^{\gamma}_{\alpha\beta}u^{\alpha}u^{\beta} = 0$$

$$u^{\alpha}u^{\gamma}_{,\alpha} + \Gamma_{\gamma,\alpha\beta}u^{\alpha}u^{\beta} = 0$$

$$u^{\alpha}u^{\gamma}_{;\alpha} = 0$$

so the covariant derivative of the four-velocity along the geodesic is zero. The four-velocity is parallel transported along the geodesic. We can generalize this to particles with zero rest mass using the four-momentum (p^{α}),

$$p^{\alpha}p^{\gamma}_{;\alpha} = 0$$

so the four-momentum is also parallel transported along the geodesic. We have chosen that the connection or covariant derivative to be compatible with the metric (that is, $g_{\mu\nu};\gamma = 0$) and symmetric. These choices define the geometry of spacetime to be pseudo-Riemannian. Many of the important theorems of general relativity rely on this property of spacetime, and in fact alternative geometries in which the connection is not symmetric have been propose that agree with the results of general relativity except in the most extreme situations such as within the event horizons of black holes, the interiors of neutron stars and around the time of the Big Bang.

Killing Vectors

Let's look at the following scalar quantity

$$\epsilon_\mu p^\mu$$

and calculate how it changes along a geodesic

$$p^\nu (\epsilon_\mu p^\mu)_{;\nu} = p^\nu \epsilon_{\mu;\nu} p^\mu + p^\nu \epsilon_\mu p^\mu_{;\nu} = p^\nu p^\mu \epsilon_{\mu;\nu} = \frac{1}{2} p^\nu p^\mu \left(\epsilon_{\mu;\nu} + \epsilon_{\nu;\mu} \right).$$

where in the last step, we use the symmetry of the expression to create a more general expression.

The vector field that satisfies the condition $\epsilon_{\mu;\nu} + \epsilon_{\nu;\mu} = 0$ is called a Killing vector. Because the L.H.S. is a tensor that equals to zero, from the definition of the tensor transformations, this condition will be satisfied in all coordinate systems; it is a property of the spacetime itself. Furthermore, if ϵ^μ is a Killing vector, then the scalar quantity $\epsilon_\mu p^\mu$ is conserved along a geodesic.

Let's draw the connection between this result and the coordinate dependent finding that if the metric does not depend on a particular coordinate, $g_{\alpha\beta,\gamma}$ then p_γ is constant along a geodesic. We will explicitly calculate $\epsilon_{\mu;\nu} + \epsilon_{\nu;\mu}$ using the expression for the covariant derivative in a particular coordinate system

$$\begin{aligned} \epsilon_{\mu;\nu} + \epsilon_{\nu;\mu} &= \epsilon_{\mu,\nu} + \epsilon_{\nu,\mu} - 2\Gamma^\alpha_{\mu\nu}\epsilon_\alpha \\ &= (g_{\beta\mu}\epsilon^\beta)_{,\nu} + (g_{\beta\nu}\epsilon^\beta)_{,\mu} - g^{\alpha\beta}\left(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\right)\epsilon_\alpha \\ &= g_{\beta\mu}\epsilon^\beta_{,\nu} + g_{\beta\nu}\epsilon^\beta_{,\mu} + \epsilon^\beta g_{\mu\nu,\beta} \end{aligned}$$

so the Killing condition is related how the metric tensor changes with the coordinates. If the metric does not depend on a particular coordinate say the zeroth, so $g_{\mu\nu,0} = 0$ then $\epsilon^\beta = \delta^\beta_0$ is a Killing vector and in particular $\epsilon^\beta u_\beta$ is conserved along a geodesic in this case u_0 is constant.

Let's look at a flat spacetime with the metric

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

The metric coefficients do not depend on the coordinates so we clearly have four Killing vectors corresponding to the four coordinate directions. That is, the metric does not change under translations in space or time. The conserved quantities is the four-momentum (energy and the linear momentum). However, we can construct other Killing vectors that are not coordinate vectors. For example

$$\epsilon_\mu = \begin{bmatrix} 0 \\ y \\ -x \\ 0 \end{bmatrix}$$

corresponds to rotation about the $z-$axis with

$$\epsilon_{\mu;\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so ϵ_μ is a Killing vector and the corresponding conserved quantity is

$$\epsilon_\mu p^\mu = \gamma m y v_x + \gamma m x v_y,$$

the $z-$component of the angular momentum. And of course, we can also construct two other Killing vectors corresponding to rotations about the $x-$axis and the $y-$axis as well, yielding the other components of the angular momentum.

Some Important Tensors

- First, we measure scalar quantities the length of one vector along the direction of another. These scalars do not depend on the coordinate system.
- Coordinate vectors - dx^α
- Four velocity and four momentum - u^α and p^α.
- Killing vectors (ϵ^α) hold the key to the symmetry of the spacetime. The value of $\epsilon^\alpha p_\alpha$ is constant along a geodesic.
- I am moving with four-velocity u^α and I detect a particle with four-momentum p^α. I would measure an energy of $g_{\alpha\beta} u^\alpha p^\beta$.
- Of course, $g_{\alpha\beta}$ is the most important tensor of all. Without it we could not construct scalars and measure anything.

The Christoffel symbols are not a tensor. From the rule for tensor transformation if a tensor is zero in one coordinate system it will be zero in all others. If I use the geodesics themselves, I can set up a coordinate system *locally* in which the Christoffels vanish. However, if the geodesics diverge the Christoffels won't be zero everywhere. The separation (v^μ) of two nearby geodesics evolves as

$$\frac{d^2 v^\mu}{ds^2} = u^\nu u^\alpha v^\beta R^\mu{}_{\nu\alpha\beta}$$

where $R^\mu{}_{\nu\alpha\beta}$ is the Riemann tensor

$$R^\mu{}_{\nu\alpha\beta} = \Gamma^\mu{}_{\nu\beta,\alpha} + \Gamma^\mu{}_{\nu\alpha,\beta} + \Gamma^\mu{}_{\sigma\alpha}\Gamma^\sigma{}_{\nu\beta} + \Gamma^\mu{}_{\sigma\beta}\Gamma^\sigma{}_{\nu\alpha}$$

The Riemann tensor is like a covariant derivative of the Christoffel symbols (or a second derivative of the metric). Geometrically it encodes how a vector v^β will change when parallel transported along geodesics (in this case tangent to u^ν and u^μ) that form a closed loop.

Let's imagine that we set up a small cube in spacetime by placing test particles at each corner of the cube. The location of these test particles

Dynamics

We can finally make contact through the Riemann tensor ($R^\mu{}_{\nu\alpha\beta}$) to the source of gravity. According to Newton, the paths of two nearby objects diverge due to gravity as

$$\frac{d^2 v^\mu}{ds^2} = v^\beta f_{\mu,\beta} = -v^\nu \phi_{,\mu,\beta}.$$

Therefore, in the weak-field limit $R^\mu{}_{00\beta} = -\phi_{,\mu,\beta}$ and $\phi_{,\beta,\beta} = 4\pi G \rho$ is related to the Ricci tensor, $R_{\nu\alpha} = R^\beta{}_{\nu\alpha\beta}$ or more precisely the zero-zero component of the Ricci tensor (R_{00}). We can write the following equation with $R = R^\alpha_\alpha$ (the Einstein equation)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

where the first two terms comprise the Einstein tensor $G_{\mu\nu}$. It is important to note that $G_{\mu\nu;\nu} = 0$ so the covariant derivative of the L.H.S. vanishes. This makes sense because we expect the trace of the covariant derivative of the energy-momentum tensor to vanish as well because energy and momentum are conserved.

We can solve the Einstein equation for the Ricci tensor explicitly

$$R_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} - \frac{4\pi G}{c^4} T^\alpha{}_\alpha g_{\mu\nu} + \Lambda g_{\mu\nu}.$$

Because the Ricci tensor is trace of the Riemann tensor, it represents how the volume spanned by a bundle of geodesics changes with time. In particular following the arguments of Baez and Bunn, we can think about surrounding an observer with a small sphere of test particles initially at rest with respect to the observer. If both the observer and the particles follow geodesics, the observer will measure the volume of the sphere to change as

$$\left.\frac{\ddot{V}}{V}\right|_{t=0} = -4\pi G \left[\rho_m + \frac{1}{c^2}\left(P_x + P_y + P_z\right)\right] + \Lambda.$$

where ρ_m and P are the mass density and pressure (potentially anisotropic) that the observer measures at the centre of the sphere.

Although this single scalar equation appears to have fewer constraints than the ten Einstein equations, it is true for all observers regardless of velocity. The result obtains by calculating $u^\mu u^\nu R_{\mu\nu}$. A key observation is that the volume of the sphere will not change in the absence of matter (assuming $\Lambda = 0$)

Static spherically symmetric spacetime

Let us propose a spacetime that is spherically symmetric and unchanging with time. In particular this means that we have a timelike Killing vector and three spacelike Killing vector fields (corresponding to rotations) and therefore conserved quantities like energy and angular momentum. Furthermore, we can propose an metric for this situation

$$ds^2 = e^\nu c^2 dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \cos^2\theta d\phi^2)$$

where ν and λ are functions of radius (r). This metric could naturally describe the spacetime within and surrounding a star which for the most part is very close to spherically symmetric (the Sun is off a circle by about 6 km out of its 700,000 km average radius).

By the perfect fluid assumption, the stress-energy tensor is diagonal (in the central spherical coordinate system), with eigenvalues of energy density and pressure:

$$T^0_0 = \rho c^2 \text{ and } T^j_i = -P\delta^j_i$$

where $\rho(r)$ and $P(r)$ is the fluid pressure.

To continue we have to solve Einstein's equations:

$$\frac{8\pi G}{c^4} T_{\mu\nu} = G_{\mu\nu}$$

and we start with G_{00}

$$\frac{8\pi G}{c^4}\rho c^2 e^\nu = \frac{e^\nu}{r^2}\left(1 - \frac{d}{dr} r e^{-\lambda}\right).$$

This yields

$$\frac{d}{dr} r e^{-\lambda} = 1 - \frac{8\pi G}{c^4}\rho(r) c^2 r^2$$

so

$$r e^{-\lambda} = r - \frac{2Gm(r)}{c^2} + C$$

where

$$m(r) = \int_0^r 4\pi r'^2 \rho(r') dr'.$$

Therefore,

$$e^\lambda = \left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1}$$

where we have chosen $C = 0$ so the metric is regular at $r = 0$.

Now let's examine the equation for G_{11}

$$-\frac{8\pi G}{c^4} P e^{\lambda} = \frac{1}{r^2}\left(-r\frac{d\nu}{dr} + e^{\lambda} - 1\right)$$

and after rearranging

$$\frac{d\nu}{dr} = \frac{1}{r}\left(1 - \frac{2Gm(r)}{c^2 r}\right)^{-1}\left(\frac{2Gm(r)}{c^2 r} + \frac{8\pi G}{c^4} r^2 P(r)\right).$$

If we knew $P(r)$ (and $m(r)$ for that matter), we could all solve this by integration. This would yield the structure of the spacetime for a star of a given structure; we wouldn't have solved for the structure of the star as well. To accomplish this we will need another condition. However, before we do this, we can examine the situation where we are beyond the surface of the star, so $\rho(r)$ and $P(r)$ vanish and $m(r)$ is constant (let's call it M). In this case we have the equation

$$\frac{d\nu}{dr} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \frac{2GM}{c^2 r^2}.$$

As the second factor is the derivative of the first factor we have

$$\nu(r) = \ln\left(1 - \frac{2GM}{c^2 r}\right) + C.$$

and

$$e^{\nu} = \left(1 - \frac{2GM}{c^2 r}\right)$$

where we have taken the constant term (C) to be zero to recover flat spacetime far from the star.

To look at the structure of the star, we need to supply some additional conditions. We have assumed that the star is static and spherically symmetry so the stress-energy tensor only depends on the radial coordinate, so let's look at the conservation of momentum in the radial direction

$$0 = T^{\mu}_{1;\mu} = -\frac{dP}{dr} - \frac{1}{2}\left(P + \rho c^2\right)\frac{d\nu}{dr}$$

or

$$\frac{dP}{dr} = -\frac{1}{2}\left(P + \rho c^2\right)\frac{d\nu}{dr}.$$

We can use the expression for $d\nu/dr$ to give an equation for the pressure

$$\frac{dP}{dr} = -\frac{1}{2}\left(P + \rho c^2\right)\frac{1}{r}\left(1 - \frac{2Gm(r)}{c^2 r}\right)^{-1}\left(\frac{2Gm(r)}{c^2 r} + \frac{8\pi G}{c^4} r^2 P(r)\right)$$

which yields upon simplification

$$\frac{dP}{dr} = -\frac{G}{r^2}\left(\rho + \frac{P(r)}{c^2}\right)\left(m(r) + 4\pi r^3 \frac{P(r)}{c^2}\right)\left(1 - \frac{2Gm(r)}{c^2 r}\right)^{-1},$$

the relativistic equation of stellar structure or the Tolman-Oppenheimer-Volkoff equation. One solves this equation by assuming an equation of state $P(\rho)$ and integrating from the centre outward until the pressure vanishes; this is the surface of the star beyond which we have the vacuum solution (also known as Schwarzschild spacetime).

We can take the limit as c goes to infinity to recover the Newtonian equation of stellar structure

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho(r).$$

The key differences are that pressure is also a source of the gravitational field and the curvature of space also plays a role in the structure through the final factor. In Newton physics if the equation of state $P(\rho)$ increases faster than $\rho^{4/3}$, a star can be arbitrarily massive and large. How about in general relativity? Let's imagine a material that is completely incompressible (unobtainium or vibranium perhaps), so ρ is constant

$$\frac{dP}{dr} = -Gr\left(\rho + \frac{P(r)}{c^2}\right)\left(\frac{4\pi}{3}\rho + 4\pi\frac{P(r)}{c^2}\right)\left(1 - \frac{8\pi G\rho r^2}{3c^2}\right)^{-1},$$

and

$$\frac{c^4}{4\pi}\int_0^{P_c} dP\left[\left(\rho c^2 + P\right)\left(\frac{\rho c^2}{3} + P\right)\right]^{-1} = \int_0^R drGr\left(1 - \frac{8\pi G\rho r^2}{3c^2}\right)^{-1}.$$

This yields

$$\frac{3c^2}{8\pi\rho}\ln\left[\frac{1+\frac{3P_c}{\rho c^2}}{1+\frac{P_c}{\rho c^2}}\right] = -\frac{3c^2}{16\pi\rho}\ln\left[1 - \frac{8\pi G\rho R^2}{3c^2}\right]$$

and

$$\left[\frac{1+\frac{3P_c}{\rho c^2}}{1+\frac{P_c}{\rho c^2}}\right]^2 = \left[1 - \frac{2GM}{Rc^2}\right]^{-1}.$$

The largest that the left-hand side can be is nine as P_c goes to infinity so we have

$$9 > \left[1 - \frac{2GM}{Rc^2}\right]^{-1}.$$

and

$$R > \frac{9}{8}\frac{2GM}{c^2}.$$

Therefore, in general relativity there is a lower limit on the radius of a star of a given mass, or if the density is fixed, there is a upper limit on the radius or mass of the object. For example, if we take the density of water (1 g cm^{-3}), the maximum radius is about 6.3 AU and the maximum mass is about 5.7×10^8 M$_\odot$. Even in the matter is entirely incompressible, a sphere of matter larger than this will collapse under its own gravity and form a black hole. The spacetime exterior to this object will be the same as we found outside a spherical star earlier, the Schwarzschild spacetime.

Schwarzschild spacetime

What do paths in the Schwarzschild spacetime look like? It turns out that that metric has enough symmetry for a solution in quadrature. Here is the metric that we found earlier

$$ds^2 = dt^2\left(1 - \frac{2M}{r}\right) - dr^2\left(1 - \frac{2M}{r}\right)^{-1} - r^2\cos^2\theta d\phi^2 - r^2 d\theta^2 \quad (5.1)$$

where we have taken $G = c = 1$.

The metric does not depend on time and the angle ϕ so we have two Killing vectors corresponding to the dt and $d\phi$ coordinate directions, and two integrals of motion: $E = p_t$ and $L = p_\phi$. There are two additional Killing vector fields that correspond to the fact that the metric is spherically symmetric. Let us make the substitution

$$r = r'\left(1 + \frac{M}{2r'}\right)^2$$

to yield the metric in isotropic coordinates

$$\begin{aligned} ds^2 &= dt^2\left(\frac{1-\frac{M}{r'}}{1+\frac{M}{r'}}\right)^2 - \left(1+\frac{M}{r'}\right)^4\left(dr'^2 + r'^2\cos^2\theta d\phi^2 + r'^2 d\theta^2\right) \\ &= dt^2\left(\frac{1-\frac{M}{r'}}{1+\frac{M}{r'}}\right)^2 - \left(1+\frac{M}{r'}\right)^4\left(dx^2 + dy^2 + dz^2\right) \end{aligned}$$

where $r'^2 = x^2 + y^2 + z^2$. It is clear that we can construct three Killing vector fields

$$\epsilon^\mu_{(1,2,3)} = \begin{bmatrix} 0 \\ y \\ -x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -z \\ 0 \\ x \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ z \\ -y \end{bmatrix}$$

corresponding to the conservation of angular momentum.

These Killing vector fields constrain the geodesic motion to a plane, so without loss of generality let's look at paths in the plane $\theta = \pi/2$. Let's calculate p^2 using the metric to give

$$m^2 = E^2\left(1 - \frac{2M}{r}\right)^{-1} - (p^r)^2\left(1 - \frac{2M}{r}\right)^{-1} - \frac{L^2}{r^2} \quad (5.2)$$

and solve for p^r in terms of the constants of motion

$$(p^r)^2 = E^2 - \left(m^2 + \frac{L^2}{r^2}\right)\left(1 - \frac{2M}{r}\right). \quad (5.3)$$

Furthermore, we know that

$$\left(p^{\phi}\right)^2 = \frac{L^2}{r^4} \tag{5.4}$$

and combining these results yields,

$$\left(\frac{p^{\phi}}{p^r}\right)^2 = \left(\frac{d\phi}{dr}\right)^2 = \frac{1}{r^2}\left[\frac{r^2}{b^2} - 1 + \frac{2M}{r}\left(1 + \frac{m^2 r^2}{L^2}\right)\right]^{-1} \tag{5.5}$$

where $b^2 = L^2/(E^2 - m^2)$. This equation is sufficient to calculate the path of any object around a static black hole. In particular for $E^2 - m^2 < 0$ (therefore, $b^2 < 0$ the expression in parentheses vanishes for two values of r and the object is bound and travels between those radii. Furthermore, the twice difference between the integral of $d\phi/dr$ and π gives the periastron advance over a single orbit. We are interested in the total angular deflection when the deflection is small and the object is not bound, so we are interested in the value of the integral between the minimum radius and infinity for small values of M. To make further progress, we can write

$$\frac{1}{b^2} = \frac{1}{r_0^2} - \frac{2M}{r_0}\left(\frac{1}{r_0^2} + \frac{m^2}{L^2}\right) \tag{5.6}$$

which gives the value of the minimum radius implicitly terms of E, m and L and substitute this expression into the previous equation to yield

$$\left(\frac{d\phi}{dr}\right)^2 = \frac{1}{r^2}\left[\frac{r^2}{r_0^2} - 1 - \frac{2Mr^2}{r_0}\left(\frac{1}{r_0^2} + \frac{m^2}{L^2}\right) + \frac{2M}{r}\left(1 + \frac{m^2 r^2}{L^2}\right)\right]^{-1} \tag{5.7}$$

and define $x = r/r_0$ to give

$$\left(\frac{d\phi}{dx}\right)^2 = \frac{1}{x^2}\left[x^2 - 1 - \frac{2M}{r_0}x^2\left(1 + \frac{m^2 r_0^2}{L^2}\right) + \frac{2M}{r_0}\left(\frac{1}{x} + x\frac{m^2 r_0^2}{L^2}\right)\right]^{-1} \tag{5.8}$$

and

$$\left(\frac{d\phi}{dx}\right)^2 = \frac{1}{x^2(x^2-1)}\left[1 + \frac{2M}{r_0}\frac{1}{x^2-1}\left(\frac{1}{x} - x^2 + (x - x^2)\frac{m^2 r_0^2}{L^2}\right)\right]^{-1}. \tag{5.9}$$

Now we will expand $d\phi/dx$ in terms of M/r_0 to first order

$$\frac{d\phi}{dx} \approx \frac{1}{x\sqrt{x^2-1}} - \frac{M}{r_0}\frac{1}{x\left(x^2-1\right)^{3/2}}\left[\frac{1}{x} - x^2 + \frac{m^2 r_0^2}{L^2}\left(x - x^2\right)\right]. \tag{5.10}$$

The angle through which the path is scattered is given by

$$2\int_1^{\infty}\frac{d\phi}{dx}dx - \pi \approx \pi + 4\frac{M}{r_0} + 2\frac{M}{r_0}\frac{m^2 r_0^2}{L^2} - \pi = \frac{4GM}{r_0 c^2} + 2\frac{GM}{r_0 c^2}\frac{m^2 r_0^2 c^2}{L^2} \tag{5.11}$$

where in the final step we have inserted the powers of G and c to make contact with the non-relativistic limit

$$\Delta\theta \approx \frac{2GM}{r_0 c^2} + \frac{2GM}{r_0 v^2} \tag{5.12}$$

where v is the velocity at closest approach relative to a static observer. The expression is valid for small angles and all velocities.

Periastron Advance

Let's start with the equation

$$\left(\frac{d\phi}{dr}\right)^2 = \frac{1}{r^2}\left[\frac{r^2}{r_0^2} - 1 - \frac{2Mr^2}{r_0}\left(\frac{1}{r_0^2} + \frac{m^2}{L^2}\right) + \frac{2M}{r}\left(1 + \frac{m^2 r^2}{L^2}\right)\right]^{-1} \tag{5.13}$$

and define L in terms of the maximum radius r_1 as

$$\frac{1}{L^2} = \frac{r_1 r_0 (r_1 + r_0) - 2M\left(r_1^2 + r_0^2 + r_1 r_0\right)}{2m^2 M r_1^2 r_0^2} \tag{5.14}$$

yielding

$$\left(\frac{d\phi}{dr}\right)^2 = \frac{1}{r^2}\frac{r_1 r_0}{(r_1 - r)(r - r_0)}\left[1 - 2M\left(\frac{1}{r_0} + \frac{1}{r_1} + \frac{1}{r}\right)\right]^{-1}. \tag{5.15}$$

Let us expand $d\phi/dr$ to first order in M to yield

$$\frac{d\phi}{dr} \approx \frac{1}{r}\sqrt{\frac{r_1 r_0}{(r_1 - r)(r - r_0)}}\left[1 + M\left(\frac{1}{r_0} + \frac{1}{r_1} + \frac{1}{r}\right)\right]. \tag{5.16}$$

We require the following integrals

$$\int \frac{1}{r}\sqrt{\frac{r_1 r_0}{(r_1 - r)(r - r_0)}}dr = \arcsin\left[\frac{(r_1 + r_0)r - 2r_1 r_0}{r(r_1 - r_0)}\right] + C \tag{5.17}$$

and

$$\begin{aligned}\int \frac{1}{r^2}\sqrt{\frac{r_1 r_0}{(r_1 - r)(r - r_0)}}dr &= \frac{(r_1 - r)(r - r_0)}{r_1 r_0} \\ &\quad + \frac{1}{2}\frac{r_1 + r_0}{r_1 r_0}\arcsin\left[\frac{(r_1 + r_0)r - 2r_1 r_0}{r(r_1 - r_0)}\right] + C.\end{aligned}$$

These yield the following expression for the orbit,

$$\begin{aligned}\phi(r) &\approx \left[1 + \frac{3M}{2}\left(\frac{1}{r_1} + \frac{1}{r_0}\right)\right]\arcsin\left[\frac{(r_1 + r_0)r - 2r_1 r_0}{r(r_1 - r_0)}\right] \\ &\quad + \frac{M(r_1 - r)(r - r_0)}{r_1 r_0}\end{aligned}$$

The argument of the arcsine is 1 for $r = r_1$ and -1 for $r = r_0$, yielding the following result

$$\int_{r_0}^{r_1} \frac{d\phi}{dr}dr \approx \pi + \frac{3\pi}{2}M\left(\frac{1}{r_1} + \frac{1}{r_0}\right) \tag{5.18}$$

so the advance of periastron through one orbit is

$$\Delta\theta \approx \frac{3\pi GM}{c^2}\left(\frac{1}{r_1}+\frac{1}{r_0}\right) = \frac{6\pi GM}{c^2 a(1-e^2)} \tag{5.19}$$

where a is the semi-major axis of the elliptical orbit and e is its eccentricity.

If we specialize to Mercury, we have $GM_\odot/c^2 = 1.48$ km, $a = 5.791 \times 10^7$ km and $e = 0.2056$ so

$$\Delta\theta \approx 5.03 \times 10^{-7} \tag{5.20}$$

or 43 arcseconds per century.

Looking back at the expression for the orbit, let us define a new angle that accounts for the precession of the orbit,

$$\phi(r) = \left[1 + \frac{3M}{2}\left(\frac{1}{r_1}+\frac{1}{r_0}\right)\right]\psi(r) \tag{5.21}$$

so

$$\begin{aligned}\psi(r) \quad\approx\quad & \arcsin\left[\frac{(r_1+r_0)\,r - 2r_1 r_0}{r\,(r_1-r_0)}\right] \\ & + \frac{M(r_1-r)(r-r_0)}{r_1 r_0}\left[1+\frac{3M}{2}\left(\frac{1}{r_1}+\frac{1}{r_0}\right)\right]^{-1}\end{aligned}$$

The exact solution is also tractable in terms of elliptic integrals. Looking at $(d\phi/dr)^2$ again yields

$$\left(\frac{d\phi}{dr}\right)^2 = \frac{c^2}{2GM}\frac{1}{r}\frac{r_0 r_1 r_2}{(r-r_0)\,(r_1-r)\,(r-r_2)} \tag{5.22}$$

where

$$r_2 = \left[\frac{1}{2M} - \frac{1}{r_1} - \frac{1}{r_0}\right]^{-1} = \left[\frac{c^2}{2GM} - \frac{1}{r_1} - \frac{1}{r_0}\right]^{-1}. \tag{5.23}$$

The Newtonian limit obtains as $c \to \infty$ so the exact relativistic expression is the minimal modification to the Newtonian result. For the value of $(d\phi/dr)$ to make sense we must have $r_1 \geq r_0 \geq r_2$, so the limiting circular orbit has $r_1 = r_0 = r_2$ which allows us to conclude that $r_0 = r_1 = r_2 = 6M$ is the limiting stable circular orbit. There are circular orbits down to $r_0 = r_1 = 3M$. However, these orbits do not have neighbouring non-circular orbits; therefore, they are unstable. For elongated orbits we have $r_1 > r_0 = r_2$ so $r_1 > 2Mr_0/(r_0 - 4M)$. For example the innermost orbit with $r_1 = 2r_0$ has $r_0 = 5M$, but more elongated orbits are allowed all the way down to $r_0 = 4M$. For the most extreme elongated orbits with $r_1 = 2Mr_0/(r_0 - 4M)$, the rate of periastron advance diverges. The resulting orbit consists of many revolutions near the inner radius, a rapid excursion to the outer radius and then many further revolutions near the inner radius. Fig. 5.1 depicts a series of orbits with $r_1 = 2r_0$ or $e = 1/3$. Although the orbits with $r_0 \gg 2GM/c^2$ appear nearly like ellipses with a constant rate of precession, the smaller orbits are significantly distorted.

Time Evolution

We have a very compact expression for the shape of the orbit. Can we develop an expression for how the position of the particle changes with time? We have

$$p^t = \left(1 - \frac{2M}{r}\right)^{-1} E \tag{5.24}$$

so

$$\begin{aligned}\left(\frac{dt}{dr}\right)^2 &= \left(\frac{p^t}{p^r}\right)^2 = \left(\frac{p^\phi}{p^r}\right)^2 \left(\frac{p^t}{p^\phi}\right)^2 \\ &= \frac{E^2}{\left(1-\frac{2M}{r}\right)^2} \frac{r^4}{L^2} \frac{1}{2M} \frac{1}{r} \frac{r_0 r_1 r_2}{(r-r_0)(r_1-r)(r-r_2)}\end{aligned}$$

We can clean this up a bit to yield

$$\left(\frac{dt}{dr}\right)^2 = \frac{E^2}{2ML^2} \frac{r^5 r_0 r_1 r_2}{(r-2M)^2 (r-r_0)(r_1-r)(r-r_2)} \tag{5.25}$$

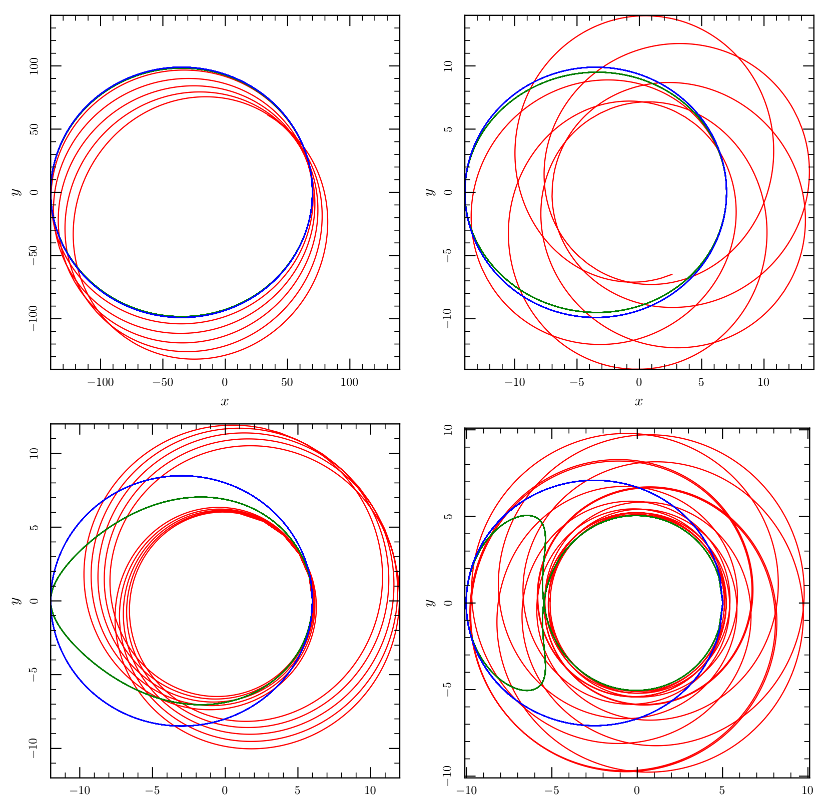


Figure 5.1: Orbits in General Relativity with $2GM/c^2 = 2$. In each panel, red traces the orbit, green traces the orbit with the mean periastron advance $\dot\omega$ subtracted, and blue traces the ellipse with the same values of r_0 and r_1 Upper-left panel: $r_0 = 70, r_1 = 140$ and $\dot\omega P = 0.21$. Upper-right panel: $r_0 = 7, r_1 = 14$ and $\dot\omega P = 1.02$. Lower-left panel: $r_0 = 6, r_1 = 12$ and $\dot\omega P = 6.56$. Lower-right panel: $r_0 = 5.01, r_1 = 10.02$ and $\dot\omega P = 26.68$. Because as the orbits get smaller and smaller the mean periastron advance increases, the orbits beome successively more pointy toward the apastron and for the most extreme orbits develop hour-glass shapes. The periastron advance per orbit of the orbit in the lower-left panel is approximately 2π more than than of the upper-left panel.

Is space curved?

Let's look at this in the context of an alternative metric with a different amount of spatial curvature

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2\gamma M}{r}\right)^{-1} dr^2 - r^2 \cos^2\theta d\phi^2 - r^2 d\theta^2. \tag{5.26}$$

Here γ is the PPN parameter that quantifies how much space curvature is produced by unit rest mass.

Again without loss of generality let's look at paths in the plane $\theta = \pi/2$. The metric does not depend on time and the angle ϕ so we have two integrals of motion in addition to the invariant mass of the particle; these are $E = p_t$ and $L = p_\phi$. Let's calculate p^2 using the metric to give

$$m^2 = E^2 \left(1 - \frac{2M}{r}\right)^{-1} - (p^r)^2 \left(1 - \frac{2\gamma M}{r}\right)^{-1} - \frac{L^2}{r^2} \tag{5.27}$$

and solve for p^r in terms of the constants of motion

$$(p^r)^2 = \left[E^2 - \left(m^2 + \frac{L^2}{r^2}\right)\left(1 - \frac{2M}{r}\right)\right]\left(1 - \frac{2\gamma M}{r}\right)\left(1 - \frac{2M}{r}\right)^{-1}. \tag{5.28}$$

This expression is a bit more complicated than the general relativistic result, Eq. 5.3. In general relativity the two terms outside the brackets cancel. However, the derivation of the equation for the shape of the orbit can proceed almost identically.

Again, we know that

$$\left(p^\phi\right)^2 = \frac{L^2}{r^4} \tag{5.29}$$

and combining these results yields

$$\left(\frac{d\phi}{dr}\right)^2 = \frac{1}{r^2}\left[\frac{r^2}{b^2} - 1 + \frac{2M}{r}\left(1 + \frac{m^2 r^2}{L^2}\right)\right]^{-1}\left(1 - \frac{2M}{r}\right)\left(1 - \frac{2\gamma M}{r}\right)^{-1} \tag{5.30}$$

where $b^2 = L^2/(E^2 - m^2)$.

Again we look at the minimum radius of r_0 for an unbound trajectory. We can write E^2 in terms of the value of r_0 as before, Eq. 5.6, we have

$$\frac{1}{b^2} = \frac{1}{r_0^2} - \frac{2M}{r_0}\left(\frac{1}{r_0^2} + \frac{m^2}{L^2}\right) \tag{5.31}$$

which gives the value of the minimum radius implicitly terms of E, m and L and substitute this expression into the previous equation to yield

$$\left(\frac{d\phi}{dr}\right)^2 = \frac{1}{r^2}\left[\frac{r^2}{r_0^2} - 1 - \frac{2Mr^2}{r_0}\left(\frac{1}{r_0^2} + \frac{m^2}{L^2}\right) + \frac{2M}{r}\left(1 + \frac{m^2 r^2}{L^2}\right)\right]^{-1} \frac{r - 2M}{r - 2\gamma M} \tag{5.32}$$

as before but with the extra factor. Again we can define $x = r/r_0$ and expand $d\phi/dx$ to lowest order in M/r_0 to yield

$$\frac{d\phi}{dx} \approx \frac{1}{x\sqrt{x^2-1}} - \frac{M}{r_0}\frac{1}{x\left(x^2-1\right)^{3/2}}\left[\frac{1}{x} - x^2 + \frac{m^2 r_0^2}{L^2}\left(x - x^2\right)\right] + \frac{M}{r_0}\frac{\gamma-1}{x\sqrt{x^2-1}}. \tag{5.33}$$

We can calculate the deflection as before but with the extra term to yield

$$\Delta\theta \approx \frac{2GM}{c^2 r_0} + \frac{2GM}{v^2 r_0} + \frac{2(\gamma-1)GM}{c^2 r_0} = \frac{2\gamma GM}{c^2 r_0} + \frac{2GM}{v^2 r_0} \tag{5.34}$$

so the fact that the bending of light is twice the Newtonian value due to the curvature of space as well as time.

We can also look at the perihelion advance, starting with the modification of Eq. 5.15 to become

$$\left(\frac{d\phi}{dr}\right)^2 = \frac{1}{r^2}\frac{r_1 r_0}{\left(r_1 - r\right)\left(r - r_0\right)}\left[1 - 2M\left(\frac{1}{r_0} + \frac{1}{r_1} + \frac{1}{r}\right)\right]^{-1}\frac{r - 2M}{r - 2\gamma M} \tag{5.35}$$

and to first order in M we have

$$\frac{d\phi}{dr} \approx \frac{1}{r}\sqrt{\frac{r_1 r_0}{\left(r_1 - r\right)\left(r - r_0\right)}}\left[1 + M\left(\frac{1}{r_0} + \frac{1}{r_1} + \frac{\gamma}{r}\right)\right]. \tag{5.36}$$

This yields the following expression for the orbit,

$$\begin{aligned}\phi(r) \quad \approx \quad & \left[1 + M\left(1 + \frac{\gamma}{2}\right)\left(\frac{1}{r_1} + \frac{1}{r_0}\right)\right]\arcsin\left[\frac{\left(r_1 + r_0\right) r - 2 r_1 r_0}{r\left(r_1 - r_0\right)}\right] \\ & + \frac{\gamma M (r_1 - r)(r - r_0)}{r_1 r_0}\end{aligned} \tag{5.37}$$

and the advance of periastron through one orbit of

$$\Delta\theta \approx \frac{2\pi GM}{c^2}\left(1 + \frac{\gamma}{2}\right)\left(\frac{1}{r_1} + \frac{1}{r_0}\right) = \frac{4\pi GM}{c^2 a(1-e^2)}\left(1 + \frac{\gamma}{2}\right) \tag{5.38}$$

so one third of the periastron advance is due to the curvature of space and two-thirds from the curvature of time.

The $\gamma = 0$ values of the periastron advance and the equivalent deflection of a distant passer-by are called the classical values. The gravitational redshift is independent of γ because it only depends on the g_{00} part of the metric.

The Kerr Metric

The spacetime surrounding an astrophysical black hole is thought to be described by the Kerr metric which carries angular momentum as well. The Kerr metric is somewhat more complicated than the

Schwarzschild metric, but it too admits the constants of motion $E = p_t$ and $L = p_\phi$ because the metric does not depend on time or the azimuthal angle

$$\begin{aligned} ds^2 &= \left(1-\frac{2Mr}{\Sigma}\right)dt^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 \\ &\quad - \left(r^2+a^2+\frac{2Mra^2}{\Sigma}\sin^2\theta\right)\sin^2\theta\, d\phi^2 \\ &\quad + \frac{4Mra\sin^2\theta}{\Sigma}c\,dt\,d\phi \end{aligned} \tag{5.39}$$

where $a = J/Mc$, $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. It is not strictly speaking a static spacetime because of the mixture of the time and azimuthal coordinate; it is classified as a stationary spacetime because it still has a timelike Killing vector.

Again we will first look at paths in the plane $\theta = \pi/2$. However, in the case of the Kerr metric, we will have to consider other paths as well because the spacetime is not spherically symmetric, just cylindrically symmetric. In the Schwarzschild metric, the orbit is restricted to a plane which is effectively an additional constant of the motion. The Kerr metric also has this additional constant of the motion but because the the angular momentum of the spacetime it does not admit such a clear geometric interpretation.

To examine the orbits near to the equatorial plane, we will write the metric as

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 + 2g_{t\phi}dtd\phi \tag{5.40}$$

and again define $L = p_\phi$ and $E = p_t$. Because the metric is no longer diagonal the relationships are somewhat more complicated

$$p^\phi = \frac{g_{tt}L - g_{t\phi}E}{g_{\phi\phi}g_{tt} - g_{t\phi}^2} \text{ and } p^t = \frac{g_{\phi\phi}E - g_{t\phi}L}{g_{\phi\phi}g_{tt} - g_{t\phi}^2} \tag{5.41}$$

and

$$m^2 = \frac{g_{\phi\phi}E^2 - 2g_{t\phi}EL + g_{tt}L^2}{g_{\phi\phi}g_{tt} - g_{t\phi}^2} + g_{rr}\left(p^r\right)^2 + g_{\theta\theta}\left(p_\theta\right)^2. \tag{5.42}$$

Again this equation can be solved by quadrature as we did for the Schwarzschild metric if one focuses on motion in the equatorial plane. And there is an additional constant of the motion known as the Carter constant which makes the equations of motion integrable.

Epicyclic frequencies

However, as we have not yet made precision tests of orbits in the Kerr metric, we will take a different tack and examine the frequencies of perturbations on circular orbits in the Kerr metric. Of course, the most basic frequency is the Kepler (Ω_ϕ) frequency, the rate at which a distant observer would see the particle complete a circular orbit. It is defined as the value of $d\phi/dt$ for a geodesic with a constant value of r and $\theta = \pi/2$ (the equatorial plane). The two additional frequencies are the vertical and radial epicyclic frequencies. The differences between these frequencies and the Kepler frequency are the nodal precession ($\Omega_{\rm node}$), and the periastron precession ($\Omega_{\rm peri}$) frequencies.

The three observed frequencies can be represented as follows

$$\begin{aligned} \Omega_\phi &= \frac{u^\phi}{u^t} && (5.43)\\ \Omega_{\rm node} &= \Omega_\phi - \omega_\theta && (5.44)\\ \Omega_{\rm peri} &= \Omega_\phi - \omega_r\,. && (5.45) \end{aligned}$$

We begin the derivations for a general stationary, axisymmetric spacetime by finding the Kepler frequency, Ω_ϕ, using the geodesic equation with respect to radius, r. Since $u_r = g_{rr}u^r$, and $u^r = \frac{dr}{d\tau}$, we can re-write the geodesic equation using:

$$\frac{du_r}{d\tau} = \frac{d}{d\tau}g_{rr}u^r = \frac{d}{d\tau}\left(g_{rr}\frac{dr}{d\tau}\right)\,. \qquad (5.46)$$

However, since the object's radial velocity is constant over time, we can write the change in this velocity as $\frac{d}{d\tau}\left(g_{rr}\frac{dr}{d\tau}\right) = 0$, simplifying the geodesic equation to:

$$\frac{1}{2}(g_{\mu\nu,r}u^\mu u^\nu) = 0\,. \qquad (5.47)$$

For the Kepler frequency, we assume that the orbit is at a constant radius in the equatorial plane, so the sum reduces to three terms

$$\frac{1}{2}(g_{tt,r}u^t u^t + 2g_{t\phi,r}u^t u^\phi + g_{\phi\phi,r}u^\phi u^\phi) = 0\,. \qquad (5.48)$$

Because the Kepler frequency is $\Omega_\phi = u^\phi/u^t$, dividing the equation by $u^t u^t/2$ yields a quadratic equation for the Kepler frequency

$$(g_{tt,r} + 2g_{t\phi,r}\Omega_\phi + g_{\phi\phi,r}\Omega_\phi^2) = 0\,. \qquad (5.49)$$

Using the coefficients from the metric and solving the quadratic equation for the Kepler frequency gives

$$\Omega_\phi = \pm\frac{\sqrt{M}}{r\sqrt{r} \pm a\sqrt{M}}\,. \qquad (5.50)$$

If we take $a = 0$, we recover Kepler's third law

$$\Omega_\phi^2 = \frac{M}{r^3}\,. \qquad (5.51)$$

For the remaining two frequencies, the value of u^t is needed for the circular orbit. We begin by noting that

$$u \cdot u = u_\alpha u^\beta = g_{\alpha\beta} u^\alpha u^\beta = 1\,. \tag{5.52}$$

Because the nodal and periastron precession frequencies can be thought of as perturbations to the Kepler frequency, a circular orbit is assumed in the derivation of u^t. This assumption results in only the t and ϕ components playing a role, such that

$$1 = g_{tt}(u^t)^2 + 2g_{t\phi}u^\phi u^t + g_{\phi\phi}(u^\phi)^2\,. \tag{5.53}$$

Remembering that $\Omega_\phi = u^\phi / u^t$, and therefore $u^\phi = \Omega_\phi u^t$:

$$1 = g_{tt}(u^t)^2 + 2g_{t\phi}\Omega_\phi(u^t)^2 + g_{\phi\phi}(\Omega_\phi u^t)^2\,, \tag{5.54}$$

where the Kepler frequency, Ω_ϕ, was previously derived. This leaves the above equation with only one unknown, u^t, and hence:

$$u^t = (g_{tt} + 2g_{t\phi}\Omega_\phi + g_{\phi\phi}\Omega_\phi^2)^{-\frac{1}{2}}\,. \tag{5.55}$$

To find the radial epicyclic frequency, ω_r, we use the conserved quantities for the orbit. The quantity u_ϕ is the angular momentum (L) and u_t is the energy (E). With these new definitions, and a corresponding change of indices from lowered to raised on all metric coefficients, we can rewrite Eq. 5.53 as follows:

$$1 = g^{tt}E^2 + 2g^{t\phi}EL + g^{\phi\phi}L^2\,. \tag{5.56}$$

To perturb the orbit, additional terms are added to account for movement in the radial direction.:

$$1 + E_2 = g^{tt}E^2 + 2g^{t\phi}EL + g^{\phi\phi}L^2 + g_{rr}(u^r)^2\,, \tag{5.57}$$

where u^r is the radial component of the four velocity, E_2 is the additional energy of the perturbation and the values of E and L are held fixed at the values for a circular orbit. We perturb the position about the circular orbit which has $r = r_0$ and use Eq. 5.53 and the geodesic equation of the circular orbit (Eq. 5.49) to yield the second-order perturbation

$$E_2 = \frac{(r - r_0)^2}{2}\left(g^{tt}_{,rr}E^2 + 2g^{t\phi}_{,rr}EL + g^{\phi\phi}_{,rr}L^2\right) + g_{rr}(u^r)^2\,. \tag{5.58}$$

By dividing the above equation by two, it becomes clear that the $\frac{1}{2}g_{rr}(u^r)^2$ term behaves as kinetic energy (classically of the form $\frac{1}{2}mv^2$). The above equation is therefore a conservation equation, with the remaining terms accounting for the effective potential energy,

$$E_2 = \frac{1}{2}k\,(r - r_0)^2 + \frac{1}{2}mv^2 \tag{5.59}$$

where $v = u^r = dr/d\tau$, $m = g_{rr}$ and

$$k = \frac{1}{2}\left(g^{tt}_{,rr}E^2 + 2g^{t\phi}_{,rr}EL + g^{\phi\phi}_{,rr}L^2\right). \tag{5.60}$$

In the frame of the orbiting particle, the oscillation frequency is given by

$$\omega^2 = \frac{k}{m} = \frac{g^{tt}_{,rr}E^2 + 2g^{t\phi}_{,rr}EL + g^{\phi\phi}_{,rr}L^2}{2g_{rr}}. \tag{5.61}$$

The next step is to convert everything from proper time to time in the observer's reference frame. After converting reference frames, we are left with our radial solution:

$$\omega_r^2 = \frac{g^{tt}_{,rr}E^2 + 2g^{t\phi}_{,rr}EL + g^{\phi\phi}_{,rr}L^2}{2g_{rr}(u^t)^2}. \tag{5.62}$$

This equation can be simplified further by recognizing that $u^\phi = \Omega_\phi u^t$ and dividing out the $(u^t)^2$ term in the denominator

$$\omega_r^2 = \frac{g_{tt,rr} + 2g_{t\phi,rr}\Omega_\phi + g_{\phi\phi,rr}\Omega_\phi^2}{2g_{rr}}. \tag{5.63}$$

Because no assumptions were made in the derivation of ω_r that would not also apply to ω_θ, the solution for the final frequency ω_θ can be carried out in the exact same manner but holding r constant and perturbing θ. For brevity, only the final result is included here:

$$\omega_\theta^2 = \frac{g_{tt,\theta\theta} + 2g_{t\phi,\theta\theta}\Omega_\phi + g_{\phi\phi,\theta\theta}\Omega_\phi^2}{2g_{\theta\theta}}. \tag{5.64}$$

Linearized Gravity

Up to now we have focused on spacetimes with a lot of symmetry, in particular a large set of Killing vectors (including a timelike Killing vector). Of course, in astrophysics this does not exhaust the possibilities, and the completely general situation is very complicated and must be treated numerically. However, we can make some progress to understand a special subset of dynamic spacetimes, those without matter and that deviate from flatness by a small amount, that is spacetimes such that

$$ds^2 = cdt^2 - dx^2 - dy^2 - dz^2 + h_{\mu\nu}dx^\mu dx^\nu$$

or equivalently $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu} \ll 1$. Although we have not yet discussed the point in detail, the metric contains many degrees of freedom that do not have physical consequences. We have seen this in electromagnetism where the underlying equations are most simply written using the potential A^μ or $\phi, \mathbf{A}$ but the physical quantities of interest are the electric and magnetic field. In electromagnetism this freedom is called gauge invariance. The resulting physics is invariant under gauge transformations. In general relativity, the physical consequences do not depend on the choice of coordinates that we use, and we ensure this through the tensor transformation rules. For example, we performed a coordinate transformation on the Schwarzschild metric to highlight the symmetries of the spacetime that were not obvious in the original coordinate system.

This freedom in general relativity is called diffeomorphism invariance (or gauge invariance as well). The situation in general relativity is quite a bit more difficult than in electromagnetism because there is not a local quantity that one can claim is diffeomorphism invariant (in analogy to the electric and magnetic field). In fact one can set up a coordinate system locally in which the effects of gravity apparently vanish, that is $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^{\gamma}{}_{\mu\nu} = 0$. Thus far we have avoided issues with diffeomorphism invariance by restricting our attention to observables (e.g. what is the rate that a distant observer will see the particle orbit the star?) that we calculate using scalar quantities (which are necessarily diffeomorphism invariant). Although the answers that we give are functions of the coordinates that we use to measure the spacetime, the physical content would be independent of the coordinate system. We haven't checked, but the tensor transformation rules guarantee it.

With all of this talk of gauge transformations and diffeomorphisms, one might be prompted to ask is the split of the metric into a flat portion and the small perturbation invariant with respect to diffeomorphisms. The answer is of course no, so we will have to be very careful in working with these perturbations. In particular we are interested in applying changes of coordinates that leave the metric with the structure of a small perturbation to $\eta_{\mu\nu}$. Let us take a coordinate transformation of the form

$$x'^{\mu}(x^{\nu}) = x^{\nu} + \epsilon\xi^{\mu}(x^{\nu}),$$

where $\xi^{\mu}(x^{\nu})$ is an arbitrary vector field. We use the ϵ to designate the order of smallness of the term. As we perform the calculations, we will be interested in keeping terms up to order ϵ or $h_{\mu\nu}$. Let's apply it to the metric

$$\begin{aligned} g^{(\epsilon)}_{\alpha\beta} &= \frac{\partial x'^{\gamma}}{\partial x^{\alpha}}\frac{\partial x'^{\delta}}{\partial x^{\beta}} g'_{\gamma\delta} = (\delta^{\gamma}_{\alpha} + \epsilon\xi^{\gamma}{}_{,\alpha})(\delta^{\delta}_{\beta} + \epsilon\xi^{\delta}{}_{,\beta})(\eta_{\mu\nu} + h_{\mu\nu}) \\ &= \eta_{\alpha\beta} + h_{\alpha\beta} + \epsilon(\partial_{\alpha}\xi_{\beta} + \partial_{\beta}\xi_{\alpha}) \end{aligned}$$

where in the second line we have only retained terms to first order.

This yields the transformed metric perturbation

$$h_{\mu\nu}^{(\epsilon)} = h_{\mu\nu} + \epsilon(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu).$$

so we see that we have the freedom to add the symmetrized gradient of a vector field to the perturbation of the metric while keeping the spacetime fixed. We have not proven here that such a transformation exhausts the set of the diffeomorphisms that preserve the structure of a small perturbation to a flat metric, but this is the case.

So how can we use this linearization and the gauge invariance to help us? First of all, the linearization of the metric reduces the number of terms in the Einstein tensor from about fifty to

$$G_{\mu\nu} = \frac{1}{2}(\partial_\sigma \partial_\mu h_\nu^\sigma + \partial_\sigma \partial_\nu h_\mu^\sigma - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu} \partial_\rho \partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu} \Box h),$$

a linear second order differential equation for $h_{\mu\nu}$. In this equation $h = h^\mu{}_\mu$. We can simplify this further if

$$\partial_\mu h_\nu^\mu = \frac{1}{2} \partial_\nu h.$$

We can achieve this by using the gauge invariance of the equations and selecting a particular gauge

$$\Box \xi_\mu = -\partial_\nu h_\mu^\nu + \frac{1}{2} \partial_\mu h.$$

This may be reminiscent of the Lorenz gauge in electromagnetism, and in fact it is called the Lorenz or harmonic gauge in general relativity. This simplifies the Einstein tensor further to

$$G_{\mu\nu} = -\frac{1}{2} \Box \left(h_{\mu\nu}^{(\epsilon)} - \frac{1}{2} h^{(\epsilon)} \eta_{\mu\nu} \right)$$

where the superscript (ϵ) indicates that we have chosen a particular gauage.

We call the expression in the parenthesis the "trace-reversed metric," $\bar{h}_{\mu\nu}^{(\epsilon)} = h_{\mu\nu}^{(\epsilon)} - \frac{1}{2} h^{(\epsilon)} \eta_{\mu\nu}$ (and $h_{\mu\nu}^{(\epsilon)} = \bar{h}_{\mu\nu}^{(\epsilon)} - \frac{1}{2} \bar{h}^{(\epsilon)} \eta_{\mu\nu}$), yielding

$$\Box \bar{h}_{\mu\nu}^{(\epsilon)} = -16\pi G T_{\mu\nu}.$$

In terms of the "trace-reversed metric" the gauge condition is

$$\bar{h}_{\mu\nu}^{(\epsilon),\nu} = 0$$

in analogy to the Lorenz gauge condition.

We have already developed the machinery to solve this equation and intuition about the solutions as well. In particular the value of $\bar{h}^{(\epsilon)}_{\mu\nu}$ at a particular location is proportional to the integral of the energy-momentum tensor over the past light cone,

$$\bar{h}^{(\epsilon)}_{\mu\nu}(\mathbf{r}, t) = 4G \int d^3r' \frac{T_{\mu\nu}\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}$$

There is no ambiguity about the structure of the past light cone because in the linearization process we neglect terms past first order, so it is the light cone of flat spacetime. And we will have gravitational radiation and approximately Newtonian gravity. However, we do know that this equation is necessarily an approximation to the full picture, and the simple and easy to interpret expression is the result of a particular gauge choice. These ambiguities led physicists to question the reality of gravitational radiation, in particular whether it can be produced or detected for nearly half a century. Of course, the measurement of the orbital decay of the binary pulsar in line with the predictions of linearized general relativity in the 1970s put these questions to rest.

Gravitational Waves

Let's first look at the solution outside of matter so we have

$$\Box \bar{h}^{(\epsilon)}_{\mu\nu} = 0$$

and we can propose a solution

$$\bar{h}^{(\epsilon)}_{\mu\nu} = \Re\left[A_{\mu\nu}\exp(k_\alpha x^\alpha)\right]$$

which yields two conditions on the solution

$$k^\alpha k_\alpha = 0, A_{\mu\alpha}k^\alpha = 0.$$

The first results from the wave equation and the second from the gauge choice. As a symmetric tensor $A_{\mu\nu}$ has ten independent components. We have imposed four additional constraints from our gauge choice, leaving six independent components. There is still additional gauge freedom. If we define a vector field

$$\xi^\mu = \Re\left[-iC^\mu \exp(k_\alpha x^\alpha)\right],$$

we can fix the gauge further with four more constraints, leaving two independent components for $\bar{h}^{(\epsilon)}_{\mu\nu}$.

It is customary to fix impose the four additional constraints by selecting a particular frame of reference, a four-velocity u^μ, everywhere in the spacetime, perhaps the four-velocity of the observer. This is fine because we are assuming that the spacetime is flat and insisting that

$$A_{\mu\nu}u^\mu = 0 \text{ (transverse)}; A^\mu{}_\mu = 0 \text{ (traceless)}.$$

With these choices $\bar{h}^{(\epsilon)}_{\mu\nu} = h^{(\epsilon)}_{\mu\nu}$, which we will call $h^{TT}_{\mu\nu}$. In order to put the field into the TT-gauge we exploited the wavelike solution that we imposed. Non-radiative components of the metric cannot necessarily be put into the TT-gauge.

To make things explicit let us take the wave to be propagating in the $z-$direction which means that

$$h^{TT}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{xx} & h_{xy} & 0 \\ 0 & h_{xy} & -h_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where the components depend only on z and t. We know that because the metric does not depend on x or y that u_x and u_y are constant as the wave passes. Furthermore, if we assume that we are measuring the distance between two objects located at x^μ_A and x^μ_B and that both were not moving in the frame before the wave arrives, then the spatial coordinates will be remain constant while the wave is passing and in particular $\Delta x^\mu = x^\mu_A - x^\mu_B$ will be constant.

Polarization

Let's calculate the proper length corresponding to Δx^μ. We will focus on the $x-$ and $y-$components and the set the others to zero so we have

$$ds^2 = [\Delta x^x \;\; \Delta x^y] \begin{bmatrix} 1 + h_{xx} & h_{xy} \\ h_{xy} & 1 - h_{xx} \end{bmatrix} \begin{bmatrix} \Delta x^x \\ \Delta x^y \end{bmatrix}$$

and rewrite this in a special way

$$ds^2 = [\Delta x^x \;\; \Delta x^y] \begin{bmatrix} 1 + \frac{1}{2}h_{xx} & \frac{1}{2}h_{xy} \\ \frac{1}{2}h_{xy} & 1 - \frac{1}{2}h_{xx} \end{bmatrix} \begin{bmatrix} 1 + \frac{1}{2}h_{xx} & \frac{1}{2}h_{xy} \\ \frac{1}{2}h_{xy} & 1 - \frac{1}{2}h_{xx} \end{bmatrix} \begin{bmatrix} \Delta x^x \\ \Delta x^y \end{bmatrix}.$$

It looks like we look the square root of the matrix in the middle, which is what we did to first order in h^{TT}. What we have done is construct an orthogonal basis from the metric. Let's define

$$\Delta\vec{x} = \begin{bmatrix} 1 + \frac{1}{2}h_{xx} & \frac{1}{2}h_{xy} \\ \frac{1}{2}h_{xy} & 1 - \frac{1}{2}h_{xx} \end{bmatrix} \begin{bmatrix} \Delta x^x \\ \Delta x^y \end{bmatrix}.$$

This is a regular three vector (two components of a three vector to be precise) and its length is given by $\sqrt{\Delta\vec{x} \cdot \Delta\vec{x}}$ using the usual dot product. We can now use our usual three dimensional reasoning to see what would happen to a bunch of masses arranged in a circle. We have assumed that the values of h^{TT} vary sinusoidally with an amplitude given by $A_{\mu\nu}$, and to start we will assume that $A_{xy} = 0$ and $A_{xx} > 0$. The circle will initially be a bit stretched in the $x-$direction and squished in the $y-$direction while preserving its area. A quarter of a period later it will be a circle again. At one-half period, it will be stretched in the $y-$direction and squished in the $x-$direction, and at three-quarters it will be a circle. Because the variation lies along the $x-$ and $y-$axes, we call this mode the $+$ (plus) polarization.

On the other hand if $A_{xx} = 0$ and $A_{xy} > 0$, we have the $\times$ (cross) polarization. At the start of the period the circle will be stretched toward 45 degrees above the $x-$axis, and at half period, the circle will be stretched toward 45 degrees below the $x-$axis. We can also given $A_{\mu\nu}$ a complex amplitude that will change the phase of the oscillation and also generate circular polarization, in which the squishing and stretching will be constant in amplitude but rotate with phase as shown in Fig. 5.2. In the case of the circular polarization, it appears that the ring itself is rotating, but this is not the case. Rather each mass executes traces out a small circle with a different phase which creates observed patterns in the ring (as shown by the small vanes attached to the masses in the figure).

Production

Gravitational radiation carries energy, and understanding the amount of energy carried helps to pinpoint the important production channels from gravitational radiation in the Universe. One cannot pinpoint the energy density of a gravitational wave to a particular point, but when averaged over several wavelengths or periods one obtains the mean energy-momentum tensor

$$T^{GW}_{\mu\nu} = \frac{c^4}{32\pi G} \langle h^{TT}_{ij,\mu} h^{TT}_{ij,\nu} \rangle$$

where h^{TT} denotes the transverse-traceless portion of $h_{\mu\nu}$ or $h_{\mu\nu}$ in its entirety in the transverse-traceless gauge. Let us express this in terms of the amplitude of the wave

$$T^{GW}_{\mu\nu} = \frac{c^4}{64\pi G} k_\mu k_\nu |A_{ij}|^2.$$

The units are included to give a sense of how much energy is carried even by a small amplitude wave. We will take GW170817 as an example with a frequency of 400 Hz and A_{ij} of about 10^{-21} which gives $T^{GW}_{\mu\nu}$ as an energy flux of about 0.2 W/m^2, larger by a factor of a million than any star upon the Earth except for the Sun.

To start we will compare the solution for the h^{TT} with the vector

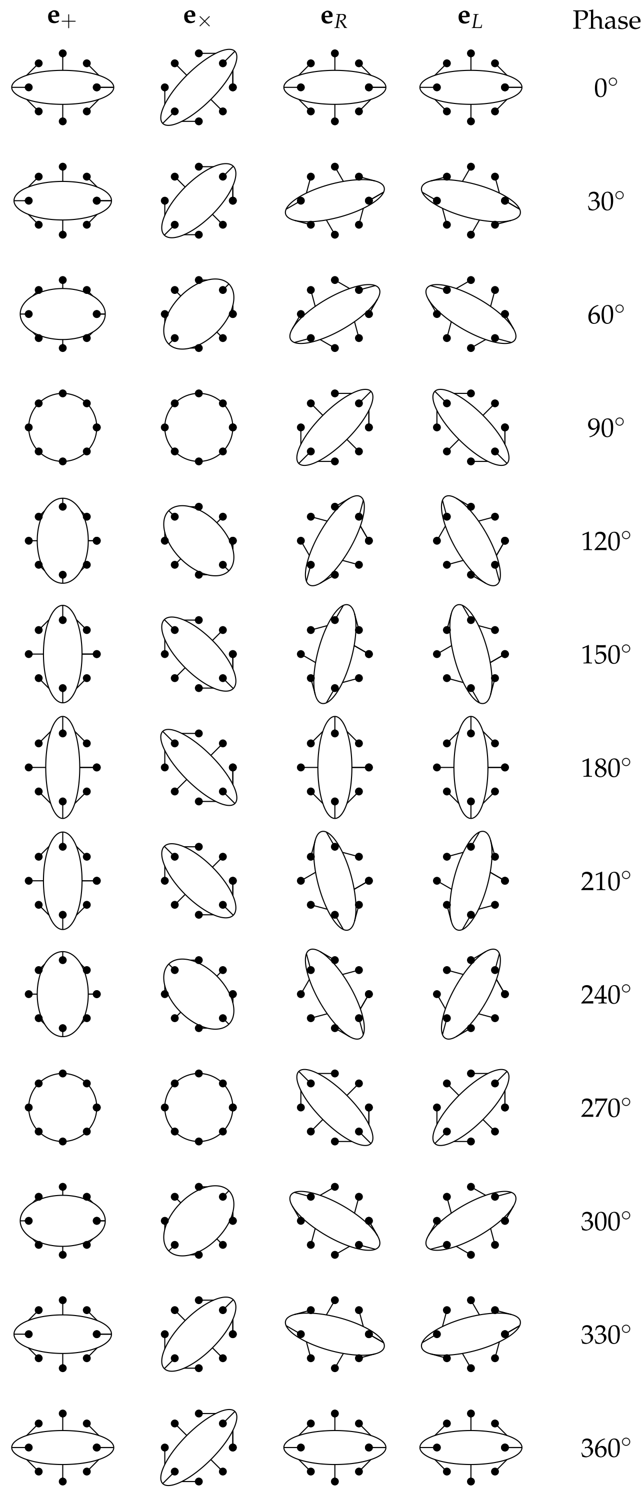


Figure 5.2: The figure depicts how a ring of masses will be distorted by a passing gravitional wave in the given polarization. The amplitude of each non-zero component of h^{TT} in each case is unity. The vanes depict the movement of each mass. In the case of linear polarization, each mass moves back and forth along a line of length unity. In the case of circular polarizatione each mass traces a circle of radius one half.

potential in electrodynamics. We have

$$h_{\mu\nu}^{TT}(\mathbf{r},t) = \frac{4G}{c^4}\int d^3r' \frac{T_{\mu\nu}^{TT}\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}$$

and

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int d^3r' \frac{\mathbf{J}\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}.$$

Because of the similarity in the two expressions, it is natural to proceed in analogy with electromagnetism.

In electromagnetism, electric dipole radiation is the dominant channel because monopole radiation is impossible because of conservation of charge. The power emitted is proportional to the square of the second derivative of the electric dipole with respect to time. If we define a mass dipole of a system of particles as

$$\mathbf{d} = \sum_i m_i \mathbf{x}_i.$$

The rate of change of the mass dipole is the total momentum of the system $\mathbf{P} = \sum_i m_i \mathbf{v}_i$. As the total momentum of the system is conserved, the second derivative of the mass dipole will vanish and so too does the analogue of electric dipole radiation in a gravitational radiation.

The next strongest term in the multipole expansion is typically the magnetic dipole. The magnetic dipole of a system proportional to the current and the area of the loop, so we have

$$\mu \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i = \mathbf{L}$$

where $\mathbf{L}$ is the total angular momentum of the system which is also conserved, so there is no magnetic dipole gravitational radiation.

If we look at electric quadrupole radiation we have for the total power

$$P_{\text{electric quadrupole}} = \frac{1}{20}\dddot{Q}_{ik}\dddot{Q}_{ik}$$

where

$$Q_{ik} \equiv \sum_A q_A \left(x_{Ai}x_{Ak} - \frac{1}{3}\delta_{ik}r_A^2\right).$$

We can define the mass quadrupole similarly as

$${\rlap{-}I}_{ik} \equiv \sum_A m_A \left(x_{Ai}x_{Ak} - \frac{1}{3}\delta_{ik}r_A^2\right)$$

which yields

$$P_{\text{gravitational quadrupole}} = \frac{1}{5}\frac{G}{c^5}\dddot{{\rlap{-}I}}_{ik}\dddot{{\rlap{-}I}}_{ik}.$$

To make things concrete let us calculate the mass quadrupole for two stars of mass m_1 and m_2 orbiting each other in a circular orbit of radius a with

$$G(m_1 + m_2) = \Omega^2 a^3$$

to get

$$\rlap{-}I_{xx} = \mu a^2 \left(\cos^2 \Omega t - \frac{1}{3} \right), \rlap{-}I_{yy} = \mu a^2 \left(\sin^2 \Omega t - \frac{1}{3} \right),$$

$$\rlap{-}I_{xy} = \mu a^2 \sin \Omega \cos \Omega t$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass. Notice that each term varies at an angular frequency of 2Ω, so the frequency of the gravitational wave will be twice that of the orbit and

$$P_{\text{gravitational quadrupole}} = \frac{32}{5} \frac{G}{c^5} \mu^2 a^4 \Omega^6 = \frac{32}{5} \frac{G}{c^5} \mu^2 \left(G(m_1 + m_2) \right)^{4/3} \Omega^{10/3}.$$

The energy that is emitted in gravitational radiation is lost from the orbit

$$E = -\frac{1}{2} \mu a^2 \Omega^2 = -\frac{1}{2} \mu \left(G(m_1 + m_2) \right)^{2/3} \Omega^{2/3}$$

and

$$\frac{dE}{dt} = -\frac{1}{3} \mu \left(G(m_1 + m_2) \right)^{2/3} \Omega^{-1/3} \frac{d\Omega}{dt}$$

so

$$\frac{d\Omega}{dt} = \frac{96}{5} \frac{G}{c^5} \mu \left(G(m_1 + m_2) \right)^{2/3} \Omega^{11/3} = \frac{96}{5} \left(\frac{G\mathcal{M}}{c^5} \right)^{5/3} \Omega^{11/3}$$

where $\mathcal{M}$ is the chirp mass

$$\mathcal{M} = \mu^{3/5} (m_1 + m_2)^{2/5} = \frac{m_1^{3/5} m_2^{3/5}}{(m_1 + m_2)^{1/5}}.$$

The orbital frequency evolves with time as

$$\Omega = \left[\frac{288}{15} (t_0 - t) \right]^{-3/8} \left(\frac{G\mathcal{M}}{c^5} \right)^{-5/8}.$$

Fig. 5.3 depicts the chirp frequency formula on the time dependent power spectrum of the binary neutron star merger GW 170817.

We have focussed on circular orbits, but the average power (averaged over a complete orbit) emitted by an elliptical orbit is given by

$$-\left\langle \frac{dE}{dt} \right\rangle = \frac{32 G^4 m_1^2 m_2^2 (m_1 + m_2)}{5 c^5 a^5 \left(1 - e^2\right)^{7/2}} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right)$$

where e is the orbital eccentricity and a is the semimajor axis of the elliptical orbit. The angular brackets on the left-hand side of the

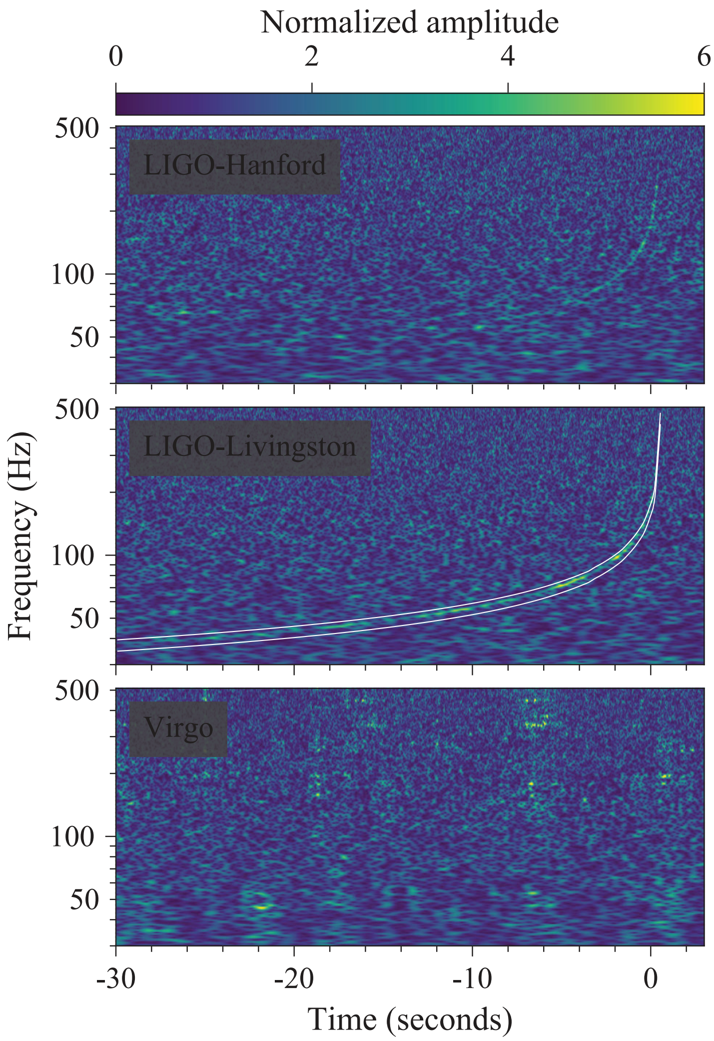


Figure 5.3: The normalized amplitude as a function of frequency and time of GW170817 with the chirp frequency formula superimposed on the middle panel.

equation represent the averaging over a single orbit. Similarly, the average rate of losing angular momentum equals

$$-\left\langle \frac{dL_z}{dt} \right\rangle = \frac{32G^{7/2}m_1^2m_2^2\sqrt{m_1+m_2}}{5c^5a^{7/2}\left(1-e^2\right)^2}\left(1+\frac{7}{8}e^2\right).$$

The rate of period decrease is given by

$$-\left\langle \frac{dP_b}{dt} \right\rangle = \frac{192\pi G^{5/3}m_1m_2(m_1+m_2)^{-1/3}}{5c^5\left(1-e^2\right)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)\left(\frac{P_b}{2\pi}\right)^{-5/3}$$

where P_b is the orbital period. The power increases dramatically with increasing eccentricity and the bulk of the emission occurs at pericenter when the accelerations are largest. Furthermore, the ratio of energy loss to angular momentum loss typically damps the eccentricity on the save timescale as the period decreases, so typically during the final moments of the inspiral (where LIGO is sensitive), the orbits are typically very close to circular — LISA is expected to measure elliptical orbits more often.

Detection

So far we have focussed on the effects of the gravitational radiation on the orbiting bodies. Gravitational wave detectors such as LISA and LIGO measure the strain h_{ij}^{TT} (or at least components of the strain) direct. We can also give a result for the strain. The formula reads

$$h_{ij}^{TT}(t,r) = \frac{2G}{c^4r}\ddot{I}_{ij}^{TT}(t-r/c).$$

Although the angular dependence is not explicit, the tensorial nature of the transverse-traceless gauge means that the transverse-traceless projection of $\ddot{I}_{ij}$ depends on where the strain is measured relative to the motion of the source. Operationally we measure the acceleration of the material in the plane of the sky and depending on the orientation of our detector relative to the location of the source on the sky, it will be more or less sensitive to the source. For a circular orbit the polarization will be circular if the orbit lies in the plane of the sky and as the inclination increases the polarization becomes elliptical and finally linear for edge-on orbits. In the edge-on case, the polarization will be + lined up with the position angle of the orbit. Ground-based interferometers such as LIGO and Virgo are most sensitive to sources directly above or directly below them, and measure the + polarization as aligned with the arms of the interferometer. For sources near the horizon the sensitivity depends on azimuth relative to the arms, and there are nulls for regions of the sky at 45 degrees relative to the arms. The two LIGO detectors and the Virgo detector are well separated on the Earth so the planes defined by their arms are different, yielding sensitivity throughout the sky for the network as a whole, and the capacity to determine the complete polarization state of the radiation in concert with each other. LISA on the other hand has a equilateral geometry so it senses both the $+$ and $\times$ polarizations equally, and can in principle determine the polarization state of the incoming radiation on its own. It also is more sensitive perpendicular to the plane, but does not exhibit nulls near the plane defined by the detectors. In principle the optimal configuration would a tetrahedron which would yield closer to uniform all-sky coverage.

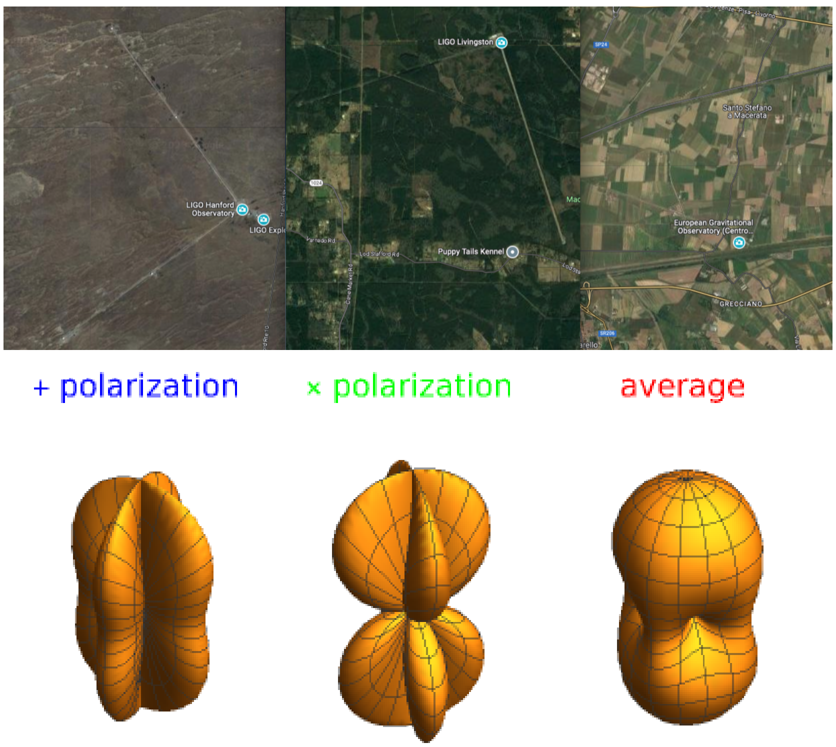


Figure 5.4: Upper: From West to East, The two LIGO detectors and the Virgo detector to scale with North toward the top. Lower: The antenna pattern of a two-arm interferometer. Here the polarization states are defined relative to the zenith.

Further Reading

To learn more about general relativity, consult

- Misner, C.W.,Thorne, K, and Wheeler, J.A., *Gravitation*.

Problems

1. **Circular Orbit** The equation for a geodesic (an orbit) is given by

$$\frac{du^\mu}{ds} + \Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta = 0$$

where u^μ is the four-velocity. When an index in an expression is repeated you are supposed to sum over the index. The indices run through t, r, θ and ϕ.

(a) Let's suppose that the particle at one moment is just going around the center of the black hole so the velocities in the r and θ directions vanish and we'll take $\theta = \pi/2$ (the equatorial plane).

In this situation u^t and u^ϕ are the only components of the four velocity that don't vanish and $\Gamma^r{}_{tt}$ and $\Gamma^r{}_{\phi\phi}$ are the only Christoffel symbols that don't have vanishing coefficients. Write out the geodesic equations in terms of the Christoffel symbols (don't calculate the Christoffel symbols).

(b) We would like for the velocity to be constant around the circular orbit so we would like the first term in the geodesic equation to vanish. Solve for $\Omega = u^\phi/u^t$ in terms of the Christoffel symbols.

(c) The two Christoffel symbols that play a role are

$$\Gamma^r{}_{tt} = \frac{(r-2M)M}{r^3} \text{ and } \Gamma^r{}_{\phi\phi} = (2M - r)\sin^2\theta.$$

What is Ω in terms of M and r?

(d) Substitute your value of Ω into the Schwarzschild metric and calculate ds^2 along the circular orbit. Over what range of radii can a material object (a toaster, UBC undergrad etc.) travel in a circular orbit around a Schwarzschild black hole.

2. **Photon Orbit**

We are going to find a radius at which a light will orbit a black hole.

(a) Start with the Schwarzschild metric. We want a circular orbit so we will set $dr = 0, d\theta = 0$ and $\theta = \pi/2$. What is ds^2 for a photon (a photon travels along a null geodesic)? Solve for $(d\phi/dt)^2$.

(b) $d\phi/dt$ is simply Ω for the photon orbit. Kepler's third law works in the Schwarzschild spacetime for circular orbits. Solve for R.

3. **Thermodynamics and General Relativity:**

In general relativity if two bodies are in thermodynamic equilibrium,

$$\frac{T_1}{1+z_1} = \frac{T_2}{1+z_2}$$

We can exploit this relationship along with Kirchoff's law to derive some interesting facts about how light travels from a neutron star to our telescopes.

(a) Because everything is in thermodynamic equilibrium, we can safely assume that the neutron star of mass M and radius R emits as a blackbody at a temperature T. Calculate the total power emitted from the neutron star surface in the frame of the neutron star surface.

(b) Calculate the total power received at infinity. Let the redshift of the surface be z.

(c) Let the space surrounding the star be filled with blackbody radiation in thermal equilibrium with the surface of the neutron star. You can imagine that the neutron star is in a gigantic thermos bottle. Let T_∞ be the temperature of this blackbody radiation measured at infinity (i.e. $z = 0$). What is T_∞?

(d) Now here comes Kirchoff's law: in thermodynamic equilibrium a body emits as much as it receives. How much power does the neutron star absorb from the blackbody at infinity? This is the product of the surface area of the neutron star with the flux per unit area of the blackbody radiation.

(e) A conundrum: compare the answer to (b) with the answer to (f). They differ. Does the neutron star cool down because (b) is greater than (d)?

(f) The neutron star can't cool down because it is already in equilibrium, so one of our assumptions must be wrong. It turns out that the most innocuous sounding assumption is incorrect. The power that the neutron star absorbs is the product of its apparent surface area with the flux per unit area of the blackbody radiation. Let the apparent radius be R_∞ and recalculate the answer to (d).

(g) Equate (b) and (e) and solve for R_∞.

$R_\infty \neq R$ because in the vicinity of a neutron star light does not travel in a straight line. One can also derive the value of R_∞ by solving for a null geodesic that is tangent to the surface of the neutron star. What is the minimum value of R_∞ for a constant value of M? You will need to know that

$$1 + z = \frac{1}{\sqrt{1 - \frac{2GM}{Rc^2}}}$$

What is the value of R? Call this radius R_γ.

(h) Prove the size of the image of the neutron star must decrease or remain the same as the radius of the neutron star decreases.

Use the fact that rays that we ultimately see remain outgoing throughout their journey to us (otherwise by symmetry they would hit the surface a second time).

(i) What happens to the size of the image of the star if the radius of the star is less that R_γ?

(j) The calculation of the apparent radius of the star from thermodynamics hinges on the assumption that the outgoing flux from the surface reaches infinity. For $R < R_\gamma$, the size of the image no longer increases while thermodynamics says it should, so we must conclude that for radii less than R_γ initially outgoing photons can become incoming photons.

From these arguments and spherical symmetry speculate what might happen to a photon emitted precisely at R_γ tangentially, i.e. neither ingoing or outgoing.

(k) Calculate how much radiation a star whose radius is less than R_γ will absorb.

The answer to (k) falls short of (b) again. We know that the star can't heat up, so an assumption must be wrong. Within R_γ not every photon emitted can escape to infinity, many photons return and hit the surface.

(l) Using the answers to (b) and (k), calculate the fraction of the outgoing photon flux emitted from the surface that manages to escape.

(m) Use the fact that a blackbody emits isotropically to determine the opening angle of the cone into which the escaping photons are emitted. This region is symmetric around the radial direction.

(n) Pat yourself on the back. You have derived many of the quirky things about the Schwarschild metric (the metric that surrounds a spherically symmetric mass distribution). List the key assumptions that you have made to make this derivation work.

Part II

Radiative Processes

6 Bremsstrahlung

Bremsstrahlung or braking radiation is the radiation that the charged particle emits while being accelerated in the electric field of another particle. Because the amount of radiation produced is proportional to the square of the acceleration, the less massive particle generally dominates the emission. Typically, we are talking about an electron and an ion, so the mass ratio is greater than 1,800 to one.

Why is it called bremsstrahlung? X-rays were first produced in the laboratory by accelerating electrons along a strong electric field (a typical potential difference of 10kV) from an anode to a cathode in vacuum. When the electrons hit the thick metal cathode and stop (brake), they emit cathode rays or X-rays.

The Physics of Bremsstrahlung

If we ignore the effect of radiation reaction of the trajectory of the charged particle, we can solve for its path exactly (at least in the classical limit) and then use the formulae for the radiation field that we derived a few weeks back. You can check out the Padmanabhan text if you would like to see an exact treatment.

Something to think about. What is the exact classical trajectory of the charged particle?

We will approximate the exact trajectories shown in the left-hand panel of Fig. 1 by a simple straight line trajectory in which the acceleration of the particle lies mainly normal to the direction of the particle's motion.

Something to think about. When does the straight line approximation fail?

We will also use the dipole approximation, so $\mathbf{d} = -e\mathbf{R}$ where $\mathbf{R}$ is the position of the particle. By taking the time derivative on both

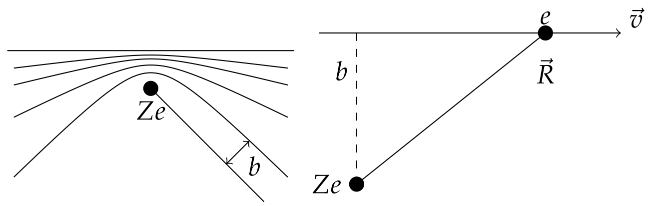


Figure 6.1: Bremsstrahlung. The left panel gives the exact trajectory excluding radiation reaction, and the right panel shows how we will approximate the trajectory (i.e. as a straight line such that the change in velocity is much smaller than the velocity).

sides twice we find,

$$\ddot{\mathbf{d}} = -e\dot{\mathbf{v}} \tag{6.1}$$

Ultimately we will be interested in the Fourier transform of the radiation field to understand the spectrum so we have,

$$-\omega^2 \hat{\mathbf{d}}(\omega) = -\frac{e}{2\pi}\int_{-\infty}^{\infty} \dot{\mathbf{v}} e^{i\omega t} dt. \tag{6.2}$$

Most of the acceleration of the particle occurs during a short time in which $R \sim b$. This *collision time* is approximately

$$\tau = \frac{b}{v} \tag{6.3}$$

so the bulk of the contribution to the integral happens for $-\tau \lesssim t \lesssim \tau$. If $\omega\tau \gg 1$, the integrand will oscillate rapidly so the integral will be small. On the other hand if $\omega\tau \ll 1$ the exponential is essentially unity so

$$\hat{\mathbf{d}}(\omega) \sim \begin{cases} \frac{e}{2\pi\omega^2}\Delta\mathbf{v}, & \omega\tau \ll 1 \\ 0, & \omega\tau \gg 1 \end{cases} \tag{6.4}$$

where $\Delta\mathbf{v}$ is the change of velocity during the collision.

The energy spectrum is given by

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}\left|\hat{d}(\omega)\right|^2 = \begin{cases} \frac{2e^2}{3\pi c^3}\left|\Delta\mathbf{v}\right|^2, & \omega\tau \ll 1 \\ 0, & \omega\tau \gg 1 \end{cases} \tag{6.5}$$

Let's estimate the value of Δv,

$$\Delta v = \int_{-\infty}^{\infty} \frac{b}{R}\frac{1}{m}\frac{Ze^2}{R^2} dt = \frac{Ze^2}{m}\int_{-\infty}^{\infty}\frac{b}{\left(b^2+v^2t^2\right)^{3/2}} dt = \frac{2Ze^2}{mbv} \tag{6.6}$$

so we can estimate the emission from a single collision

$$\frac{dW(b)}{d\omega} = \begin{cases} \frac{8Z^2e^6}{3\pi c^3 m^2 v^2 b^2}, & b \ll v/\omega \\ 0, & b \gg v/\omega \end{cases} \tag{6.7}$$

We would like to integrate over all impact parameters. We know that for a particular frequency, ω, the contribution to the spectrum vanishes for $b \gg v/\omega$. Let's assume that there is a mininum value of the impact parameter $b_{\min}$ below which our analysis fails. Looking at the left-hand panel of Fig. 1, you may be able to figure out when this is the case, so we have

$$\frac{dW}{d\omega dVdt} = n_e n_i 2\pi v \int_{b_{\min}}^{b_{\max}} \frac{8Z^2e^6}{3\pi c^3 m^2 v^2 b^2} b db \tag{6.8}$$

$$= \frac{16e^6}{3c^3m^2v} n_e n_i Z^2 \int_{b_{\min}}^{b_{\max}} \frac{db}{b} \tag{6.9}$$

$$= \frac{16e^6}{3c^3m^2v} n_e n_i Z^2 \ln\left(\frac{b_{\max}}{b_{\min}}\right) \tag{6.10}$$

We can see that our particular choice of the minimum and maximum impact parameters is not particularly important because they enter logarthimically. We can take

$$b_{\max} \equiv \frac{v}{\omega}. \tag{6.11}$$

There are two ways to get a value of $b_{\min}$. First is to estimate at what impact parameter does the trajectory strongly differ from a straight line, so $\Delta v \sim v$, we get

$$v \sim \Delta v(b_{\min}^{(1)}) = \frac{2Ze^2}{mb_{\min}^{(1)}v} \to b_{\min}^{(1)} \sim \frac{2Ze^2}{mv^2}. \tag{6.12}$$

We could have gotten this same value if we had compared the initial kinetic energy of the particle with its potential energy at point of closest approach. The standard value is slightly different

$$b_{\min}^{(1)} = \frac{4Ze^2}{\pi mv^2}. \tag{6.13}$$

The second estimate comes from our assumption that the path is classical. Typically over distances less that the de Broglie length of the electron one must treat the problem quantum mechanically,

$$b_{\min}^{(2)} = \frac{h}{mv} \tag{6.14}$$

$b_{\min}^{(1)} \approx b_{\min}^{(2)}$ for $mv^2/2 \sim Z^2(13.6\text{eV})$, *i.e.* when the kinetic energy of the particle is comparable to the binding energy of the ion.

Generally, the result for the bremsstrahlung spectrum is expressed as

$$\frac{dW}{d\omega dVdt} = \frac{16\pi e^6}{3\sqrt{3}c^3m^2v} n_e n_i Z^2 g_{ff}(v,\omega) \tag{6.15}$$

where the Gaunt factor

$$g_{ff}(v,\omega) = \frac{\sqrt{3}}{\pi} \ln\left(\frac{b_{\max}}{b_{\min}}\right) \tag{6.16}$$

shifts the uncertainities about the values of the minimum and maximum impact parameters into some function of order unity.

Thermal Bremsstrahlung Emission

The most important case astrophysically is thermal bremsstrahlung where the electrons have a thermal distribution so the probablility of a particle having a particular velocity is

$$dP \propto e^{-E/kT} d^3\mathbf{v} = \exp\left(\frac{-mv^2}{2kT}\right) d^2\mathbf{v} \tag{6.17}$$

We would like to integrate the emission over all the velocities of the electrons to get the total emission per unit volume,

$$\frac{dW(T,\omega)}{d\omega dV dt} = \frac{\int_{v_{\min}}^{\infty} \frac{dW(v,\omega)}{d\omega dV dt} v^2 e^{-\beta m v^2/2} dv}{\int_0^{\infty} v^2 e^{-\beta m v^2/2} dv} \tag{6.18}$$

If we look at the emission for a particular velocity, the emisision rate diverges as $v \to 0$, but the phase space vanishes faster; however, it is stll reasonable to cut off the integral at some minimum velocity. We know that radiation comes in bunches of energy $\hbar\omega$ so for a particular frequency $mv^2/2 > h\nu$ for the electron to have enough energy to emit a photon.

The integral in the numerator is straightforward (the one in the denominator is also possible) and we get,

$$\epsilon_\nu^{ff} \equiv \frac{dW}{dV dt d\nu} \tag{6.19}$$

$$= \frac{2^5 \pi e^6}{3mc^3}\left(\frac{2\pi}{3km}\right)^{1/2} T^{-1/2} Z^2 n_e n_i e^{-h\nu/kT} \bar{g}_{ff} \tag{6.20}$$

$$= 6.8 \times 10^{-38} Z^2 n_e n_i T^{-1/2} e^{-h\nu/kT} \bar{g}_{ff} \tag{6.21}$$

where everything is in c.g.s. units. $\bar{g}_{ff}$ is the thermally averaged Gaunt factor.

We can integrate ϵ_ν^{ff} over frequency to obtain,

$$\epsilon^{ff} = \frac{2^5 \pi e^6}{3hmc^3}\left(\frac{2\pi kT}{3m}\right)^{1/2} Z^2 n_e n_i \bar{g}_B \tag{6.22}$$

$$= 1.4 \times 10^{-27} Z^2 n_e n_i T^{1/2} \bar{g}_B \tag{6.23}$$

Use 1.2 or so for $\bar{g}_B$.

Thermal Bremsstrahlung Absorption

If we assume that the photon field is in thermal equilibrium with the electrons and ion we can obtain an expression for the corresponding absorption,

$$\frac{\epsilon_\nu^{ff}}{4\pi} = j_\nu^{ff} = \alpha_\nu^{ff} B_\nu(T) \tag{6.24}$$

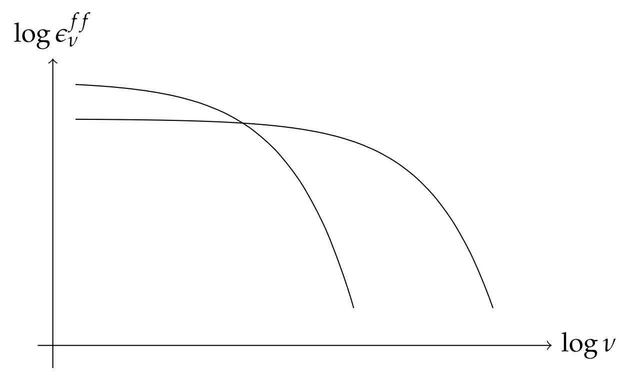


Figure 6.2: Thermal bremsstrahlung spectra for two temperatures that differ by a factor of ten

Something to think about. Why does this equation hold?

Using the form of the Planck function we obtain

$$\alpha_\nu^{ff} = \frac{4e^6}{3mhc}\left(\frac{2\pi}{3km}\right)^{1/2} T^{-1/2} Z^2 n_e n_i \nu^{-3}\left(1 - e^{-h\nu/kT}\right)\bar{g}_{ff} \tag{6.25}$$

For $h\nu \gg kT$ the exponential is negligible so $\alpha_\nu \propto \nu^{-3}$. For $h\nu \ll kT$ we have

$$\alpha_\nu^{ff} = \frac{4e^6}{3mkc}\left(\frac{2\pi}{3km}\right)^{1/2} T^{-3/2} Z^2 n_e n_i \nu^{-2}\bar{g}_{ff} \tag{6.26}$$

We can also integrate α_ν^{ff} over all photon energies to get the Rosseland mean absorption coefficient which is

$$\alpha_R^{ff} = 1.7 \times 10^{-25} T^{-7/2} Z^2 n_e n_i \bar{g}_R \tag{6.27}$$

Relativistic Bremsstrahlung

We are essentially going to redo the whole bremsstrahlung calculation in an entirely different way. This is called the method of virtual quanta, and it gives hints about how one does calculations in quantum field theory.

We spent a lot of time looking at the consequences of the electromagnetic fields of a moving particle, specifically the so-called acceleration field. Now we will focus on the velocity field,

$$\mathbf{E}(r,t) = q\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3 R^2}\right] \tag{6.28}$$

$$\mathbf{B}(r,t) = [\mathbf{n} \times \mathbf{E}(r,t)]. \tag{6.29}$$

where

$$\kappa = 1 - \mathbf{n}\cdot\boldsymbol{\beta} \tag{6.30}$$

If you remember, the brackets mean that the value inside is taken at the retarded time. Let's assume that the charged particle is moving

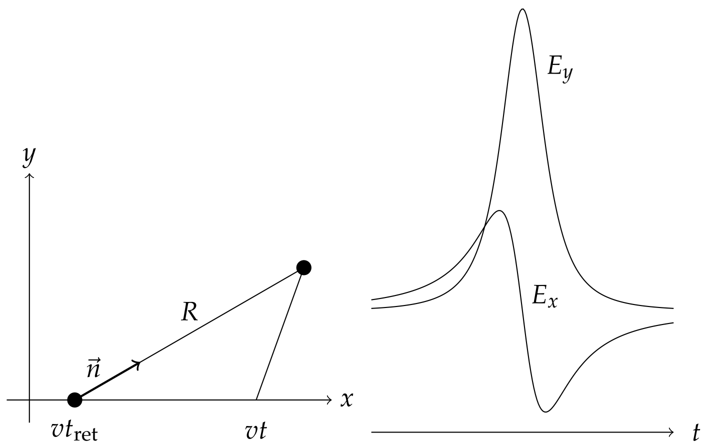


Figure 6.3: Geometry of a moving charge

along the $x-$axis at a constant velocity $\mathbf{v}$ and passed through the origin at $t = 0$. First, the retarded time for the particle is

$$t_{\rm ret} = t - \frac{R}{c} \tag{6.31}$$

and

$$\begin{aligned} R^2 &= y^2 + (x - vt_{\rm ret})^2 & (6.32) \\ &= y^2 + \left(x - vt + \frac{vR}{c}\right)^2 & (6.33) \\ 0 &= \left(\beta^2 - 1\right) R^2 + 2\beta \left(x - vt\right) R + \left(x - vt\right)^2 + y^2 & (6.34) \end{aligned}$$

so

$$R = \gamma^2 \beta (x - vt) + \gamma \left(y^2 + \gamma^2 (x - vt)^2\right)^{1/2} \tag{6.35}$$

We can also write the unit vector

$$\mathbf{n} = \frac{y\hat{\mathbf{y}} + (x - vt + vR/c)\hat{\mathbf{x}}}{R} \tag{6.36}$$

so

$$\begin{aligned} \mathbf{n} - \boldsymbol{\beta} &= \frac{y\hat{\mathbf{y}} + (x - vt + vR/c - vR/c)\hat{\mathbf{x}}}{R} & (6.37) \\ &= \frac{y\hat{\mathbf{y}} + (x - vt)\hat{\mathbf{x}}}{R}. & (6.38) \end{aligned}$$

Let's calculate

$$\begin{aligned} \kappa &= 1 - \mathbf{n} \cdot \boldsymbol{\beta} & (6.39) \\ &= 1 - \frac{v(x - vt + vR/c)}{Rc} = \frac{(1 - \beta^2)R - \beta(x - vt)}{R} & (6.40) \\ &= \frac{R - \gamma^2 \beta (x - vt)}{\gamma^2 R} = \frac{\left(y^2 + \gamma^2 (x - vt)^2\right)^{1/2}}{\gamma R} & (6.41) \end{aligned}$$

Let's get the components of the electric field

$$E_x = q(x-vt)(1-\beta^2)\frac{\gamma^3}{\left[y^2+\gamma^2(x-vt)^2\right]^{3/2}} \quad (6.42)$$

$$= \frac{q\gamma(x-vt)}{r^3} \quad (6.43)$$

$$E_y = \frac{q\gamma y}{r^3} \quad (6.44)$$

$$E_z = \frac{q\gamma z}{r^3} \quad (6.45)$$

where we get the z component and dependence by symmetry and

$$r^3 = \left[z^2+y^2+\gamma^2(x-vt)^2\right]^{3/2}. \quad (6.46)$$

Let's assume a second charged particle is located a distance b from the origin along the $y-$axis, it will experience an electric and magnetic field given by

$$E_x = -\frac{qv\gamma t}{(\gamma^2v^2t^2+b^2)^{3/2}} \quad B_x = 0 \quad (6.47)$$

$$E_y = \frac{q\gamma b}{(\gamma^2v^2t^2+b^2)^{3/2}} \quad B_y = 0 \quad (6.48)$$

$$E_z = 0 \quad B_z = \beta E_y \quad (6.49)$$

The electric field in the $y-$direction is typically larger by a factor of γ also the electric field in the $x-$direction changes direction so its effects are less.

Let's imagine that the charge is moving at nearly the speed of light, then the $y-$component of the field really dominates and the perpendicular magnetic field is nearly the same magnitude, so the field of the moving charge looks a lot like a transverse electromagnetic wave. We can imagine that the second charge Thomson scatters some of this "virtual" wave to form a real wave.

We need to calculate the Fourier transform of this virtual wave to get the spectrum of scattered radiation

$$\hat{E}_x(\omega) = \frac{1}{2\pi}\int E_x(t)e^{i\omega t}dt = -\frac{q\gamma v}{2\pi}\int_{-\infty}^{\infty} t\left(\gamma^2v^2t^2+b^2\right)^{-3/2} e^{i\omega t}dt \quad (6.50)$$

$$\hat{E}_y(\omega) = \frac{1}{2\pi}\int E_y(t)e^{i\omega t}dt = \frac{q\gamma b}{2\pi}\int_{-\infty}^{\infty}\left(\gamma^2v^2t^2+b^2\right)^{-3/2} e^{i\omega t}dt. \quad (6.51)$$

One can see that because $E_x = -vt/bE_y$ that

$$\hat{E}_x(\omega) = i\frac{v}{b}\frac{d}{d\omega}\hat{E}_y(\omega) \quad (6.52)$$

After the change of variable $x = \gamma vt/b$, these integrals can be expressed as modified Bessel functions

$$\hat{E}_x(\omega) = i\frac{q}{\pi\gamma bv}\left[\frac{\omega b}{\gamma v}K_0\left(\frac{\omega b}{\gamma v}\right)\right] \quad (6.53)$$

$$\hat{E}_y(\omega) = \frac{q}{\pi bv}\left[\frac{\omega b}{\gamma v}K_1\left(\frac{\omega b}{\gamma v}\right)\right]. \quad (6.54)$$

From Fig. 6.4 we see that the energy flux carried by the electric field in the $x-$direction is suppressed by a factor of γ relative to that in $y-$direction and that it peaks around $\omega = \gamma v/b$.

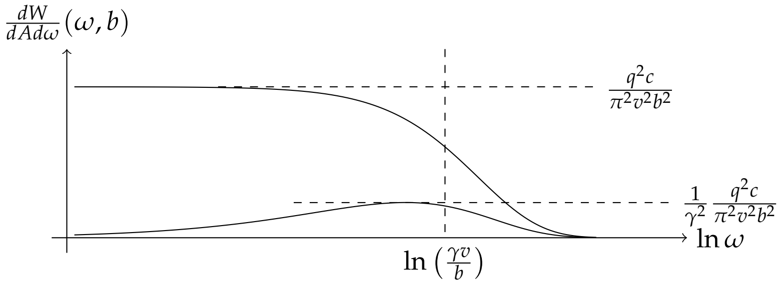


Figure 6.4: The frequency spectra for the electric field in the y-direction (upper) and $x-$direction for a fast moving charge

What remains is to integrate this spectrum over all possible impact parameters from $b_{\rm min}$ to infinity. Because the expressions in Eq. 6.53 and 6.54 cut off exponetially for large values of b, we do not need to consider a maximum impact parameter. This yields

$$\frac{dW}{dWd\omega}(\omega) = \frac{2}{\pi}\frac{q^2}{c}\left(\frac{c}{v}\right)^2\left[xK_0(x)K_1(x) - \frac{1}{2}\frac{v^2}{c^2}x^2\left(K_1^2(x) - K_0^2(x)\right)\right] \tag{6.55}$$

where $x = \omega b_{\rm min}/(\gamma v)$.

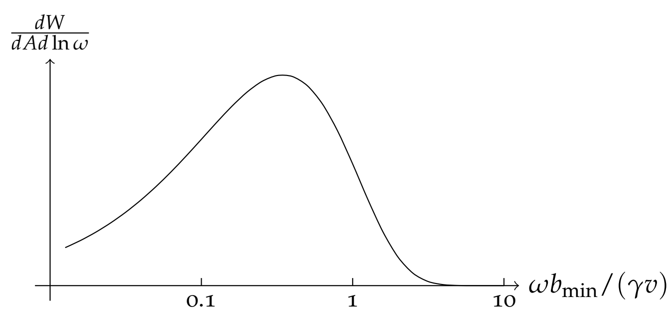


Figure 6.5: The frequency spectrum for the total electric field of a rapidly moving charge averaged over impact parameter.

We could also have used the same assumptions as in the non-relativistic case to perform the integral. The bulk of the contribution to the integral is for $\gamma v t \sim b$, so if $\omega \gg \gamma v/b$ we expect the integral to be really small, on the other hand if $\omega \ll \gamma v/b$ we have

$$\hat{E}(\omega) \approx \frac{1}{2\pi}\int E_y(t)e^{i\omega t}dt = \frac{q\gamma b}{2\pi}\int_{-\infty}^{\infty}\left(\gamma^2 v^2 t^2 + b^2\right)^{-3/2} dt = \frac{q}{vb\pi} \tag{6.56}$$

The energy flux carried by the virtual wave is

$$\frac{dW}{dAd\omega} = c|\hat{E}(\omega)|^2 = \begin{cases} \frac{q^2 c}{\pi^2 v^2 b^2}, & b \ll \gamma v/\omega \\ 0, & b \gg \gamma v/\omega \end{cases} \tag{6.57}$$

It's quite straightforward to calculate the flux of virtual radiation scattered by the electron,

$$\frac{dW}{d\omega} = \sigma_T \frac{\sigma(\omega)}{\sigma_T} \frac{dW}{dAd\omega} = \begin{cases} \frac{8\pi Z^2 e^6}{3v^2 m^2 c^3 b^2} \frac{\sigma(\omega)}{\sigma_T}, & b \ll \gamma v/\omega \\ 0, & b \gg \gamma v/\omega \end{cases} \tag{6.58}$$

which for $\gamma \rightarrow 1$ is *exactly* what we got before. The extra bit with $\sigma(\omega)/\sigma_T$ is to include the fact that the cross-section for electrons to scatter light differs from σ_T for photons with $\hbar\omega > mc^2$.

So bremsstrahlung comes down the Thomson scattering of the virtual photons of the electromagnetic field of an ion.

Further Reading

To learn more about bremsstrahlung and virtual quanta, consult Chapter 15 of

- Jackson, J. D., *Classical Electrodynamics*.

Problems

1. **Bremsstrahlung:**

   Consider a sphere of ionized hydrogen plamsa that is undergoing spherical gravitational collapse. The sphere is held at uniform temperature, T_0, uniform density and constant mass M_0 during the collapse and has decreasing radius R_0. The sphere cools by emission of bremsstrahlung radiation in its interior. At $t = t_0$ the sphere is optically thin.

   (a) What is the total luminosity of the sphere as a function of $M_0, R(t)$ and T_0 while the sphere is optically thin?

   (b) What is the luminosity of the sphere as a function of time after it becomes optically thick in terms of $M_0, R(t)$ and T_0?

   (c) Give an implicit relation in terms of $R(t)$ for the time t_1 when the sphere becomes optically thick.

   (d) Draw a curve of the luminosity as a function of time.

7
Synchrotron Radiation

Syhchrotron radiation, *a.k.a.* magnetic bremmstrahlung, is produced by relativistic charged particles travelling through a magnetic field.

Motion in a magnetic field

The Lorentz force equation relates the rate of change of the four-momentum to the electric and magnetic field,

$$\frac{dp^\mu}{d\tau} = \frac{q}{c} F^{\mu\nu} U_\nu \tag{7.1}$$

If the electric field vanishes, we get the following two equations

$$\frac{d}{dt}(\gamma m \mathbf{v}) = \frac{q}{c}\mathbf{v} \times \mathbf{B} \text{ and } \frac{d}{dt}(\gamma m c^2) = 0 \tag{7.2}$$

The second equation tells us the that the magnitude of the velocity does not change. The first equation tells us that the magnitude of the velocity ($v_\parallel$) along the field **B** is also constant. Because both the magnitude of the velocity and the parallel component are constant, we find that the magnitude of the perpendicular component is constant too, so we find

$$\frac{d\mathbf{v}_\perp}{dt} = \frac{q}{\gamma m c}\mathbf{v}_\perp \times \mathbf{B} \tag{7.3}$$

and the particle gyrates around the magnetic field with a frequency

$$\omega_B = \frac{qB}{\gamma m c}. \tag{7.4}$$

The acceleration ($\omega_B v_\perp$) is perpendicular to the motion of the particle so we can use formula (44) from Unit 3 to get the total power,

$$P = \frac{2}{3}\frac{q^2 \dot{u}^2}{c^3}\gamma^4 = \frac{2q^2}{3c^3}\gamma^2 \frac{q^2 B^2}{m^2 c^2} v_\perp^2 = \frac{2}{3} r_0^2 c \beta_\perp^2 \gamma^2 B^2 \tag{7.5}$$

Let's assume that the particles have a random distribution of velocities relative to the direction of the magnetic field, so we need the mean value of

$$\left\langle \beta_\perp^2 \right\rangle = \frac{\beta^2}{4\pi} \int \sin^2 \alpha d\Omega = \frac{2\beta^2}{3} \tag{7.6}$$

where α is the pitch angle.

Spectrum of Synchrotron radiation

If the electron is non-relativistic its dipole moment varies as $e^{i\omega_B t}$ so we would expect radiation at a single frequency ω_B. The relativistic case is somewhat more complicated. The electron still travels in the circular with a particular frequency but the electric field essentially vanishes except for a small region $\Delta\theta \sim 1/\gamma$. near the direction of the electron's motion (remember relativistic beaming).

We know that the electric field vanishes everywhere except within a cone of opening angle $1/\gamma$, so a distance observer will only detect a significant electric field while the electron is within an angle $\Delta\theta/2 \sim 1/\gamma$ of the point where the path is tangent to the line of sight. How long does it take the particle to pass through this angle?

Using the equation of motion we have

$$\gamma m \frac{\Delta \mathbf{v}}{\Delta t} = \frac{q}{c} \mathbf{v} \times \mathbf{B} \tag{7.7}$$

Let's use $\Delta \mathbf{v} = v\Delta\theta = 2v/\gamma$ to get

$$\gamma m \frac{2v}{\gamma \Delta t} = \frac{q}{c} v \sin \alpha B \tag{7.8}$$

$$\Delta t = \frac{2mc}{qB \sin \alpha} = \frac{2}{\gamma \omega_B \sin \alpha} \tag{7.9}$$

We also need to calculate how long between when the radiation emitted at t and $t + \Delta t$ arrives at the distant observer. The difference between the observed times is less than Δt by $v\Delta t/c$ so we get

$$\Delta t^A = \frac{2}{\gamma \omega_B \sin \alpha} \left(1 - \frac{v}{c}\right) \tag{7.10}$$

In the ultrarelativistic limit, $1 - v/c \approx 1/(2\gamma^2)$ so we have

$$\Delta t^A \approx \left(\gamma^3 \omega_B \sin \alpha\right)^{-1}. \tag{7.11}$$

The distant observer will see zero electric field most of the time with blips of electric field lasting for a time Δt^A every time the electron loops $2\pi/\omega_B$. Let's define a critical frequency,

$$\omega_c \equiv \frac{3}{2} \gamma^3 \omega_B \sin \alpha. \tag{7.12}$$

We expect that the spectrum will cut off at frequencies similar to ω_c.

Qualitative Spectrum

We earlier found that for a relativistic particle the intensity of the radiation field depended almost entirely on the combination $\gamma\theta$ where θ is the angle between the line of sight and the direction of the particle's motion, so

$$E(t) = F(\gamma\theta) \tag{7.13}$$

If we take t to be the time measured in the observer's frame after $\theta = 0$ we find that

$$\gamma\theta \approx 2\gamma\left(\gamma^2\omega_B \sin\alpha\right) t \propto \omega_c t \tag{7.14}$$

from equation (11), so we find that

$$E(t) = g(\omega_c t). \tag{7.15}$$

To find the spectrum we are interested in Fourier transform of $E(t)$,

$$\hat{E}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega_c t)e^{i\omega t}dt. = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi)e^{i\xi\omega/\omega_c}\frac{d\xi}{\omega_c}. = h(\omega/\omega_c) \tag{7.16}$$

so the average power per unit frequency is a function of ω/ω_c,

$$\frac{dW}{dtd\omega} = T^{-1}\frac{dW}{d\omega} \equiv P(\omega) = C_1 F\left(\frac{\omega}{\omega_c}\right). \tag{7.17}$$

We already know from equation (5) what the total power emitted by the charged particle so we have

$$P = \frac{2}{3}r_0^2 c\beta_\perp^2\gamma^2 B^2 = C_1\int_0^\infty F\left(\frac{\omega}{\omega_c}\right)d\omega = \omega_c C_1\int_0^\infty F(x)dx \tag{7.18}$$

We would like the function $F(x)$ to be dimensionless which sets the value of C_1 up to a dimensionless number. If we take $\beta \approx 1$ we obtain

$$P(\omega) = \frac{\sqrt{3}}{2\pi}\frac{q^3 B\sin\alpha}{mc^2}F\left(\frac{\omega}{\omega_c}\right) \tag{7.19}$$

where

$$\omega_c = \frac{3\gamma^2 qB\sin\alpha}{2mc} \tag{7.20}$$

Spectral Index for a Power-Law Distribution of Particle Energies

Even before calculating the form of $F(\omega/\omega_c)$, we can determine some interesting properties of the radiation spectrum. We found from the homework that a shock often gives the electrons that bounce across it a power-law distribution of energies, such that

$$N(E)dE = CE^{-p}dE \text{ or } N(\gamma)d\gamma = C\gamma^{-p}d\gamma \tag{7.21}$$

over a particular range of particle energy. Let's use formula (19) to calculate the total spectrum from these particles,

$$P_{\rm tot}(\omega) = C\int_{\gamma_1}^{\gamma_2} P(\omega)\gamma^{-p} d\gamma \propto \int_{\gamma_1}^{\gamma_2} F\left(\frac{\omega}{\omega_c}\right)\gamma^{-p} d\gamma. \qquad (7.22)$$

Let's change variables to $x \equiv \omega/\omega_c$. Remember that $\omega_c = A\gamma^2$ so $\gamma^2 \propto \omega/x$, we get

$$P_{\rm tot}(\omega) \propto \omega^{-(p-1)/2} \int_{x_1}^{x_2} F(x) x^{(p-3)/2} dx \qquad (7.23)$$

If the range of the power-law distribution is sufficiently large (at least an order of magnitude) we can take $x_1 \rightarrow 0$ and $x_2 \rightarrow \infty$ in (23) so that the integral is simply a constant and we find that the spectral distribution is also a power-law ω^{-s} with a power-law index of $s = (p-1)/2$.

This power-law spectrum is valid essentially between $\omega_c(\gamma_1)$ and $\omega_c(\gamma_2)$. To understand the spectrum for frequencies outside this range and other details as well we must calculate the function $F(x)$.

Spectrum and Polarization of Synchrotron emission — Details

The spectrum of the observed radiation will depend on the Fourier transform with respect to the observed time of the electric field. The radiation field is given by the expression

$$\mathbf{E}(r,t) = \frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R} \times \left[(\mathbf{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]\right] \qquad (7.24)$$

In chapter three we manipulated this expression under the assumption that we were really far from the particle so the value of R doesn't change much as the particle moves to get

$$\frac{dW}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) \exp\left[i\omega\left(t' - \mathbf{n}\cdot\mathbf{r}_0(t')/c\right)\right] dt'\right|^2 \qquad (7.25)$$

With no loss of generality we can assume that the particle gyrates in the $x-y-$plane and our line of sight is in the $x-z-$plane and makes an angle with the $x-$axis. With this geometric setup we can calculate

$$\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) = -\boldsymbol{\epsilon}_\perp \sin\left(\frac{vt'}{a}\right) + \boldsymbol{\epsilon}_\parallel \cos\left(\frac{vt'}{a}\right)\sin\theta \qquad (7.26)$$

The term in the exponential is the observed time in terms of the retarded time,

$$t' - \frac{\mathbf{n}\cdot\mathbf{r}(t')}{c} = t' - \frac{a}{c}\cos\theta\sin\left(\frac{vt'}{a}\right) \qquad (7.27)$$

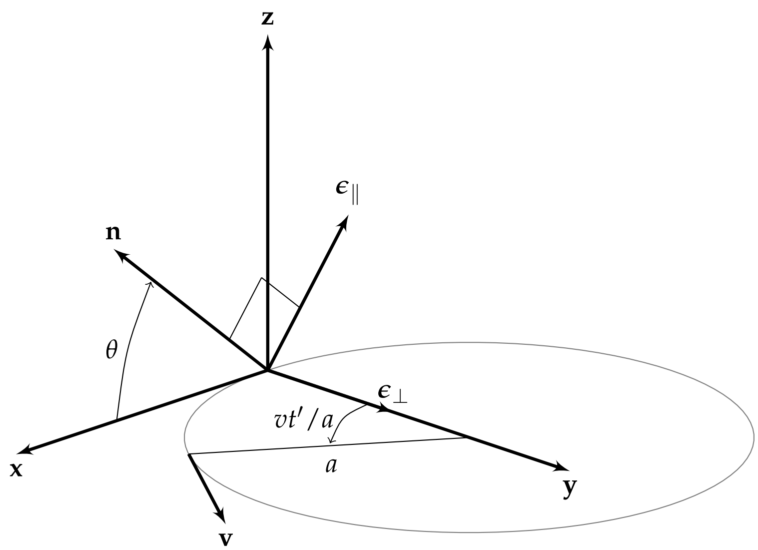


Figure 7.1: Geometry of synchrotron emission

$$\approx \quad t' - \frac{a}{c}\left(1-\frac{\theta^2}{2}\right)\left[\frac{vt'}{a} - \frac{1}{6}\left(\frac{vt'}{a}\right)^3\right] \qquad (7.28)$$

$$\approx \quad t' - \beta t' + \frac{\theta^2}{2}\beta t' + \frac{c^2\beta^3 t'^3}{6a^2} \qquad (7.29)$$

$$\approx \quad \left(2\gamma^2\right)^{-1}\left[\left(1+\gamma^2\theta^2\right)t' + \frac{c^2\gamma^2 t'^3}{3a^2}\right] \qquad (7.30)$$

To get the final equation we took $1-\beta \rightarrow (2\gamma)^{-1}$ and $\beta \rightarrow 1$. Substituting these results into equation (25) and taking the small-angle approximation for the polarization vectors as well yield

$$\frac{dW}{d\omega d\Omega} \equiv \frac{dW_\parallel}{d\omega d\Omega} + \frac{dW_\perp}{d\omega d\Omega} \qquad (7.31)$$

$$\frac{dW_\perp}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int \frac{ct'}{a}\exp\left[\frac{i\omega}{2\gamma^2}\left(\theta_\gamma^2 t' + \frac{c^2\gamma^2 t'^3}{3a^2}\right)\right]dt'\right|^2 \qquad (7.32)$$

$$\frac{dW_\parallel}{d\omega d\Omega} = \frac{q^2\omega^2\theta^2}{4\pi^2 c}\left|\int \exp\left[\frac{i\omega}{2\gamma^2}\left(\theta_\gamma^2 t' + \frac{c^2\gamma^2 t'^3}{3a^2}\right)\right]dt'\right|^2 \qquad (7.33)$$

where $\theta_\gamma^2 = 1+\gamma^2\theta^2$.

Let's make the following change of variables,

$$y \equiv \gamma\frac{ct'}{a\theta_\gamma} \text{ and } \eta \equiv \frac{\omega a\theta_\gamma^3}{3c\gamma^3} \approx \frac{\omega}{2\omega_c} \qquad (7.34)$$

to yield

$$\frac{dW_\perp}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left(\frac{a\theta_\gamma^2}{\gamma^2 c}\right)^2\left|\int_{-\infty}^{\infty} y\exp\left[\frac{3}{2}i\eta\left(y+\frac{1}{3}y^3\right)\right]dt'\right|^2 \qquad (7.35)$$

$$\frac{dW_\parallel}{d\omega d\Omega} = \frac{q^2\omega^2\theta^2}{4\pi^2 c}\left(\frac{a\theta_\gamma}{\gamma c}\right)^2\left|\int_{-\infty}^{\infty} y\exp\left[\frac{3}{2}i\eta\left(y+\frac{1}{3}y^3\right)\right]dt'\right|^2 \qquad (7.36)$$

(7.37)

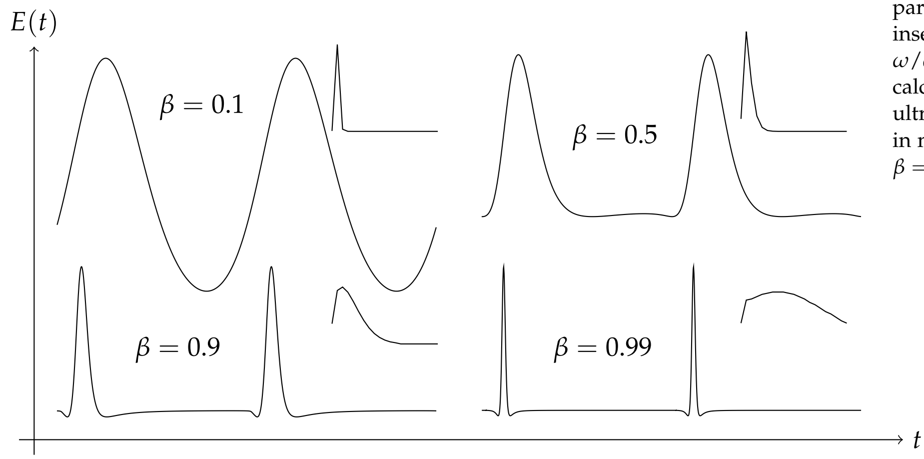


Figure 7.2: Electric field from a gyrating particle $\beta = 0.1, 0.5, 0.9$ and 0.99. The insets show the power spectrum with ω/ω_c along the $x-$axis. These were calculated without the small-angle or ultrarelativistic approximations. Keep in mind a particle with $\gamma = 10^3$ has $\beta = 0.9999995$.

It turns out that these integrals can be performed with some special functions called Airy integrals or modified Bessel functions to yield,

$$\frac{dW_\perp}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left(\frac{a\theta_\gamma^2}{\gamma^2 c}\right)^2 K_{\frac{2}{3}}^2(\eta) \tag{7.38}$$

$$\frac{dW_\parallel}{d\omega d\Omega} = \frac{q^2\omega^2\theta^2}{4\pi^2 c}\left(\frac{a\theta_\gamma}{\gamma c}\right)^2 K_{\frac{1}{3}}^2(\eta) \tag{7.39}$$

We would like to find the total emission per orbit over all solid angles. The emission lies within an angle $1/\gamma$ of a cone of half-angle α centered on the magnetic field, so we can take the element of solid angle $d\Omega = 2\sin\alpha d\theta$ and integrate,

$$\frac{dW_\perp}{d\omega} = \frac{2q^2\omega^2 a^2 \sin\alpha}{3\pi c^3\gamma^4}\int_{-\infty}^{\infty}\theta_\gamma^4 K_{\frac{2}{3}}^2(\eta)d\theta \tag{7.40}$$

$$\frac{dW_\parallel}{d\omega} = \frac{2q^2\omega^2 a^2 \sin\alpha}{3\pi c^3\gamma^4}\int_{-\infty}^{\infty}\theta_\gamma^2\theta^2 K_{\frac{1}{3}}^2(\eta)d\theta \tag{7.41}$$

These integrals can be written in terms of Bessel functions to yield,

$$\frac{dW_\perp}{d\omega} = \frac{\sqrt{3}q^2\gamma\sin\alpha}{2c}\left[F(x)+G(x)\right] \tag{7.42}$$

$$\frac{dW_\parallel}{d\omega} = \frac{\sqrt{3}q^2\gamma\sin\alpha}{2c}\left[F(x)-G(x)\right] \tag{7.43}$$

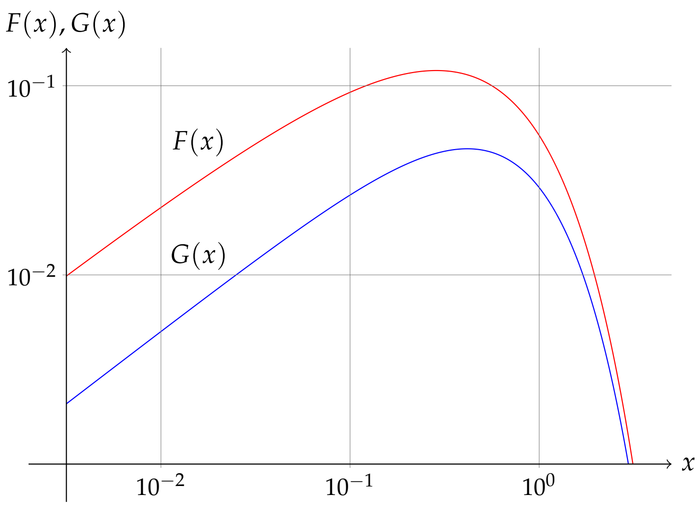
where

$$F(x) = x\int_x^\infty K_{\frac{5}{3}}(\xi)d\xi, \qquad G(x) = xK_{\frac{2}{3}}(x) \tag{7.44}$$

and x = ω/ω_c .

Something to think about. Why could we take the limits of integration in equations (35) and (36) and (39) and (40) to be infinite?

Figure 7.3: Synchrotron Functions



To convert these values in a power per frequency we have to divide by the orbital period of the charge $T = 2\pi/\omega_B$ to give

$$P_\perp = \frac{\sqrt{3}q^3 B \sin\alpha}{4\pi mc^2}\left[F(x) + G(x)\right] \tag{7.45}$$

$$P_\parallel = \frac{\sqrt{3}q^3 B \sin\alpha}{4\pi mc^2}\left[F(x) - G(x)\right] \tag{7.46}$$

The total power is proportional to $F(x)$ that has the following asymptotic values,

$$F(x) \sim \frac{4\pi}{\sqrt{3}\Gamma\left(\frac{1}{3}\right)}\left(\frac{x}{2}\right)^{1/3}, \quad x \ll 1 \tag{7.47}$$

$$F(x) \sim \left(\frac{\pi}{2}\right)^{1/2} e^{-x} x^{1/2}, \quad x \gg 1 \tag{7.48}$$

Synchrotron Absorption

We are particularly interested in the form of the spectrum from a power-law distribution of particles for frequencies where the region is optically thick. We know from the formal soltuion of radiative transfer that the spectrum approaches the source function at large optical depth. Furthermore Eq. 1.86 yields a relationship between the number of particles of various energies and the volume of phase space at each energy

$$S_\nu = \frac{2h\nu^3}{c^2}\left(\frac{g_2 n_1}{g_1 n_2} - 1\right)^{-1}. \tag{7.49}$$

For free ultrarelativistic particles the density of states is proportional to E^2, and we have assumed that their number is proportional to E^{-p}. Let us imagine that synchrotron emission is a transition between the upper state at an energy $E + h\nu$ and the lower state at energy E, so

$$\frac{g_2 n_1}{g_1 n_2} - 1 = \frac{(E+h\nu)^2 E^{-p}}{E^2 (E+h\nu)^{-p}} - 1 = \left(1 + \frac{h\nu}{E}\right)^{p+2} - 1 \approx (p+2)\frac{h\nu}{E} \tag{7.50}$$

where the final result obtains in the classical limit where $h\nu \ll E$, so we have

$$S_\nu \approx \frac{2h\nu^3}{c^2}\frac{E}{(p+2)\,h\nu} \approx \frac{2}{(p+2)}\left(\frac{4\pi}{3}\frac{m^3 c}{qB\sin\alpha}\right)^{1/2}\nu^{5/2} \tag{7.51}$$

where we have related the energy of the state (E) to the frequency through Eq. 7.12. Notice that the spectral index does not depend on the power-law index of the particle distribution but rather results from the power-law relationship between particle energy and frequency. Because the optically thin emission spectrum increases more slowly with frequency than the source function (or even decreases), we expect synchrotron absorption to be important at low frequencies where the the integrated optically thin emission exceeds the source function.

A Complete Synchrotron Spectrum

The complete spectrum from synchrotron radiation must account for the evolution of the electron energies, absorption, the minimum electron energy and the age of the source. We have all the necessary ingredients to figure this out. First, § 7 calculates the shape of the photon spectrum for a given power-law distribution of electron energies. The second ingredient is to calculate the distribution of electron energies as a function of the time since the electrons were accelerated (or *injected*). Let us assume that the electrons initially have a power-law distribution of energies

$$\frac{dN}{dtd\gamma_0} = C\gamma_0^{-p} \text{ for } \gamma_0 > \gamma_m. \tag{7.52}$$

Furthermore the electron energies decrease with time according to

$$\gamma = \frac{\gamma_0}{1 + A\gamma_0 t'}, A = \frac{2q^4 B_\perp^2}{3m^3c^5} \tag{7.53}$$

where t' is the time since the particles were accelerated By inverting this equation we obtain the initial electron energy in terms of the final electron energy

$$\gamma_0 = \frac{\gamma}{1 - A\gamma t'} \tag{7.54}$$

and

$$\frac{d\gamma_0}{d\gamma} = \frac{1}{(1 - A\gamma t')^2} \tag{7.55}$$

so

$$\frac{dN}{d\gamma dt} = C\gamma^{-p}\left(1 - A\gamma t'\right)^{p-2}. \tag{7.56}$$

It is crucial to understand the validity of this distribution. Clearly we must have

$$\frac{\gamma_m}{1 + A\gamma_m t'} < \gamma < \frac{1}{At'}. \tag{7.57}$$

Otherwise the number of electrons vanishes because either they have not yet had time to cool to such a low energy or they have already cooled below this energy. The original power-law distribution is truncated at high energies and is extended below γ_m by cooling.

If the source were only active instantaneously or for a short time long ago $\Delta t \ll t'$, this would be sufficient, but if we are interested in a source that has been active continually from some time ago we must intergrate this distribution over the injection times in the past. The resulting distribution is

$$\frac{dN}{d\gamma} = C\gamma^{-p} \int_{t_0}^{t_1} \left(1 - A\gamma t'\right)^{p-2} dt' = C\gamma^{-p} \left. \frac{(1 - A\gamma t')^{p-1}}{(p-1)(-A\gamma)} \right|_{t_0}^{t_1} \tag{7.58}$$

where the conditions in Eq. 7.57 determine the values of t_0 and t_1. We have

$$t_0 = \max\left[0, \frac{1}{A}\left(\gamma^{-1} - \gamma_m^{-1}\right)\right] \text{ and } t_1 = \min\left(t, \frac{1}{A\gamma}\right). \tag{7.59}$$

The latter expression in Eq. 7.59 encourages us to define

$$\gamma_c = \frac{1}{At}. \tag{7.60}$$

We will use this quantity to eliminate A from the equations. The energy divides the electrons into those that have had a chance to cool significantly since the source turned on a time t ago and those that have not cooled significantly. There are two possibilities for the resulting distribution depending on whether γ_c exceeds γ_m. We will examine the distribution for $\gamma_c > \gamma_m$ first; this is known as the *slow cooling* regime. There are three possibly ranges for γ. First at the highest energies we have

$$\frac{dN}{d\gamma} = \frac{Ct\gamma_c}{p-1}\gamma^{-(p+1)} \text{ for } \gamma_m < \gamma_c < \gamma. \tag{7.61}$$

For intermediate energies we have

$$\frac{dN}{d\gamma} = \frac{Ct\gamma_c}{p-1}\gamma^{-(p+1)} \left[1 - \left(1 - \frac{\gamma}{\gamma_c}\right)^{p-1}\right] \text{ for } \gamma_m < \gamma < \gamma_c \tag{7.62}$$

For $\gamma \ll \gamma_c$ this reduces to

$$\frac{dN}{d\gamma} \approx Ct\gamma^{-p} \text{ for } \gamma_m < \gamma \ll \gamma_c. \tag{7.63}$$

For the lowest energies the following expression obtains

$$\frac{dN}{d\gamma} = \frac{Ct\gamma_c}{p-1}\gamma^{-(p+1)} \left[\left(\frac{\gamma}{\gamma_m}\right)^{p-1} - \left(1 - \frac{\gamma}{\gamma_c}\right)^{p-1}\right] \text{ for } \gamma < \gamma_m < \gamma_c. \tag{7.64}$$

The other cooling regmine is called *fast cooling* where $\gamma_c < \gamma_m$. Is this regime for the largest energies $\gamma > \gamma_m > \gamma_c$, Eq. 7.61 also holds. For intermediate energies we have

$$\frac{dN}{d\gamma} = \frac{Ct\gamma_c}{(p-1)\gamma_m^{p-1}}\gamma^{-2} \text{ for } \gamma_c < \gamma < \gamma_m. \tag{7.65}$$

and for the smallest energies $\gamma < \gamma_c < \gamma_m$ Eq. 7.64 again obtains.

Both of the distributions either for slow or fast cooling vanish for

$$\gamma < \gamma_{\text{cut-off}} = \left(\frac{1}{\gamma_m} + \frac{1}{\gamma_c}\right)^{-1}. \tag{7.66}$$

Well into the slow cooling regime we have $\gamma_m \ll \gamma_c$ so $\gamma_{\text{cut-off}} \approx \gamma_m$. Well into the fast cooling regime we have $\gamma_{\text{cut-off}} \approx \gamma_c$. To find the distribution of photon energies we have to convolve the electron distribution with the function $F(x)$. For $\omega > \omega_{\max}$ where

$$\omega_{\max} = \frac{3\gamma^2_{\text{cut-off}} qB \sin\alpha}{2mc} \tag{7.67}$$

the results of § 7 apply. On the other hand below this frequency, the radiation results from the low-frequency limit of the function $F(x)$ (Eq. 7.47), i.e. $F_\omega \propto \omega^{1/3}$. A second complication is the role of synchrotron absorption outlined in § 7. We can combine the various results from this section to derive a schematic of the emission spectrum from a synchrotron cooling population of electrons with constant particle injection. Figs. 7.4 and 7.5 depict the spectrum for slow and fast cooling.

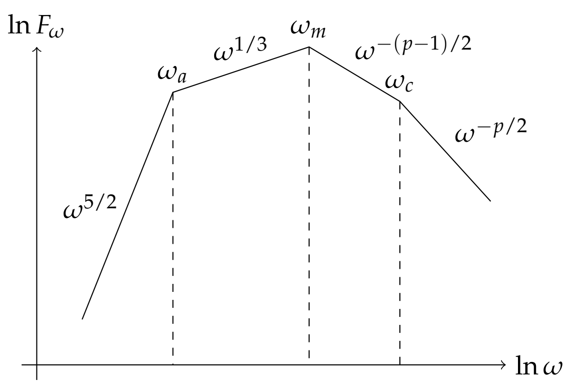


Figure 7.4: Complete synchrotron spectrum for an age less than the maximum cooling time (slow cooling).

To learn more about the frequency spectrum of synchrotron emission from mono-energetic electrons, consult Chapter §14.6 of

- Jackson, J. D., *Classical Electrodynamics*.

Problems

1. **Synchrotron Radiation:**

   An ultrarelativistic electron emits synchrotron radiation. Show that its energy decreases with time according to

$$\gamma = \gamma_0 \left(1 + A\gamma_0 t\right)^{-1}, A = \frac{2e^4 B_\perp^2}{3m^3c^5}. \tag{7.68}$$

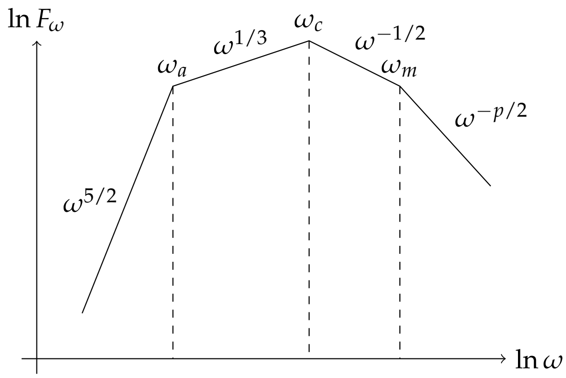


Figure 7.5: Complete synchrotron spectrum for an age greater than the maximum cooling time (fast cooling).

Here γ_0 is the initial value of γ and $B_\perp = B \sin \alpha$. Show that the time for the electron to lose half its energy is

$$t_{1/2} = (A\gamma_0)^{-1} \tag{7.69}$$

How do you reconcile the decrease of γ with the result of constant γ for motion in a magnetic field?

2. **Synchrotron Cooling More Precisely:**

   Derive the evolution of the energy of the electron (or γ) evolves in time without making the ultrarelativistic approximation.

3. **Power-Law Distribution More Precisely:**

   Calculate the photon spectrum for a power-law distribution of electron energies as in § 7 including the normalization and polarization.

8
Compton Scattering

When we looked at the scattering of light by electrons we assumed that the energy of photon was not changed by the scattering and that the electron was not relativistic. Compton scattering involves dropping these two assumptions.

The Kinematics of Photon Scattering

We assumed that the light carries only energy but it also carries momentum so when an electron scatters light some momentum may be transferred between the light and the electron. Let's consider that the electron is at rest (we can always move into the frame of the electron). Initially we have

$$p^\mu_{ei} = \begin{bmatrix} mc \\ \mathbf{0} \end{bmatrix} \text{ and } p^\mu_{\gamma i} = \frac{E_i}{c} \begin{bmatrix} 1 \\ \mathbf{n}_i \end{bmatrix} \tag{8.1}$$

and after the scattering we have

$$p^\mu_{ef} = \begin{bmatrix} \frac{E}{c} \\ \mathbf{p} \end{bmatrix} \text{ and } p^\mu_{\gamma i} = \frac{E_f}{c} \begin{bmatrix} 1 \\ \mathbf{n}_f \end{bmatrix} \tag{8.2}$$

The conservation of energy-momentum tells us that

$$p^\mu_{ei} + p^\mu_{\gamma i} = p^\mu_{ef} + p^\mu_{\gamma f} \tag{8.3}$$

or

$$p^\mu_{ef} = p^\mu_{ei} + p^\mu_{\gamma i} - p^\mu_{\gamma f} \tag{8.4}$$

Let's calculate the square of both sides,

$$p^\mu_{ef} p_{\mu,ef} = (p^\mu_{ei} + p^\mu_{\gamma i} - p^\mu_{\gamma f})(p_{\mu,ei} + p_{\mu,\gamma i} - p_{\mu,\gamma f}) \tag{8.5}$$

$$m^2c^2 = m^2c^2 + 2p^\mu_{ei} p_{\mu,\gamma i} - 2p^\mu_{ei} p_{\mu,\gamma f} - 2p^\mu_{\gamma f} p_{\mu,\gamma i} \tag{8.6}$$

$$0 = 2mE_i - 2mE_f - 2\frac{E_i E_f}{c^2} (1 - \cos\theta) \tag{8.7}$$

$$E_f = \frac{E_i}{1 + \frac{E_i}{mc^2} (1 - \cos\theta)} \tag{8.8}$$

We can write this really compactly by using the relationship between the energy and wavelength of a photon ($E = hc/\lambda$),

$$\lambda_f = \lambda_i + \frac{h}{mc}\left(1 - \cos\theta\right) \tag{8.9}$$

A cute thing to ask is what is the final energy of the photon if the initial energy is much greater than the electron's rest-mass. We get

$$E_f \approx \frac{mc^2}{1 - \cos\theta} \tag{8.10}$$

There is a second important change. If the energy of the photon changes dramatically, *i.e.* $E_f \ll E_i$, the cross section for the scattering is reduced from σ_T. Specifically

$$\frac{d\sigma}{d\Omega} = \frac{r_0^2}{2}\left(\frac{E_f}{E_i}\right)^2\left(\frac{E_i}{E_f} + \frac{E_f}{E_i} - \sin^2\theta\right) \tag{8.11}$$

for unpolarized radiation.

Inverse Compton Scattering

In Compton scattering the photon always loses energy to an electron initially at rest. Inverse Compton scattering corresponds to the situation where the photon gains energy from the electron because the electron is in motion.

Let's imagine that the electron is travelling along the $x-$axis with Lorentz factor γ. Furthermore, let's think about the lab frame (unprimed) and the electron's rest frame (primed). The initial and final energies

$$E_i' = E_i\gamma\left(1 - \beta\cos\theta_i\right) \text{ and } E_f = E_f'\gamma\left(1 - \beta\cos\theta_f'\right) \tag{8.12}$$

where θ is the angle that the photon makes with the $x-$axis in the lab frame. Furthermore, we know that

$$E_f' = \frac{E_i'}{1 + \frac{E_i'}{mc^2}\left(1 - \cos\Theta\right)} \tag{8.13}$$

where Θ is the angle between the incident and scattered photon in the rest-frame of the electron.

Let's consider the case we $E_i' \ll mc^2$ so $E_i' \approx E_f'$. If we look at the redshift formulae we find that

$$E_f = E_i\gamma^2\left(1 - \beta\cos\theta\right)\left(1 + \beta\cos\theta_f'\right) \tag{8.14}$$

Let's consider the case of relativistic electrons. If we assume that the photon distribution is isotropic, the angle $\langle\cos\theta\rangle = 0$. $\langle\cos\theta_f'\rangle$ is also zero because the scatter photon is forward-backward symmetric in the rest-frame of the electron so we find that

$$E_f = \gamma^2 E_i \tag{8.15}$$

when averaged over angle.

Inverse Compton Power - Single Scattering

Let's consider an isotropic distribution of photons and derive the total power emitted by an electron passing through.

We will first make use on some of the transformation rules that we derived for the phase-space density of photons. Let $v dE$ be the density of photons having energy in the range dE. The number of photons in a box over the energy range is a Lorentz invariant

$$v dE d^3x = v' dE' d^3x' \tag{8.16}$$

Remember that $d^3x = \gamma^{-1} d^3x'$ and that $E = \gamma E'$ (with forward-backward symmetry) so we find that

$$\frac{v dE}{E} = \frac{v' dE'}{E'} = \text{Lorentz Invariant} \tag{8.17}$$

Let's switch to the rest-frame of the electron. The total power scattered in the electron's rest frame is

$$\frac{dE_f}{dt} = \frac{dE'_f}{dt'} = c\sigma_T \int E'_f v' dE' \tag{8.18}$$

where we have assumed that $E'_i \ll mc^2$. The first equality holds because the emitted power is a Lorentz invariant. **Why is this true?**

Let's assume that the change in the energy of the photon in the rest frame of the electron is negligible compared to the change in the energy of the photon in the lab frame, *i.e.* $\gamma^2 - 1 \gg E/(mc^2)$, so $E'_f = E'$, so we have

$$\frac{dE_f}{dt} = c\sigma_T \int E'^2 \frac{v' dE'}{E'} = c\sigma_T \int E'^2 \frac{v dE}{E} \tag{8.19}$$

The redshift formula for photons is

$$E' = E\gamma (1 - \beta \cos\theta) \tag{8.20}$$

so we have

$$\frac{dE_f}{dt} = c\sigma_T \gamma^2 \int (1 - \beta\cos\theta)^2 E v dE = c\sigma_T \gamma^2 \left(1 + \frac{1}{3}\beta^2\right) U_{\text{ph}} \tag{8.21}$$

where

$$U_{\text{ph}} = \int E v dE \tag{8.22}$$

The rate of decrease of the total initial photon energy is

$$\frac{dE}{dt} = -c\sigma_T \int E v dE = -\sigma_T c U_{\rm ph} \tag{8.23}$$

so the total change in the energy of the electron and converted into the increased energy of the radiation field is

$$\frac{dE_{\rm rad}}{dt} = \frac{dE_f}{dt} + \frac{dE}{dt} = c\sigma_T U_{\rm ph}\left[\gamma^2\left(1+\frac{1}{3}\beta^2\right)-1\right] = \frac{4}{3}\sigma_T c\gamma^2\beta^2 U_{\rm ph}. \tag{8.24}$$

Let's compare this with the synchrotron power of the same electron,

$$P = \frac{2}{3}r_0^2 c\beta_\perp^2\gamma^2 B^2 = \frac{2}{3}\left(\frac{3}{8\pi}\sigma_T\right)c\left(\frac{2}{3}\beta^2\right)(8\pi U_B) = \frac{4}{3}\sigma_T c\gamma^2\beta^2 U_B \tag{8.25}$$

Inverse Compton Spectra - Single Scattering

Let's suppose that we have an isotropic distribution of photons of a single energy E_0 and a beam of electrons travelling along the $x-$axis with energy γmc^2 and density N. Let's also use an intensity that counts the number of photons not their energies so

$$I(E) = \frac{I_\nu}{h\nu} = F_0\delta(E-E_0). \tag{8.26}$$

What does the intensity look like in the rest frame of the electrons. Remember that I_ν/ν^3 was a Lorentz invariant so we have

$$I'(E',\mu') = F_0\left(\frac{E'}{E}\right)^2\delta(E-E_0) \tag{8.27}$$

$$= F_0\left(\frac{E'}{E_0}\right)^2\delta(\gamma E'(1+\beta\mu')-E_0) \tag{8.28}$$

$$= \frac{F_0}{\gamma\beta E'}\left(\frac{E'}{E_0}\right)^2\delta\left(\mu'-\frac{E_0-\gamma E'}{\gamma\beta E'}\right) \tag{8.29}$$

In the rest frame of the electrons the emission coefficient is simply proportional to the mean intensity,

$$j'(E'_f) = N'\sigma_T\frac{1}{2}\int_{-1}^{1} I'(E'_f,\mu')d\mu' \tag{8.30}$$

where we have assumed that $E'_f = E'$. Because I' is proportional to a delta function, the integral is trivial giving

$$j'(E'_f) = \frac{N'\sigma_T E'_f F_0}{2E_0^2\gamma\beta} \text{ if } \frac{E_0}{\gamma(1+\beta)} < E'_f < \frac{E_0}{\gamma(1-\beta)} \tag{8.31}$$

and zero otherwise. Now we can transform into the lab frame, using the fact that j_ν/ν^2 is a Lorentz invariant. We have

$$j(E_f,\mu_f) = \frac{E_f}{E'_f}j'(E'_f) \tag{8.32}$$

$$= \frac{N\sigma_T E_f F_0}{2E_0^2\gamma^2\beta} \quad (8.33)$$

$$\text{if } \frac{E_0}{\gamma(1+\beta)(1-\beta\mu_f)} < E_f < \frac{E_0}{\gamma(1-\beta)(1-\beta\mu_f)}$$

and zero otherwise. **Where did the extra γ come from?**.

Let's assume that there are many beams isotropically distributed, so we need to find the mean value of $j(E_f, \mu_f)$ over angle,

$$j(E_f) = \frac{1}{2}\int_{-1}^{1} j(E_f, \mu_f) d\mu_f \quad (8.34)$$

Depending on the value of E_f/E_0 this integral may vanish. Specifically the integrand is non-zero only if μ_f lies in the range

$$\frac{1}{\beta}\left[1 - \frac{E_0}{E_f}(1+\beta)\right] < \mu_f < \frac{1}{\beta}\left[1 - \frac{E_0}{E_f}(1-\beta)\right]. \quad (8.35)$$

Putting this together and integrating yields,

$$j(E_f) = \frac{N\sigma_T F_0}{4E_0\gamma^2\beta^2}\begin{cases} (1+\beta)\frac{E_f}{E_0} - (1-\beta), & \frac{1-\beta}{1+\beta} < \frac{E_f}{E_0} < 1 \\ (1+\beta) - \frac{E_f}{E_0}(1-\beta), & 1 < \frac{E_f}{E_0} < \frac{1+\beta}{1-\beta} \\ 0, & \text{otherwise} \end{cases} \quad (8.36)$$

If $\gamma \gg 1$, the second portion of the emission dominates (many more photons gain energy than lose) and we can derive a simple approximation. Let

$$x \equiv \frac{E_f}{4\gamma^2 E_0} \quad (8.37)$$

and we find that

$$j(E_f) = \frac{3N\sigma_T F_0}{4\gamma^2 E_0}\underbrace{\left[\frac{2}{3}(1-x)\right]}_{f_{\text{iso}}(x)} \quad (8.38)$$

The mean energy of the scattering photon has $x = 1/2$ or $E_f = 2\gamma^2 E_0$ [see equation (15)].

To be more precise, we could have relaxed the assumption that the scattering is isotropic and we would have found

$$f(x) = 2x\ln x + x + 1 - 2x^2, \quad 0 < x < 1. \quad (8.39)$$

Here the mean energy of the scattered photon is slightly lower $4/3\gamma^2 E_0$.

Now we have all of the ingredients to determine the spectrum of radination scattered off of a power-law distribution of electrons, $dN = C\gamma^{-p}d\gamma$. We have

$$\frac{dE}{dVdtdE_f} = 4\pi E_f j(E_f) \quad (8.40)$$

$$= \frac{3}{4} c\sigma_T C \int dE \left(\frac{E_f}{E}\right) v(E) \int_{\gamma_1}^{\gamma_2} d\gamma \gamma^{-p-2} f\left(\frac{E_f}{4\gamma^2 E}\right) \quad (8.41)$$

$$= 3\sigma_T c C 2^{p-2} E_f^{-(p-1)/2} \times \quad (8.42)$$
$$\int dE E^{(p-1)/2} v(E) \int_{x_1}^{x_2} x^{(p-1)/2} f(x) dx$$

so we find that the scattered photons have an energy distribution E^{-s} where $s = (p-1)/2$. This is the same index as for synchrotron radiation. **Can you say why?**

This power-law distribution is valid over a limited range of photon energies. If the initial photon distribution peaks at $\bar{E}$ the power-law will work between $4\gamma_1^2 \bar{E}$ and $4\gamma_2^2 \bar{E}$

Repeated Scattering

Let's now look at the case where a photon might scatter off of the electrons many times before it manages to tranverse the hot plasma. Let's define the Compton y-parameter to be

$$y \equiv \left[\begin{array}{c} \text{average fractional} \\ \text{energy change per} \\ \text{scattering} \end{array} \right] \times \left[\begin{array}{c} \text{mean number of} \\ \text{scatterings} \end{array} \right] \quad (8.43)$$

The second part of this expression is simply related to the optical depth. Specifically, a good heuristic is that it is Max(τ_{es}, τ_{es}^2) where

$$\tau_{es} = \rho \kappa_{es} R = \rho \frac{\sigma_T}{m_p} R. \quad (8.44)$$

if we neglect absorption.

The first term requires a bit more thought. First let's do the non-relativistic limit, Let's imagine that to lowest order the electron is not moving, so we can use Eq. (8.8) to lowest order

$$E_f \approx E_i \left[1 - \frac{E_i}{mc^2}(1-\cos\theta)\right] = E_f \left[1 - \frac{E_i}{mc^2}\right] \quad (8.45)$$

where the second equality holds after averaging over $\cos\theta$. However, the electrons have some thermal motion so we would expect that there would be an additional term proportional to the thermal energy of the electrons,

$$\frac{E_f - E_i}{E_i} = -\frac{E_i}{mc^2} + \frac{\alpha k T}{mc^2}. \quad (8.46)$$

Let's suppose that the photons and electrons are in thermal equilibrium with each other but only scattering is important. In this case, the number of photons cannot change. Also let's assume that the number of photons is small so the number of photons of a particular energy is

$$dN = K E^2 e^{-E/kT} dE \quad (8.47)$$

Because the photons and electrons are in equlibrium the average change in the energy of a photon must be zero

$$\langle E_f - E_i \rangle = -\frac{\langle E_i^2 \rangle}{mc^2} + \frac{\alpha kT}{mc^2}\langle E_i \rangle = 0. \tag{8.48}$$

$$= -\frac{12(kT)^2}{mc^2} + \frac{3\alpha(kT)^2}{mc^2} = 0 \tag{8.49}$$

so $\alpha = 4$ and we find that the fractional change in the photon's energy per scattering is

$$\frac{E_f - E_i}{E_i} = \frac{1}{mc^2}\left(4kT - E_i\right) \tag{8.50}$$

We have already worked through the ultrarelatistic case, from equation (24), we find that

$$E_f - E_i \approx \frac{4}{3}\gamma^2 E_i \tag{8.51}$$

If the electrons are ultrarelativistic, they follow the distribution in Eq. (8.47), so we have

$$\frac{E_f - E_i}{E_f} \approx \frac{4}{3}\left[12\left(\frac{kT}{mc^2}\right)^2\right] = 16\left(\frac{kT}{mc^2}\right)^2. \tag{8.52}$$

Combining these results we can calculate the Compton $y-$parameter in the two regimes

$$y_{NR} = \frac{4kT}{mc^2}\text{Max}(\tau_{es}, \tau_{es}^2) \tag{8.53}$$

$$y_R = \left(\frac{4kT}{mc^2}\right)^2 \text{Max}(\tau_{es}, \tau_{es}^2) \tag{8.54}$$

Essentially the Compton $y-$parameter tracks how the energy of a photon changes as it passes through a cloud of hot electrons. Specifically, the energy of a photon will be $E = e^y E_i$ after passing through a cloud of non-relativistic electrons with $kT \gg E$

Repeated Scattering with Low Optical Depth

We saw how a power-law energy distribution of electrons can yield a power-law energy distribution of photons. This is not too surprising. However, it is also possible to produce a power-law distribution of photons from a thermal distribution of electrons if the optical depth to scattering is low. This will also give some insight about how one gets power-law energy distributions in general.

Let A be the mean amplification per scattering,

$$A \equiv \frac{E_f}{E_i} \sim \frac{4}{3}\langle\gamma^2\rangle = 16\left(\frac{kT}{mc^2}\right)^2. \tag{8.55}$$

The probability that a photon will scatter as it passes through a medium is simply τ_{es} if the optical depth is low, and the probability that it will undergo k scatterings $p_k \sim \tau_{es}^k$ and its energy after k scatterings is $E_k = A^k E_i$, so we have

$$\frac{E_k}{E_i} = A^k \text{ and } p_k = \tau_{es}^k \tag{8.56}$$

The intensity after scattering looks like

$$I(E_k) = I(E_i) p_k = I(E_i) \tau_{es}^k \tag{8.57}$$

To make sense of this let's take the logarithm of the first expression in Eq. 8.56 to get

$$k = \frac{\ln \frac{E_k}{E_i}}{\ln A} \tag{8.58}$$

and substitute this into the intensity formula

$$I(E_k) = I(E_i) \exp\left(\frac{\ln \tau_{es} \ln \frac{E_k}{E_i}}{\ln A}\right) = I(E_i)\left(\frac{E_k}{E_i}\right)^{-\alpha} \tag{8.59}$$

where

$$\alpha = \frac{-\ln \tau_{es}}{\ln A} \tag{8.60}$$

The total Compton power in the output spectrum is

$$P = \int_{E_i}^{A^{1/2} mc^2} I(E_k) dE_k = I(E_i) E_i \left[\int_1^{A^{1/2} mc^2 / E_i} x^{-\alpha} dx\right]. \tag{8.61}$$

If $\alpha \leq 1$ the factor in the brackets can get really large so we find that the amplification is important when

$$\ln \frac{1}{\tau_{es}} \gtrsim \ln A \tag{8.62}$$

so

$$A\tau_{es} \approx 16 \left(\frac{kT}{mc^2}\right)^2 . \tau_{es} \gtrsim 1 \tag{8.63}$$

which is equivalent to $y_R \gtrsim 1$ but $\tau_{es} < 1$

Problems

1. **The Sunyaev-Zeldovich Effect**

   (a) Let's say that you have a blackbody spectrum of temperature T of photons passing through a region of hot plasma (T_e). You can assume that $T \ll T_e \ll mc^2/k$

   What is the brightness temperature of the photons in the Rayleigh-Jeans limit after passing through the plasma in terms of the Compton $y-$parameter?

(b) Let's suppose that the gas has a uniform density ρ and consists of hydrogen with mass-fraction X and helium with mass-fraction Y and other stuff Z. You can assume that $Z/A = 1/2$ is for the other stuff. What is the number density of electrons in the gas?

(c) If you assume that the gas is spherical with radius R, what is the value of the Compton $y-$parameter as a function of b, the distance between the line of sight and the center of the cluster? You can assume that the optical depth is much less than one.

(d) Let's assume that the sphere contains 10^{15} $M_\odot$ of gas and that the radius of the sphere is 1 Mpc, $X = 0.7, Y = 0.27$ and $Z = 0.03$ what is the value of the $y-$parameter?

(e) Let's suppose that the blackbody photons are from the cosmic microwave background. What is the difference in the brightness temperature of the photons that pass through the cluster and those that don't (including the sign)? How does this difference compare with the primordial fluctuations in the CMB? How can you tell this change in the spectrum due to the cluster from the primordial fluctuations?

2. **Synchrotron Self-Compton Emission Blazars**

(a) What is the synchrotron emission from a single electron passing through a magnetic field in terms of the energy density of the magnetic field and the Lorentz factor of the electron?

(b) The number density of the electrons is n_e and they fill a spherical region of radius R. What is the energy density of photons within the sphere, assuming that it is optically thin?

(c) What is the inverse Compton emission from a single electron passing through a gas of photons field in terms of the energy density of the photons and the Lorentz factor of the electron?

(d) What is the total inverse Compton emission from the region if you assume that the synchrotron emission provides the seed photons for the inverse Compton emission?

Part III

Quantum Mechanics

9
Atomic Structure

So far we have used classical and semi-classical approaches to understand how radiation interacts with matter. We have generally treat the electrons (the lightest charged particle so the biggest emitter) classically and the radiation either classically or as coming in quanta (i.e. semi-classically). We also derived some important relationships between how atoms emit and absorb radiation, but to understand atomic processes in detail we will have to treat the electrons quantum mechanically.

In quantum mechanics we characterize the state of a particles (or group of particles) by the wavefunction (Ψ). The wavefunction evolves forward in time according to the *time-dependent Schrodinger equation*

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi \tag{9.1}$$

where H is the Hamiltonian operator. If the Hamiltonian is independent of time we can solve this equation by

$$\Psi(\mathbf{r}, t) = \psi(\mathbf{r}) e^{-iEt/\hbar} \tag{9.2}$$

where ψ satisfies the *time-independent Schrodinger equation*,

$$H\psi = E\psi \tag{9.3}$$

where E is the energy and ψ is the wave function of the corresponding energy state. We can imagine the operator H as a matrix that multiplies the state vector ψ, so this equation is an eigenvalue equation with E as the eigenvalue and ψ as an eigenvector (or eigenfunction) of the matrix (or operator) H.

The Hamiltonian classically is the sum of the kinetic energy and the potential energy of the particles. This realization allows us to write the equation that the wavefunction of an atom must satisfy

$$\left(-\frac{\hbar^2}{2m} \sum_j \nabla_j^2 - E - Ze^2 \sum_j \frac{1}{r_j} + \sum_{i>j} \frac{e^2}{r_{ij}} \right) \psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_j) = 0 \tag{9.4}$$

We have neglect the spin of the electrons, relativistic and nuclear effects. For most atomic states, these effects can be treated at perturbations. We can simplify these equations by using

$$a_0 = \frac{\hbar}{me^2} = 0.529 \times 10^{-8} \text{ cm}w \text{ and } \frac{e^2}{a_0} = 4.36 \times 10^{-11}\text{erg} = 27.\text{ eV} \tag{9.5}$$

and the unit of length and energy respectively. This gives the following dimensionless equation

$$\left(\frac{1}{2} \sum_j \nabla_j^2 + E - Z \sum_j \frac{1}{r_j} + \sum_{i>j} \frac{1}{r_{ij}} \right) \psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_j) = 0 \tag{9.6}$$

A single electron in a central field

Let's first treat the case of a single electron in a central field. Although in principle this approximation will only apply accurately to hydrogen, it is extremely powerful (it explains the periodic table for example). We can imagine that when we focus on a single electron in an atom, the sum of all the other electrons averages out to a spherical distribution. This assumption isn't perfect for all atoms, but the imperfections can be treated as perturbations.

As the electron gets really far from the atom, the potential approaches

$$V(r) \to \frac{Z - N + 1}{r} \tag{9.7}$$

where N is the total number of electrons. On the other hand near the nucleus the potential looks like

$$V(r) \to \frac{Z}{r}. \tag{9.8}$$

This effect is called shielding.

If the potential is a function of the radial distance from the nucleus alone the Schrodinger equation is separable,

$$\psi(r, \theta, \phi) = r^{-1} R(r) Y(\theta, \phi) \tag{9.9}$$

where the functions $Y(\theta, \psi)$ are the *spherical harmonics*

$$Y = Y_{lm}(\theta, \phi) = \left[\frac{(l - |m|)!}{(l + |m|)!} \frac{2l + 1}{4\pi} \right]^{1/2} (-1)^{(m+|m|)/2} P_l^{|m|}(\cos\theta) e^{im\phi} \tag{9.10}$$

where $P_l^m(x)$ are the Legendre polynomials.

The functions Y_{lm} are eigenfunctions of the angular momentum operator. If the potential only depends on the radius, angular momentum is conserved classically. This means quantum-mechanically that the Hamiltonian commutes with the angular momentum operator, and that the wavefunctions that satify the Hamiltonian also are eigenfunction of the angular momentum operator ($\mathbf{L} = \mathbf{r} \times \mathbf{p}$). We have

$$\mathbf{L}^2 Y_{lm} = l(l+1) Y_{lm} \text{ and } L_z Y_{lm} = m Y_{lm} \tag{9.11}$$

so the total angular momentum of the state is related to l and the $z-$component of the angular momentum is related to m. Both l and m take on integral values with $-l < m < l$. The states with different values of l have special letters associated with them we have s, p, d, f and g for $l = 0, 1, 2, 3, 4$ respectively.

The angular eigenfunctions are orthronormal so

$$\int d\Omega Y^*_{lm}(\theta, \phi) Y_{l'm'}(\theta, \phi) = \delta_{l,l'} \delta_{m,m'}. \tag{9.12}$$

The angular eigenfunctions take this form regardless of the form of the central potential. They are simply the eigenfunctions of the angular momentum operator.

The radial part of the wavefunction satisfies the equation

$$\frac{1}{2} \frac{d^2 R_{nl}}{dr^2} + \left[E - V(r) - \frac{l(l+1)}{2r^2} \right] R_{nl} = 0 \tag{9.13}$$

This equation is pretty straightforward to understand. The first term is simply the kinetic term (like the Laplacian in the 3-D Schrodinger equation). The next term is the energy eigenvalue. $V(r)$ is the radial potential and the term proportional to $l(l+1)$ is the centripetal potential.

Because the equation does not depend on m, the radial wavefunction only depends on l. Because it is an eigenvalue equation we also expect each l value to have several solutions labeled by n.

As we have defined them in Eq. 9.9, the radial eigenfunctions have the following normalization.

$$\int_0^\infty R_{nl}(r) R_{n'l}(r) dr = \delta_{n,n'} \tag{9.14}$$

Because the radial eigenfunctions for different values of l satisfy different equations, there is no orthogonality relation for the radial wavefunctions with different l values.

If $V(r) = -Z/r$, we have the following solutions

$$R_{nl}(r) = -\left\{ \frac{Z(n-l-1)!}{n^2 \left[(n+1)! \right]^3} \right\} e^{-\rho/2} \rho^{l+1} L^{2l+1}_{n+1}(\rho) \tag{9.15}$$

$$E_n = -\frac{Z^2}{2n^2} \tag{9.16}$$

$$\rho = \frac{2Zr}{n} \tag{9.17}$$

where L_{n+l}^{2l+1} are the associated Laguerre polynomials.

To make things concrete some of the radial functions are

$$R_{10} = 2Z^{3/2} r e^{-Zr} \tag{9.18}$$

$$R_{20} = \left(\frac{Z}{2}\right)^{3/2} (2 - Zr) r e^{-Zr/2} \tag{9.19}$$

$$R_{21} = \left(\frac{Z}{2}\right)^{3/2} \frac{Zr^2}{\sqrt{3}} e^{-Zr/2} \tag{9.20}$$

The electrons themselves have spin, so we have an additional quantum number $m_s = \pm\frac{1}{2}$ to denote the spin of an electron.

Energies of Electron States

The energy levels of the hydrogen atom at this level of approximation simply depend on the quantum number n. For atoms with more than one electron the picture is more complicated. The most important effect is that when an electron is far from the nucleus the charge of the nucleus is shielded by the other electrons, so wavefunctions that get closer to the nucleus see the full charge of the nucleus and lie lower in energy.

If we look at Eq. 9.13, we see that the centripetal term is proportional to $l(l+1)$, so we would expect that wavefunctions with larger values of l typically stay further from the nucleus, so we have the rule that for a given value of n states with smaller values of l are more bound. *N.B.* This result only applies to atoms with more than one electron, so at this level of approximation, the $2s$ ($n = 2, l = 0$) and $2p$ ($n = 2, l = 1$) are degenerate. Actually, it is relativistic effects that remove this degeneracy.

Sometimes this shielding effect is stronger than the change in the principal quantum number so we have the following ordering of states

$$1s2s2p3s3p[4s3d]4p[5s4d]5p[6s4f5d]6p[7s5f6d]7p \ldots \tag{9.21}$$

The energies of the levels in brackets is really close so sometimes the filling order varies from atom to atom because of the *Hund's rules* below. I have used the letters $s, p, d, f, g \ldots$ to denote $l = 0, 1, 2, 3, 4 \ldots$ of the single electron states.

A second important fact is that because electrons are indistinguishable, the wave function of more than one electron must be antisymmetric with respect to interchange of any two electrons (within the axioms of non-relativistic QM it could have be symmetric, but one can prove in relativistic QM that the wavefunction must be antisymmetric – the spin-statistics theorem).

This has several important consequences. Two electrons cannot occupy the same state. We can label the states by their quantum numbers, n, l, m_l, m_s where $l < n$, $-l \leq m_l \leq l$, $m_s = \pm\frac{1}{2}$. This is quite important. If electrons were bosons (particles that can occupy the same state), atoms would not have the structure that they do. All of the electrons would simply drop down to the lowest energy state available.

Furthermore, we generally try to solve the multielectron problem by assuming that the wavefunction of all the electrons is the antisymmetrized product of single electron wavenfunctions,

$$\psi(\mathbf{r}_1, \mathbf{r}_2 \ldots \mathbf{r}_n) = \frac{1}{\sqrt{N!}} \begin{vmatrix} u_a(\mathbf{r}_1) & u_a(\mathbf{r}_2) & \cdots u_a(\mathbf{r}_n) \\ u_b(\mathbf{r}_1) & u_b(\mathbf{r}_2) & \cdots u_b(\mathbf{r}_n) \\ \vdots & & \\ u_k(\mathbf{r}_1) & u_k(\mathbf{r}_2) & \cdots u_k(\mathbf{r}_n) \end{vmatrix} \tag{9.22}$$

This is called the *Slater determinant*.

When you substitute this into the multi-particle Schrodinger equation you get an equation for each electron state (the *Hartree-Fock equations*)

$$F u_i(\mathbf{r}_1) = E_i u_i(\mathbf{r}_1) \tag{9.23}$$

where

$$F = \frac{p_i^2}{2m} - \frac{Ze^2}{r} + V(\mathbf{r}_1). \tag{9.24}$$

where the potential V has two terms, one is called the *direct interaction* term and the other is called the *exchange interaction term*.

$$V(\mathbf{r}_1) = \sum_j \left[J_j(\mathbf{r}_1) + (-1)^S K_j(\mathbf{r}_1) \right] \tag{9.25}$$

where $S = m_{s,i} + m_{s,j}$ and

$$J_j(\mathbf{r}_1) u_i(\mathbf{r}_1) = \left[\int d^3\mathbf{r}_2 u_j^*(\mathbf{r}_2) \left(\frac{e^2}{r_{12}} \right) u_j(\mathbf{r}_2) \right] u_i(\mathbf{r}_1) \tag{9.26}$$

$$K_j(\mathbf{r}_1) u_i(\mathbf{r}_1) = \left[\int d^3\mathbf{r}_2 u_j^*(\mathbf{r}_2) \left(\frac{e^2}{r_{12}} \right) u_i(\mathbf{r}_2) \right] u_j(\mathbf{r}_1) \tag{9.27}$$

The term J_j is simply the potential that one electron in the state i feels from another electron in the state j. The K_j term has no classical analogue.

Let's try to understand what this means. When we solve this set of equations we imagine that all of the other electrons are fixed and we are try to solve for a single extra electron. Let's imagine that we only have two electrons. The total energy of the first electron including the effect of the second electron is

$$\int d^3\mathbf{r}_1 u_1^*(\mathbf{r}_1) \left(\frac{p_1^2}{2m} - Z\frac{e^2}{r_1} + J_2(\mathbf{r}_1) + (-1)^S K_2(\mathbf{r}_1) \right) u_2(\mathbf{r}_1) \tag{9.28}$$

Both J and K are positive, so if $S = 1$ the system has slightly lower energy.

yIn multielectron systems this result holds for any pair of electrons, so we get two rules of thumb (called *Hund's rules*) that all other things being equal

1. States with the spin of the electrons aligned have lower energies, or states with larger total spin (S) lie lower in energy.
2. Of those states with a given spin, those with the largest value of L tend to lie lower in energy.

The second rule comes about because a large value of L implies that the electrons are orbiting the nucleus in the same direction which reduces the value of the J integral.

These two rules order electron configurations (lists of the values of n and l for a set of electrons: *e.g.* $4p4d$) into terms with equal energies labels by the total orbital and spin angular momentum (L and S) *e.g.* ^3F. The superscript is the $2S+1$, the multiplicity of the spin states and the letter is the value of L using the rules described earlier.

Perturbative Splittings

Spin-Orbit Coupling

There are various fine structure splittings enter due to relativistic corrections. The simplest of these is the spin-order coupling. Let's imagine that we move into the frame of the electron, we are moving through an electric field so there is a magnetic field

$$\mathbf{B} = -\frac{1}{c}\mathbf{v}\times\mathbf{E} = \frac{\mathbf{l}}{mecr}\frac{dU}{dr} \tag{9.29}$$

where $U(r)$ is the electrostatic potential. The electron has a magnetic moment of

$$\bar{\ } = -\frac{e}{mc}\mathbf{s} \tag{9.30}$$

The magnetic energy of the electron in the field is

$$H_{so} = \frac{1}{2m^2c^2}\mathbf{s}\cdot\mathbf{l}\frac{1}{r}\frac{dU}{dr} \tag{9.31}$$

Notice that this is one-half of what you would expect. This is due to a relativistic effect called *Thomas precession*. More important to notice is that the spin-orbit term vanishes as $c \to \infty$, so it is indeed a relativistic correction. For a single electron because dU/dr is positive we find that if $\mathbf{s}\|\mathbf{l}$ the energy of the state is higher so lower values of j the total angular momentum have lower energies.

For hydrogen we have

$$1s[2s_{1/2}2p_{1/2}]2p_{3/2} \tag{9.32}$$

where the states in the brackets are still degnerate.

In multiple electron systems we find that

$$H_{so} = \xi \mathbf{S} \cdot \mathbf{L} \tag{9.33}$$

where the value of ξ depends on the configuration.

Let's focus on states with the same values of S and L but different values of J. We know that

$$\mathbf{J}^2 = (\mathbf{L} + \mathbf{S}) \cdot (\mathbf{L} + \mathbf{S}) = \mathbf{L}^2 + \mathbf{S}^2 + 2\mathbf{L} \cdot \mathbf{S} \tag{9.34}$$

so we can write

$$H_{so} = \frac{1}{2}\xi \left(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2\right) = \frac{1}{2} C\left[J(J+1) - L(L+1) - S(S+1)\right] \tag{9.35}$$

so if L and S are fixed we have

$$E_{J+1} - E_J = C(J+1) \tag{9.36}$$

The value of C can be positive (shells less than half-full) or negative (shells more than half-full). Notice that we recover the result for hydrogen; the $2p$ shell is clearly less than half-full.

We can make sense of the situtation of a nearly full shell but realizing that a completely full shell is spherically symmetric so a nearly full level acts as if it has a few holes whose charge and magnetic moment have the opposite sign of an electrons.

Zeeman Effect and Nuclear Spin

The Zeeman effect is the splitting of atomic levels on the basis of the value of the total angular momentum in the direction of the magnetic field m_J. This is why the quantum number m uses the letter m; it stands for "magnetic". The picture is similar to the spin-orbit coupling except we are looking at the interaction of the total magnetic moment of the atom with the magnetic field

$$U_B = -\text{¯} \cdot \mathbf{B} \tag{9.37}$$

where

$$\text{¯} = -\sum \left[\frac{1}{2}\left(\frac{e}{mc}\right)\mathbf{l}_i + \left(\frac{e}{mc}\right)\mathbf{s}_i\right] \tag{9.38}$$

If we average over the precession of the magnetic moments around the imposed magnetic field we get the following splitting

$$U_B = \frac{1}{2}\left(\frac{e\hbar B}{mc}\right) g M_J \tag{9.39}$$

where

$$g(J,L,S) = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)} \tag{9.40}$$

A related effect that is really important astrophysically is that the nucleus itself has a magnetic moment

$$\mu_N = g\frac{e}{2Mc}\mathbf{I} \tag{9.41}$$

that can interact with the magnetic moment of the electron. $\mathbf{I}$ is the total angular momentum of the proton and that total angular momentum of the system is $\mathbf{F} = \mathbf{J} + \mathbf{I}$

We can have transitions where the orientation of $\mathbf{J}$ changes with respect to $\mathbf{I}$ so we have a splitting depending on the value of M_J. An important case is the ground state of hydrogen which is a $^2S_{1/2}$ term. The proton has spin $1/2$ so we can have $F = 0$ and $F = 1$. The splitting between these two states corresponds to a frequency of 1420 MHz or $\lambda = 21$ cm.

It is simplest to see this effect by considering the nucleus to be stationary and averaging the effect of the electron over its wavefunction. There are two separate effects the interaction of the magnetic moment of the nucleus with that of the current induced by the electron orbital angular momentum and the interaction between the two magnetic moments themselves.

The magnetic field produced by the orbiting electron is given by

$$\mathbf{B} = -2\mu_B\frac{\mathbf{l}}{r^3} \tag{9.42}$$

where r is the distance between the electron and the nucleus and $\mathbf{l}$ is the orbital angular momentum of the electron. The situtation for the intrinsic magnetic moment of the electron is a bit more subtle. The field of a magnetic dipole is given by

$$\mathbf{B} = \frac{1}{r^3}\left[3\left(\boldsymbol{\mu}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}} - \boldsymbol{\mu}\right]. \tag{9.43}$$

However this is not the entire picture because there is the possibility that the electron and the nucleus lie right on top of each other. Let's imagine that the magnetic moment of the electron is produced by a small ring of current of radius R and integrate the total magnetic flux passing outside the ring through the plane of the ring according the formula above

$$\Phi_{\text{Outside}} = \int_{\text{Outside}}\mathbf{B}\cdot\mathbf{dA} = \mu\int_R^\infty\frac{1}{r^3}2\pi r dr = 2\pi\frac{\mu}{R} \tag{9.44}$$

and the flux clearly points in a direction opposite to the magnetic moment of the electron. Now the total flux through the entire plane that contains the current ring should vanish (the magnetic field is divergence free), so within the ring we have

$$\Phi_{\text{Inside}} = \int_{\text{Inside}} \mathbf{B}_{\text{Inside}} \cdot \mathbf{dA} = \bar{\mathbf{B}}\pi R^2 \tag{9.45}$$

and

$$\bar{\mathbf{B}} = 2\frac{\boldsymbol{\mu}}{R^3}. \tag{9.46}$$

Now let's integrate this mean field over a small sphere of radius R to yield

$$\int_{S(R)} \bar{\mathbf{B}} dV = \frac{8\pi}{3}\boldsymbol{\mu}. \tag{9.47}$$

This yields a correction to the dipole field called the *Fermi contact interaction*,

$$\mathbf{B} = \frac{1}{r^3}\left[3\left(\boldsymbol{\mu}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}} - \boldsymbol{\mu}_s\right] + \frac{8\pi}{3}\boldsymbol{\mu}\delta^3(\mathbf{r}). \tag{9.48}$$

A second way to obtain this result is to take the expression for the vector potential of a point magnetic dipole

$$\mathbf{A} = \frac{\boldsymbol{\mu}\times\mathbf{r}}{r^3} \tag{9.49}$$

and calculate the magnetic field, $\mathbf{B} = \nabla\times\mathbf{A}$.

Combining this result with the orbital contribiution yields a complete expression for the hyperfine splitting, since the energy of a magnetic dipole in a magnetic field is given by $U = -\boldsymbol{\mu}\cdot\mathbf{B}$,

$$H_{\text{HFS}} = -\frac{8\pi}{3}\boldsymbol{\mu}_e\cdot\boldsymbol{\mu}_N\delta^3(\mathbf{r}) + \frac{1}{r^3}\left[\boldsymbol{\mu}_e\cdot\boldsymbol{\mu}_N - 3\frac{\left(\mathbf{r}\cdot\boldsymbol{\mu}_e\right)\left(\mathbf{r}\cdot\boldsymbol{\mu}_N\right)}{r^2} - \frac{e}{mc}\mathbf{L}\cdot\boldsymbol{\mu}_N\right]. \tag{9.50}$$

One can observe that the first term vanishes for states with $l > 0$ and the second term vanishes for $l = 0$.

Thermal Distributions of Atoms

In thermal equilibrium the number of atoms in a particular state is proportional to $ge^{-\beta E}$ where $\beta = 1/kT$ and g is the statistical weight or degeneracy of the state (for $L-S-$coupling $g = 2(2J+1)$), so we find that

$$N_i = \frac{N}{U}g_i e^{-\beta E_i} \tag{9.51}$$

where N is the total number of atoms and U is a normalization factor

$$U = \sum g_i e^{-\beta E_i}. \tag{9.52}$$

We already run into a problem. Atoms generally have a certain ionization energy (for example, hydrogen has 13.6 eV) but there are an infinite number of states between the ground state and the ionization level so $e^{-\beta E_i}$ approaches a constant for large i and g_i typically increases so U will diverge.

In practice this is not really a problem for two reasons. First, for temperatures less than 10^4 K only the ground state is typically populated so it is okay to take $U = g_0$. Second is that atoms don't live in spendid isolation. The size of the highly excited states of atoms increases as n^2 so we only have to sum over the states until we reach

$$n_{\rm max}^2 a_0 Z^{-1} \sim N^{-1/3}, \quad n_{\rm max} \left(\frac{Z}{a_0}\right)^{1/2} N^{-1/6}. \tag{9.53}$$

Ionization Equilibrium - the Saha Equation

Let's consider a electron and ions in the ground state in equilibrium with neutral atoms also in the ground state

$$\frac{dN_0^+(v)}{N_0} = \frac{g_e g_0^+}{g_0} \exp\left[-\frac{E_I + \frac{1}{2}m_e v^2}{kT}\right] \tag{9.54}$$

where

$$g_e = \frac{2dx_1 dx_2 dx_3 dp_1 dp_2 dp_3}{h^3} \tag{9.55}$$

and v is the velocity of the electron.

The volume $dx_1 dx_2 dx_3$ contains a single electron, so $dx_1 dx_2 dx_3 = N_e^{-1}$. Furthermore, if we assume that the electron velocity distribution is isotropic we can derive

$$\frac{dN_0^+}{N_0} = \frac{8\pi m_e^3}{h^3} \frac{g_0^+}{N_e g_0} \exp\left[-\frac{E_I + \frac{1}{2}m_e v^2}{kT}\right] v^2 dv \tag{9.56}$$

Let's integrate over the electron's velocity to get,

$$\frac{N_0^+ N_e}{N_0} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \frac{2g_0^+}{g_0} e^{-E_I/kT} \tag{9.57}$$

We know that the ratio of the number of atoms in any state to those in the ground states is simply $g_0/U(T)$, so we can get *Saha's equation*

$$\frac{N^+ N_e}{N} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \frac{2U^+(T)}{U(T)} e^{-E_I/kT}. \tag{9.58}$$

We can also derive a Saha equation that connects any two stages of ionization,

$$\frac{N_{j+1} N_e}{N_j} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \frac{2U_{j+1}(T)}{U_j(T)} e^{-E_{I,j,j+1}/kT}. \tag{9.59}$$

In astrophysical contexts, there is generally a mixture of different elements. Some elements such as the alkali metals have very small values of E_I so they may dominate the number of electrons in the gas when more abundant elements such as hydrogen are completely neutral.

A bit of nomenclature: HI is neutral hydrogen, HII is ionized hydrogen, Fe XXVI has a single electron, and Fe I has all twenty six. Don't confuse HII with H_2.

A Practical Aside - Orders of Magnitude

One of the most important tools that a card-carrying astrophysics has is the order of magnitude estimate. The order of magnitude estimate combines the lack of rigour of dimensional analysis with the lack of accuracy of keeping track of only the exponents; this makes multiplication in your head easier!

The first part of the tool is the knowledge of the various constants of nature in c.g.s units but you only need to keep the exponent in your head. A glance at Table 9.1 shows that some of the physical constants are easier to remember than others, but one can exploit the relationships between them a remember only a few key numbers to obtain the the rest.

Table 9.1: Common Physical Constants in c.g.s.

Name	Value	Units	Exponent
	Mathematical Quantities		
π	3.14		0.5
Arc Second	4.86×10^{-6}		-5.5
	Astrophysical Quantities		
Mass of Sun, $M_\odot$	1.99×10^{33}	g	33.5
Luminosity of Sun, $L_\odot$	3.83×10^{33}	erg s^{-1}	33.5
Radius of Sun, $R_\odot$	6.96×10^{10}	cm	11
Mass of Earth, $M_\oplus$	5.98×10^{27}	g	28
Radius of Earth, $R_\oplus$	6.38×10^{8}	cm	9
$2\pi R_\oplus$	$40,000$	km	4.5
Year	3.16×10^{7}	s	7.5
Parsec	3.09×10^{18}	cm	18.5
Astronomical Unit	1.50×10^{13}	cm	13
	Physical Constants		
Speed of light	3.00×10^{10}	cm s^{-1}	10.5
Newton's Constant G	6.67×10^{-8}	dyn $cm^2 g^{-2}$	-7
	$(2\pi)^2$	$AU^{-3} yr^{-2} M_\odot^{-1}$	1.5
Thomson cross-section, σ_T	6.65×10^{-25}	cm^2	-24
Electron mass, m_e	9.11×10^{-28}	g	-27
	511	keV c^{-2}	2.5
Proton mass, m_p	1.67×10^{-24}	g	-27
	938	MeV c^{-2}	3
Electron-scattering opacity $\kappa_e = \sigma_T / m_p$	0.4	$cm^{-2}\ g^{-1}$	-0.5
m_p / m_e	1836.109		3
Planck constant, h	6.63×10^{-27}	erg s	-26
Reduced Planck constant, $\hbar = h/(2\pi)$	1.05×10^{-27}	erg s	-27
$\hbar c$	3.16×10^{-17}	erg cm	-16.5
Fine structure constant, $\alpha = e^2/(\hbar c)$	1/137.		-2
Electron Compton Wavelength, $\lambda\!\!\!^{-}_e = \hbar/(m_e c)$	3.86×10^{-11}	cm	-10.5
Bohr radius, $a_0 = \lambda\!\!\!^{-}_e / \alpha$	0.529	$Å = 10^{-8}$cm	-8.5
Boltzmann constant, k_B	1.38×10^{-16}	erg K^{-1}	-16
Stephan-Boltzmann constant, σ	5.67×10^{-5}	erg $cm^{-2} s^{-1} K^{-4}$	-4.5
Electron volt, eV	1.60×10^{-12}	erg	-12
	11600	K k_b^{-1}	4

Problems

1. **Particles in a Box**

   A reasonable model for the neutrons and protons in a nucleus is that they are confined to a small region. Let's take a one-dimensional model of this. The potential is $V(x)$ is zero everywhere for $0 < x < l$ and infinite otherwise. This means that

   $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E_n\psi \text{ if } 0 < x < l \tag{9.60}$$

   and $\psi = 0$ if $x < 0$ or $x > l$. What are the energy levels of this system?

2. **Hyperfine Transition - Ballpark**

   Calculate the energy and wavelength of the hyperfine transition of the hydrogen atom. You may use the following formula for the energy of two magnets near to each other

   $$E = -\frac{\vec{\ }_1 \cdot \vec{\ }_2}{r^3} \tag{9.61}$$

   We are looking for an order of magnitude estimate of the wavelength. I got 151 cm which is in the ballpark.

3. **Hyperfine Transition - Precise**

   Calculate the energy and wavelength of the transition of hydrogen with the spin of the electron and proton aligned to antialigned. Assume that the electron is in the ground state.

4. **Density and Ionization**

   Calculate the ionized fraction of pure hydrogen as a function of the density for a fixed temperature. You may take $U(T) = g_0 = 2$ and $U^+(T) = g_0^+ = 2$.

10
Radiative Transitions

Perturbation Theory

After we figured out the wavefunctions for the hydrogen atom, we examine the energy states of atoms with more than one electron. We didn't resolve Schrodinger's equation, but rather we used the spherical harmonic solutions to understand how various additional terms like the interaction between the electrons would affect the energies of the states. This powerful technique is called perturbation theory (specifically time-independent perturbation theory).

Through this process we built up a picture of the structure of atoms from two simple ideas: Schrodinger's equation and that the wavefunction of a bunch of electrons is odd under interchange of any pair of electrons. Our atoms are elegant, we know their energy levels, angular momenta . . . but they never do anything.

To understand how atoms change with time we could use the time-dependent Schrodinger equation, but like for the problem of the energies of multi-electron systems this is probably too hard and not really worth the effort. On the other hand, maybe there is something called *time-dependent perturbation theory* that will do the heavy-lifting for us.

Let's start with the *time-dependent Schrodinger equation* and add a small extra time-dependent term in the potential.

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi + \lambda H'(\mathbf{r},t)\Psi \tag{10.1}$$

If the $\lambda H'(\mathbf{r},t)$ bit weren't there we would know the solutions:

$$\Psi(\mathbf{r},t) = \psi(\mathbf{r})e^{-iEt/\hbar} \tag{10.2}$$

such that

$$H\psi = E\psi. \tag{10.3}$$

Solutions to equations like Eq. 10.3 form a complete set. This means that you can use a sum of them to represent any function, so let's imagine that the real solution to Eq. 10.1 is the sum of the solutions to Eq. 10.2 but let's allow the coefficients to be a function of time

$$\Psi(\mathbf{r},t) = \sum_j \sum_{l=0}^{\infty} A_{jl}(t)\lambda^l \psi_j(\mathbf{r}) e^{-iE_j t/\hbar} \quad (10.4)$$

and substitute into Eq. 10.1

$$i\hbar \sum_j \sum_{l=0}^{\infty} \left[-\frac{iE_j}{\hbar} A_{jl}(t) + \frac{dA_{jl}(t)}{dt} \right] \lambda^l \psi_j(\mathbf{r}) \quad e^{-iE_j t/\hbar}$$
$$= \sum_j \sum_{l=0}^{\infty} \left[E_j + \lambda H'(\mathbf{r},t) \right] \lambda^l \psi_j(\mathbf{r}) e^{-iE_j t/\hbar}. \quad (10.5)$$

To make some progress we multiply both sides by $\psi_f^*(\mathbf{r})$ and integrate over all space. We remember that the wavefunctions ψ_f are orthonormal so that the integral of a product of two wavefunctions over all space is δ_{jf}.

$$i\hbar \sum_{l=0}^{\infty} \left[-\frac{iE_f}{\hbar} A_{fl}(t) + \frac{dA_{fl}(t)}{dt} \right] \lambda^l e^{-iE_f t/\hbar}$$
$$= \sum_i \sum_{l=0}^{\infty} A_{il}(t)\lambda^l \left[\langle \psi_f | \psi_j \rangle E_j + \lambda \langle \psi_f | H'(\mathbf{r},t) | \psi_j \rangle \right] e^{-iE_j t/\hbar} \quad (10.6)$$
$$= \sum_{l=0}^{\infty} \lambda^l \left[A_{fl}(t) E_f e^{-iE_f t/\hbar} + \lambda \sum_j A_{jl}(t) \langle \psi_f | H'(\mathbf{r},t) | \psi_j \rangle e^{-iE_j t/\hbar} \right] \quad (10.7)$$

where we have used the Dirac notation

$$\langle \psi_f | H'(\mathbf{r},t) | \psi_j \rangle = \int d^3x \psi_f^* H'(\mathbf{r},t) \psi_j. \quad (10.8)$$

The integral is also over the spin coordinates if necessary.

Now we look at this summation in powers of λ. First let's do λ^0

$$i\hbar \left[-\frac{iE_f}{\hbar} A_{f0}(t) + \frac{dA_{f0}(t)}{dt} \right] e^{-iE_f t/\hbar} = A_{f0}(t) E_f e^{-iE_f t/\hbar}. \quad (10.9)$$

This equation implies that

$$\frac{dA_{f0}(t)}{dt} = 0 \Rightarrow A_{f0}(t) = A_{f0}(0). \quad (10.10)$$

Let's now look at the λ^1 term

$$i\hbar \left[-\frac{iE_f}{\hbar} A_{f1}(t) + \frac{dA_{f1}(t)}{dt} \right] e^{-iE_f t/\hbar} \quad = \quad A_{f1}(t) E_f e^{-iE_f t/\hbar} +$$
$$\sum_j A_{j0}(0) \langle \psi_f | H'(\mathbf{r},t) | \psi_j \rangle e^{-iE_j t/\hbar} \quad (10.11)$$

Canceling terms and rearranging gives

$$i\hbar \frac{dA_{f1}(t)}{dt} = \sum_j A_{j0}(0)\langle \psi_f | H'(\mathbf{r},t) | \psi_j \rangle e^{-i(E_j - E_f)t/\hbar} \tag{10.12}$$

Let's assume that $H'(\mathbf{r},t) = \frac{1}{2}H'(\mathbf{r})(e^{i\omega t} + e^{-i\omega t})$ for $t > 0$ and that at $t = 0$, $A_{j0}(0) = \delta_{ij}$

$$A_{f1}(t) = \frac{1}{2}\langle \psi_f | H'(\mathbf{r}) | \psi_i \rangle \left[\frac{e^{-i[E_i - E_f - \hbar\omega]t/\hbar} - 1}{E_i - E_f - \hbar\omega} + \frac{e^{-i[E_i - E_f + \hbar\omega]t/\hbar} - 1}{E_i - E_f + \hbar\omega} \right] \tag{10.13}$$

Let's calculate the probability that the atom is in the state f after a time t,

$$A^*_{f1}(t)A_{f1}(t) = \frac{1}{4}|\langle \psi_f | H'(\mathbf{r}) | \psi_i \rangle|^2 \left[\frac{2(1 - \cos(\omega_{fi} - \omega)t)}{\hbar^2(\omega_{fi} - \omega)^2} + 4\cos(\omega t)\frac{\cos(\omega t) - \cos(\omega_{fi} t)}{\omega_{fi}^2 - \omega^2} + \frac{2(1 - \cos(\omega_{fi} + \omega)t)}{\hbar^2(\omega_{fi} + \omega)^2} \right] \tag{10.14}$$

$$= \frac{|\langle \psi_f | H'(\mathbf{r}) | \psi_i \rangle|^2}{\hbar^2} \left\{ \frac{\sin^2\left[\frac{1}{2}(\omega_{fi} - \omega)t\right]}{(\omega_{fi} - \omega)^2} + \cos(\omega t)\frac{\cos(\omega t) - \cos(\omega_{fi} t)}{\omega_{fi}^2 - \omega^2} + \frac{\sin^2\left[\frac{1}{2}(\omega_{fi} + \omega)t\right]}{(\omega_{fi} + \omega)^2} \right\} \tag{10.15}$$

where $\omega_{fi} = (E_f - E_i)/\hbar$.

We see that if $E_f - E_i \approx \hbar\omega$, then $A^*_{f1}A_{f1}$ is big and the first term dominates if $t\omega \gg 1$. This corresponds to absorption of radiation. The frequency range over which the transition probablility is large is $\Delta\omega \approx 4\pi/t$.

Let's imagine that the perturbation $H'(\mathbf{r})$ is due to a quasimonochromatic radiation field such that the phases for different frequencies are not correlated and that the amplitude of $H'_\omega(\mathbf{r})$ is constant over a frequency range much larger than $\Delta\omega$. We can integrate the transition probability (keeping only the dominant first term) over all frequencies to get

$$A^*_{f1}(t)A_{f1}(t) = \frac{\pi}{2} t \frac{|\langle \psi_f | H'_\omega(\mathbf{r}) | \psi_i \rangle|^2}{\hbar^2} \tag{10.16}$$

so we get a transition rate of

$$W(i \to f) = \frac{\pi}{2} \frac{|\langle \psi_f | H'_\omega(\mathbf{r}) | \psi_i \rangle|^2}{\hbar^2} \tag{10.17}$$

The Perturbation to the Hamiltonian

The Hamiltonian of an electron in a external electromagnetic field is given by

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e\mathbf{A}}{c}\right)^2 + e\phi \quad (10.18)$$

$$H = \frac{p^2}{2m} - \frac{e}{mc}\mathbf{A}\cdot\mathbf{p} + \frac{e^2A^2}{2mc^2} + e\phi \quad (10.19)$$

To go from the first to the second equation we have assumed that $\nabla \cdot \mathbf{A} = 0$, the Coulomb gauge, so that the momentum commutes with the vector potential.

$$H' = -\frac{e}{mc}\mathbf{A}\cdot\mathbf{p} + \frac{e^2A^2}{2mc^2} + e\phi \quad (10.20)$$

In the Coulomb gauge, $\nabla^2\phi = 4\pi\rho$ but because there are no other charges around we can take $\phi = 0$, so we are left with

$$H' = -\frac{e}{mc}\mathbf{A}\cdot\mathbf{p} + \frac{e^2A^2}{2mc^2}. \quad (10.21)$$

The second term is generally smaller than the first for weak waves, so let's focus on the first term. We know that

$$\mathbf{E}(\mathbf{r}, t) = -\frac{1}{c}\frac{\partial A}{\partial t} = -i\frac{\omega}{c}\mathbf{A} \quad (10.22)$$

so we can write

$$H'(\mathbf{r}, t) = -i\frac{e}{mc}\frac{c}{\omega}\left(\mathbf{E}e^{i\mathbf{k}\cdot\mathbf{r}}e^{-i\omega t}\right)\cdot\mathbf{p} \quad (10.23)$$

where we can expand the exponential to yield

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 1 + \mathbf{k}\cdot\mathbf{r} + \cdots. \quad (10.24)$$

If we take the electric field to be constant in space (the dipole approximation), it is handy to write

$$H'(\mathbf{r}) = \mathbf{E}\cdot\mathbf{d} \quad (10.25)$$

where

$$\mathbf{d} = -i\frac{e}{m\omega}\mathbf{p}. \quad (10.26)$$

This expression is generally applicable. Classically however, the dipole moment of a charge is $e\mathbf{r}$. It would be nice to get a similar expression for the quantum mechanical system.

We can proceed in several ways. We intend to focus on the electric dipole approximation which is appropriate if $v \ll c$. If one looks at Eq. 10.19 one sees that if $p \ll mc$, the dominant term in the perturbation is

$$H' = e\phi + \mathcal{O}\left(\frac{v}{c}\right) \quad (10.27)$$

and we could write $\phi = \mathbf{E} \cdot \mathbf{r}$ and get

$$\mathbf{d} = e\mathbf{r}. \tag{10.28}$$

This derivation is not really valid. However, the expression above does turn out to be useful in particular situations. Let's first prove

$$\begin{aligned}
(\mathbf{r}H_0 - H_0\mathbf{r})\,\psi &= -\mathbf{r}\frac{\hbar^2}{2m}\nabla^2\psi + \mathbf{r}V(\mathbf{r})\psi + \frac{\hbar^2}{2m}\nabla^2(\mathbf{r}\psi) - V(\mathbf{r})\mathbf{r}\psi & (10.29)\\
&= -\mathbf{r}\frac{\hbar^2}{2m}\nabla^2\psi + \frac{\hbar^2}{2m}\nabla\cdot(\psi\nabla\mathbf{r} + \mathbf{r}\nabla\psi) & (10.30)\\
&= -\mathbf{r}\frac{\hbar^2}{2m}\nabla^2\psi + \frac{\hbar^2}{2m}\left(\nabla\psi\cdot\nabla\mathbf{r} + \mathbf{r}\nabla^2\psi + \nabla\psi\cdot\nabla\mathbf{r}\right) & (10.31)\\
&= -\mathbf{r}\frac{\hbar^2}{2m}\nabla^2\psi + \frac{\hbar^2}{2m}\left(2\nabla\psi + \mathbf{r}\nabla^2\psi\right) & (10.32)\\
&= \frac{\hbar^2}{m}\nabla\psi = i\frac{\hbar}{m}\mathbf{p}\psi & (10.33)
\end{aligned}$$

and substitute this result into Eq. 10.28 to get

$$\langle\psi_f|\mathbf{d}|\psi_i\rangle = -\frac{e}{\hbar\omega}\langle\psi_f|\mathbf{r}H_0 - H_0\mathbf{r}|\psi_i\rangle \tag{10.34}$$

Let's suppose that $H_0\psi_f = E_f\psi_f$ and $H_0\psi_i = E_i\psi_i$ (*i.e.* they are eigenstates of the unperturbed Hamiltonian) we have

$$\langle\psi_f|\mathbf{d}|\psi_i\rangle = -\frac{e}{\hbar\omega}(E_i - E_f)\langle\psi_f|\mathbf{r}|\psi_i\rangle = e\langle\psi_f|\mathbf{r}|\psi_i\rangle \tag{10.35}$$

so

$$\mathbf{d} = e\mathbf{r} \tag{10.36}$$

when and only when $\mathbf{d}$ operates on two eigenstates of the unperturbed Hamiltonian.

Dipole Approximation

Let's assume that $H'_\omega(\mathbf{r}) = e\mathbf{E}_\omega \cdot \mathbf{r}$. This implicitly assumes that the wavelength of the radiation is much bigger than the atom, then we get

$$W(i \to f) = \frac{\pi}{2}e^2\frac{|\langle\psi_f|\mathbf{E}_\omega\cdot\mathbf{r}|\psi_i\rangle|^2}{\hbar^2} = \frac{\pi}{2\hbar^2}\mathbf{E}^2_{\mathbf{d},\omega}|d_{if}|^2 \tag{10.37}$$

where

$$|\mathbf{d}_{if}|^2 = e^2\left(|\langle\psi_f|x|\psi_i\rangle|^2 + |\langle\psi_f|y|\psi_i\rangle|^2 + |\langle\psi_f|z|\psi_i\rangle|^2\right) \tag{10.38}$$

The energy density of the field is

$$u_\nu = (2\pi)\frac{3E^2_{\mathbf{d},\omega}}{8\pi} = 4\pi\frac{J_\nu}{c}. \tag{10.39}$$

so

$$E^2_{x,\omega} = \frac{16\pi}{3}\frac{J_\nu}{c} \tag{10.40}$$

The factor of threes arise because we assume that the radiation is isotropic so the value of E_x^2 is typically one third of E^2 Using this in the transition rate gives

$$W(i \rightarrow f) = \frac{8\pi^2}{3}\frac{|\mathbf{d}_{if}|^2}{\hbar^2 c} J_\nu. \tag{10.41}$$

We can write this result as an Einstein coefficient

$$B_{if} = \frac{8\pi^2}{3}\frac{|\mathbf{d}_{if}|^2}{\hbar^2 c} \tag{10.42}$$

Oscillator Strengths

A classical harmonic oscillator driven by electromagnetic radiation has a cross-section to absorb radiation of

$$\sigma(\omega) = \sigma_T \frac{\omega^4}{\left(\omega^2 - \omega_0^2\right)^2 + \left(\omega_0^3\tau\right)^2} \tag{10.43}$$

If we integrate this over all frequencies around the resonance we obtain

$$\int_0^\infty \sigma(\omega) d\nu = \frac{\pi e^2}{mc} = B_{if}^{\text{classical}} \frac{h\nu_{fi}}{4\pi} \tag{10.44}$$

so

$$B_{if}^{\text{classical}} = \frac{4\pi^2 e^2}{h\nu_{fi} mc} \tag{10.45}$$

We can write the Einstein coefficients in term of this classical one

$$f_{if} = \frac{B_{if}}{B_{if}^{\text{classical}}} = \frac{2m}{3\hbar^2 g_i e^2}\left(E_f - E_i\right)\sum |\mathbf{d}_{if}|^2 \tag{10.46}$$

Here we have included the possibility that the lower state has a g_f-fold degeneracy and we have summed over the degenerate upper states.

In Eq. 10.17 the final term could be important if the $E_i - E_f \approx -\hbar\omega$ this corresponds to stimulated emission of radiation. Except for the degeneracy factors for the two states, the Einstein coefficients will be the same, so we can define an oscillator strength for stimulated emission as well,

$$f_{if} = \frac{2m}{3\hbar^2 g_i e^2}\left(E_i - E_f\right)\sum |\mathbf{d}_{if}|^2 \tag{10.47}$$

Here $E_i > E_f$ so the oscillator strength is negative.

There are several summation rules that restrict the values of the oscillator strengths,

$$\sum_{n'} f_{nn'} = N \tag{10.48}$$

where N is the total number of electrons in the atom and the summation is over all the states. One must include transitions to the continuum in this summation. If there is a closed shell of electrons we can focus on just the q electrons in the open shell to get

$$\sum_{n'} f_{nn'} = q \tag{10.49}$$

where the sum is over transitions that involve the outermost electrons. We can also separate the emission from absorption oscillator strengths

$$\sum_{n',E_n'>E_n} f_{nn'} + \sum_{n',E_n'<E_n} f_{nn'} = q \tag{10.50}$$

Because the second term is for stimulated emission. All of the values of $f_{nn'}$ are negative so

$$\sum_{n',E_n'>E_n} f_{nn'} \geq q. \tag{10.51}$$

Selection Rules

We can determine the selection rules for dipole emission by examining the defintion of the dipole matrix element

$$\mathbf{d}_{fi} \equiv e \int \psi_f^* \sum_j \mathbf{r}_j \psi_i d^3x \tag{10.52}$$

where j sums over the electrons in the atom. First let's calculate the dipole matrix element after a parity transformation that takes $\mathbf{r} \rightarrow -\mathbf{r}$. Unless $\psi_f * \psi_i$ is odd under the parity transformation, the integral will vanish, so the parity of the initial and final states must be different. The parity of a particular configuration is $(-1)^{\sum l_j}$ where l_j are the orbital angular momentum quantum numbers of each electron.

We can also prove that only one electron can change its state during the transition, *the one-electron jump rule*. If we examine the integral in detail, especially the spatial part we have

$$\Delta l = \pm 1, \Delta m = 0, \pm 1 \tag{10.53}$$

for the jumping electron. For the total configuration we have

$$\Delta S = 0, \Delta L = 0, \pm 1, \Delta J = 0, \pm 1 (\text{except } J = 0 \text{ to } J = 0) \tag{10.54}$$

The first condition holds because the dipole operator does not couple to the spin of the electrons and the final condition exists because a photon carries away one unit of angular momentum.

Transitions that follow these rules are know as *allowed* and they are written using the name of the species and the wavelength. For example, HI 1219Å is Lyman-α. On the other hand transitions that don't follow these rules can proceed through magnetic dipole or higher order multipole interactions. These transitions are called *forbidden* and are designated by [OII] 3727 Å. The transition between $J = 0$ and $J = 0$ cannot proceed through the emission of a single photon but through the emission of two photons. An example is the relaxation of the $2s$ state of to the $1s$ which has a lifetime of about 0.1 s (really slow) compared to $\sim$ 1ns for the $2p$ to the $1s$ state.

Bound-Free Transitions and Milne Relations

We have covered bound-bound transitions and free-free transitions (Bremmstrahlung). We would also like to understand bound-free transitions or ionization. In this case we have to calculate the transition rate for the atom to ionize with the freed electron to be in a particular range of momentum travelling

$$dw = \frac{8\pi^2}{3\hbar^2 c}|\mathbf{d}_{if}|^2 \left[\frac{dn}{dpd\Omega}dpd\Omega\right] J_\nu \tag{10.55}$$

in the dipole approximation.

We want to calculate an ionization cross section such that

$$dw = d\sigma \frac{dN}{dAdt} \tag{10.56}$$

where $dN/(dAdt)$ is the number of incident photons per unit area per unit time. We know that

$$\frac{dN}{dAdt} = 4\pi \frac{J_\nu}{\hbar\omega} d\nu = 4\pi \frac{J_\nu}{2\pi\hbar\omega} d\omega \tag{10.57}$$

Let's divide Eq. 10.63 by Eq. 10.64 to obtain

$$d\sigma = \frac{8\pi^2}{3\hbar^2 c} \frac{\hbar\omega}{2d\omega} |\mathbf{d}_{if}|^2 \left[\frac{dn}{dpd\Omega}dpd\Omega\right] \tag{10.58}$$

We know that by conservation of energy that the electron final momentum must satisfy

$$\hbar d\omega = \frac{pdp}{m}. \tag{10.59}$$

Furthermore, we know that if the electron is localized to a volume V, the density of states is

$$\frac{dn}{dpd\Omega} = \frac{p^2 V}{h^3} \tag{10.60}$$

Combining these results yields

$$\frac{d\sigma}{d\Omega} = \frac{pVm\omega}{6\pi c\hbar^3}|\mathbf{d}_{if}|^2. \tag{10.61}$$

Let's calculate the cross-section for a photon with $\hbar\omega \gg 13.6Z^2$ eV to ionize a hydrogen-like ion from the ground state. Because the energy of the outgoing electron is much greater than the binding energy of hydrogen it is safe to assume that

$$\psi_f = \frac{1}{\sqrt{V}} e^{i\mathbf{q}\cdot\mathbf{r}} \tag{10.62}$$

where $\mathbf{q} = \mathbf{p}/\hbar$.

The initial state is

$$\psi_i = \left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} e^{-Zr/a_0} \tag{10.63}$$

The dipole operator is

$$\mathbf{d} = \frac{ie}{m\omega_{if}} \mathbf{p} = -\frac{e}{m\omega_{if}} \hbar \nabla_{\mathbf{r}}. \tag{10.64}$$

We cannot use the simpler expression $\mathbf{d} = e\mathbf{r}$ because the final plane wave is not strictly an eigenstate of the Hamiltonian.

Let's apply it to the initial and final states

$$\mathbf{d}_{if} = -\frac{e\hbar}{m\omega_{if}} \frac{1}{\sqrt{V}} \left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} \int e^{-Zr/a_0} \nabla_{\mathbf{r}} e^{i\mathbf{q}\cdot\mathbf{r}} d^3x \tag{10.65}$$

$$= -\frac{ie\hbar\mathbf{q}}{m\omega_{if}} \frac{1}{\sqrt{V}} \left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} \int e^{-Zr/a_0} e^{i\mathbf{q}\cdot\mathbf{r}} d^3x \tag{10.66}$$

$$= -\frac{ie\hbar\mathbf{q}}{m\omega_{if}} \frac{1}{\sqrt{V}} \left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} 2\pi \int_0^\infty r^2 dr \int_{-1}^1 d\mu e^{-iqr\mu} e^{-Zr/a_0} \tag{10.67}$$

$$= -\frac{ie\hbar\mathbf{q}}{m\omega_{if}} \frac{1}{\sqrt{V}} \left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} \frac{4\pi}{q} \int_0^\infty r dr e^{-Zr/a_0} \sin qr \tag{10.68}$$

$$= -\frac{ie\hbar\mathbf{q}}{m\omega_{if}} \frac{1}{\sqrt{V}} \left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} \frac{8\pi a_0^3 Z}{\left(Z^2 + q^2 a_0^2\right)^2} \tag{10.69}$$

Let's substitute the value of ω_{if}

$$\hbar\omega_{if} = \frac{Z^2 e^2}{2a_0} + \frac{\hbar^2 q^2}{2m} = \frac{e^2}{2a_0}\left(Z^2 + \frac{\hbar^2 q^2}{2m} a_0\right) \tag{10.70}$$

to get

$$\mathbf{d}_{if} = -16\pi i \frac{1}{\sqrt{V}} \left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} \frac{a_0^5 Z e \mathbf{q}}{\left(Z^2 + q^2 a_0^2\right)^3} \tag{10.71}$$

and

$$|\mathbf{d}_{if}|^2 = \frac{256\pi}{V} \left(\frac{Z}{a_0}\right)^5 \left(\frac{Z^2}{a_0^2} + q^2\right)^{-6} e^2 q^2 \approx \frac{256\pi e^2}{V} \left(\frac{Z}{a_0}\right)^5 q^{-10}. \tag{10.72}$$

Using this in the formula for the differential cross-section and multiplying by 4π gives

$$\sigma_{bf} \approx \frac{16\sqrt{2}e^2\pi Z^5}{3m^{7/2}\omega^{7/2}ca_0^5} = \frac{(2\alpha)^{9/2}\pi Z^5 c^{7/2}}{3a_0^{3/2}\omega^{7/2}} \tag{10.73}$$

Had we used the classical dipole operator $\mathbf{d} = e\mathbf{r}$ we would have twice the true value of $\mathbf{d}_{if}$ and four times the cross section, so the difference is not subtle.

We can improve upon the assumption that we made that the electron's energy is much greater than the ionization energy by using Coulomb wavefunctions which are solutions to the Schrodinger equation for positive (*i.e.* continuum) energy values.

The total cross-section for a photon of frequency ω to ionize an electron from a hydrogenic atom in state n is

$$\sigma_{bf} = \left(\frac{64\pi ng}{3\sqrt{3}Z^2}\right)\alpha a_0^2 \left(\frac{\omega_n}{\omega}\right)^3 \tag{10.74}$$

where g is a Gaunt factor and

$$\omega_n = \frac{\alpha^2 mc^2 Z^2}{2\hbar n^2}. \tag{10.75}$$

More interesting is the fact that you can relate the cross section for ionization to that of recombination, through the *Milne relations*. These are derived using the principle of detailed balance simliar to the derivation of the Einstein relations.

If we assume that the photons are in equilibrium with a set of ions and atoms we can use the blackbody formula for the photon distirbution and the Saha equation for the ions. Let $\sigma_{fb}(v)$ be the cross section for recombination for electrons with velocity v, then we have a recombination rate per unit volume of

$$N_+ N_e \sigma_{fb} f(v) v dv \tag{10.76}$$

where $f(v)$ is the Maxwellian velocity distribution, N_e is the electron density and N_+ is the ion density.

The ionization rate is given by

$$\frac{4\pi}{h\nu} N_n \sigma_{bf} (1 - e^{-h\nu/kT}) B_\nu d\nu \tag{10.77}$$

where N_n is the number density of neutrals and the factor in front of the blackbody function accounts for stimulated recombination. These two rates must be equal in equilibrium so we have

$$\frac{\sigma_{bf}}{\sigma_{fb}} = \frac{N_+ N_e}{N_n} e^{h\nu/kT} \frac{f(v)c^2 h}{8\pi m\nu^2} \tag{10.78}$$

where we have used $h\nu = \frac{1}{2}mv^2 + E_I$ to eliminate $d\nu$ and dv.

We know that

$$\frac{N^+ N_e}{N} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \frac{2U^+(T)}{U(T)} e^{-E_I/kT}. \tag{10.79}$$

and

$$f(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2kT}\right) \tag{10.80}$$

Putting all of this together give us the Milne relation:

$$\frac{\sigma_{bf}}{\sigma_{fb}} = \frac{m^2 c^2 v^2}{\nu^2 h^2} \frac{g_e g_+}{2 g_n}. \tag{10.81}$$

Line Broadening Mechanisms

There are two types of broadening mechanisms. The first type is called *inhomogeneous broadening* which results from different atoms experiencing different conditions so the energy of the transition photons that we observe is different. Some examples of this are rotation, random bulk motions, thermal motions and varying magnetic field. In all but the last of these examples the energy of the photon is shifted due to the Doppler effect.

The profile function is

$$\phi(\nu) = \frac{1}{\Delta\nu_D \sqrt{\pi}} e^{-(\nu-\nu_0)^2/\Delta\nu_D^2} \tag{10.82}$$

where

$$\Delta\nu_D = \frac{\nu_0}{c}\left(\frac{2kT}{m_a} + \xi^2\right)^{1/2}. \tag{10.83}$$

The T is the temperature of the gas and m_a is the mass of the atoms.

The second type is called *homogeneous broadening*. Here each atom emits photons over a range of energies inherently. The main source of homogeneous broadening is that the atom has a finite lifetime or it can only emit phase-connected wavetrain for a finite time. Both of these effects result in a Lorentz profile for the line of the form

$$\phi(\nu) = \frac{\Gamma/4\pi^2}{(\nu-\nu_0)^2 + (\Gamma/4\pi)^2} \tag{10.84}$$

where

$$\Gamma = \gamma_u + \gamma_l + 2\nu_{\text{col}}. \tag{10.85}$$

The first two terms are the lifetime of the upper and lower states and ν_{col} is the frequency of collisions. For example,

$$\gamma_u = \sum_{n'<n} \left(A_{un} + B_{un} J_\nu\right). \tag{10.86}$$

The Gaussian convolution of the Lorentz profile profiles has a special name: the *Voigt function*

$$H(a,u) \equiv \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{a^2 + (u-y)^2} \tag{10.87}$$

and the combined profile is

$$\phi(\nu) = (\delta\nu_D)^{-1} \pi^{-1/2} H(a,u) \tag{10.88}$$

where

$$a \equiv \frac{\Gamma}{4\pi\delta\nu_D} \text{ and } u \equiv \frac{\nu - \nu_0}{\Delta\nu_D} \tag{10.89}$$

Problems

1. **Lifetime**

   Derive the lifetime of the $n = 2, l = 1, m = 0$ state of hydrogen to emit a photon and end up in the $n = 1, l = 0, m = 0$ state.

2. **Hydrogen-Like Absorption**

   How much energy does a photon need to ionize the following atoms by removing a K-shell electron?

   Hydrogen, Helium, Carbon, Oxygen, Iron

   Using the formula that I derived in class, draw an energy diagram that shows the total cross section for one gram of gas as a function of energy between 10eV and 10keV. It would be great if you used the initial expression in Eq. 10.72 for the dipole matrix element rather than the final answer given by Eq. 10.73.

   Consider that the mass fraction of the different atoms are hydrogen (0.7), helium (0.27), carbon (0.008), oxygen (0.016) and iron (0.004).

11
Molecular Structure

We are to deal with the structure and energetics of molecules in a very heuristic fashion, deriving the energies and importance of various transitions.

The simplest molecules are the diatomic molecules such as H_2, CO, *etc.*. They are also among the most abundant in the universe, so we are going to restrict our attention to these two-atom molecules.

The Born-Oppenheimer Approximation

The problem of understanding the structure of molecules initially appears formidable. At a basic level the equations are no longer spherically symmetric. This is a real difficulty. For diatomic molecules there is still a rotational symmetry about the line connecting the two nuclei. The key simplification is that the electrons whip around a lot faster than the nuclei, so one can approximate the sitution by assuming that the electrons sit in a particular eigenstate of the potential with the two ions fixed. The ions on the other hand experience an effective potential as a function of their separation that includes the effects of the electrons (whose state we have already calculated). This in the *Born-Oppenheimer* approximation.

By looking a moleculle in terms of the electrons and the nuclei separately, we can estimate the energies of the various transitions of the molecule. Let's assume that the ions are separated by a distance $a \sim a_0$. By the uncertainty principle the momentum of the electrons will be on the order of $\hbar/a$, and the typical energy of electronic transitions will be

$$E_{\text{elec}} \sim \frac{p^2}{m} \sim \frac{\hbar^2}{ma^2} \sim \alpha^2 mc^2 \sim 1 \text{ eV} \qquad (11.1)$$

or the visibile, near-infrared and ultraviolet.

The nuclei are separated by a distance of order a as well and the typical energy change from moving nuclei over a distance a is the electronic energy (Eq. 11.1), so we can define a spring constant for the nuclei

$$k \sim \frac{E_{\text{elec}}}{a^2} \sim \frac{\hbar^2}{ma^4} \tag{11.2}$$

yielding a vibrational energy corresponding to changes in the distance between the nuclei of

$$E_{\text{vib}} \sim \hbar\omega = \hbar\left(\frac{k}{M}\right)^{1/2} = \hbar\left(\frac{\hbar^2}{mMa^4}\right)^{1/2} = \left(\frac{m}{M}\right)^{1/2}\frac{\hbar^2}{ma^2} \sim 0.01 - 0.1 \text{ eV}. \tag{11.3}$$

These energies fall in the infrared. Finally the molecule can change its rotational state. The angular momentum of the molecule is quanized in units of $\hbar$ so we would expect transitions with energies of the order of

$$E_{\text{rot}} \sim \frac{\hbar^2 L(L+1)}{2I} = \frac{\hbar^2 L(L+1)}{Ma^2} = \frac{m}{M}\frac{\hbar^2}{ma^2}L(L+1) \sim 10^{-3} \text{ eV}. \tag{11.4}$$

Because the typical energies of the various transitions are well separated we can to a good approximate consider each of them separately, justifying the Born-Oppenheimer approximation.

The H_2^+ Molecular Ion

An example which illustrates much of the physics of diatomic molecules is the hydrogen molecular ion H_2^+. The Schrodinger equation for this system is

$$\left[-\frac{\hbar^2}{2\mu_{AB}}\nabla^2_{\mathbf{R}} - \frac{\hbar^2}{2\mu_e}\nabla^2_{\mathbf{r}} - \frac{e^2}{r_A} - \frac{e^2}{r_B} + \frac{e^2}{R} - E\right]\psi(\mathbf{r},\mathbf{R}) = 0 \tag{11.5}$$

μ_{AB} is the reduced mass of the two protons, $M/2$ and μ_e is the reduced mass of the electron relative to the two protons $\approx m_e$. r_A and r_B are the distances between the electron and the two protons and R is the distance between the two protons. The key to the Born-Oppenheimer approximation is first to hold $\mathbf{R}$ fixed and neglect the first kinetic energy term and solve for the electronic wavefunction $\chi_j(\mathbf{r};\mathbf{R})$.

$$\left[-\frac{\hbar^2}{2\mu_e}\nabla^2_{\mathbf{r}} - \frac{e^2}{r_A} - \frac{e^2}{r_B} + \frac{e^2}{R} - E_j(\mathbf{R})\right]\chi_j(\mathbf{r};\mathbf{R}) = 0 \tag{11.6}$$

where the semicolon in the χ_j function encourages us to think of $\mathbf{R}$ as a parameter. We try various values of $\mathbf{R}$ and solve for $\chi_j(\mathbf{r};\mathbf{R})$ each time. The solutions to this equation are called *molecular orbitals*.

After the electronic wavefuntion is calculated as a function of R, we can determine the proton wavefunction. The proton wavefunction satisfies the one-body Schrodinger equation

$$\left[-\frac{\hbar^2}{2\mu_{AB}} \nabla^2_{\mathbf{R}} + E_j(\mathbf{R}) - E \right] F_j(\mathbf{R}) = 0 \tag{11.7}$$

Eq. 11.6 is generally to difficult to solve directly, so one generally picks a trial wavefunction and calculates the value of the energy for this function. One can prove that the ground state eigenvalue E of the Hamiltonian H

$$E \leq \langle \psi | H | \psi \rangle \tag{11.8}$$

where ψ is any normalized wavefunction. This is the basis of the *Rayleigh-Ritz variational method*.

For the case in point, we will guess that the wavefunction of the electron is a **l**inear **c**ombination of **a**tomic **o**rbitals (LCAO), specifically the 1s state of hydrogen.

$$\chi_g = \frac{1}{\sqrt{2}} \left(\psi_{1s}(r_A) + \psi_{1s}(r_B) \right) \tag{11.9}$$

and

$$\chi_u = \frac{1}{\sqrt{2}} \left(\psi_{1s}(r_A) - \psi_{1s}(r_B) \right) \tag{11.10}$$

The g and the u refer to *gerade* (even-parity) and *ungerade* (odd-parity) wavefunctions.

We can substitute these trial wavefunctions into the Hamiltonian in Eq. (11.6) to find an upper limit on the value of $E_j(\mathbf{R})$. We obtain

$$E_{gu}(R) = E_{1s} + \frac{e^2}{R} \frac{(1 + R/a_0)e^{-2R/a_0} \pm [1 - (2/3)(R/a_0)^2]e^{-R/a_0}}{1 \pm [1 + R/a_0 + (R/a_0)^2/3]e^{-R/a_0}} \tag{11.11}$$

where the upper (positive) sign corresponds to the *gerade* configuration. Fig. 11.1 depicts the energy of the electronic configuration and Fig. 11.2 shows the electron density for the two orbitals. We see that the only the gerade state binds in the case. From the picture of the electron probability density we can see why this is the case. In the gerade case, the electron lies in between the two ions so it can shield the charge of one ion from the other. In the ungerade state because it has odd parity, the electron cannot lie on the midplane between the ions so the shielding is much less effective.

For molecules with more than one electron, we find that the Hund's rule for the total spin of a system is reversed for molecules. The states with even parity (gerade) tend to bond. Because the spatial wavefunction is even with respect interchanging the electrons their spins must be antiparallel.

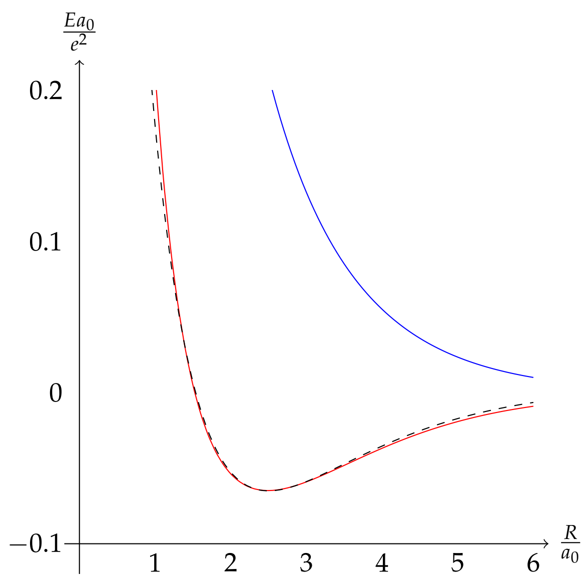


Figure 11.1: E_u (upper curve) and E_g (lower curve) for H_2^+. The dashed curve is a well-fit Morse potential for $E_g(R)$.

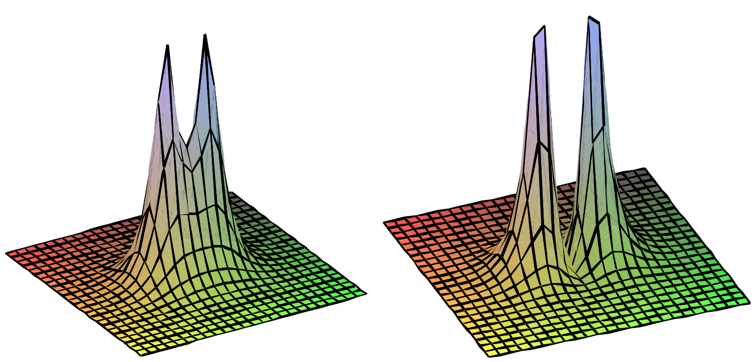


Figure 11.2: $|\psi_g|^2$ (left) and $|\psi_u|^2$ (right) for H_2^+ with $R = 2a_0$.

Molecular Excitations

The energy states of molecule may be excited in three ways: *electronic*, *vibrational*, and *rotational*. Let's start with the least energetic of these.

We can get a first-order understanding of the rotational states of a molecule simply looking at the Schrodinger equation for the ions

$$\left[-\frac{\hbar^2}{2\mu_{AB}}\left(\frac{d^2}{dR^2}-\frac{L(L+1)}{R^2}\right)+E_j(\mathbf{R})-E\right]F_j(\mathbf{R})=0 \qquad (11.12)$$

where we have solved the angular wavefunction in terms of spherical harmonics like we did for hydrogen. In Fig. 11.1 we saw that the function $E_j(\mathbf{R})$ varies over atomic distances $\sim a_0$. On the other hand because the mass of the ions is much larger than that of the electrons we expect the wavefunction of the ions to be localized in a region $\sim a_0 m/M \ll a_0$. Over this small region we can expand the function $E_j(R)$ about its minimum R_0

$$E_j(R)=E_j(R_0)+(R-R_0)\left[\frac{dE_j(R)}{dR}\right]_{R=R_0}+\frac{1}{2}(R-R_0)^2\left[\frac{d^2E_j(R)}{dR^2}\right]_{R=R_0}+\cdots \qquad (11.13)$$

where the second term vanishes because R_0 is the minimum so we have

$$\left[-\frac{\hbar^2}{2\mu_{AB}}\frac{d^2}{dR^2}-\frac{\hbar^2}{2\mu_{AB}}\frac{L(L+1)}{R^2}+E_j(R_0)+\frac{1}{2}k(R-R_0)^2-E\right]F_j(\mathbf{R})=0 \qquad (11.14)$$

so we have

$$E=E_j(R_0)+\hbar\omega_0\left(v+\frac{1}{2}\right)+\frac{\hbar^2}{2\mu_{AB}}\frac{L(L+1)}{R_0^2} \qquad (11.15)$$

where $\omega_0=(k/\mu_{AB})^{1/2}$ and we have treated the rotational motion of the molecule perturbatively. We have effectively ignored the possible centrifugal stretching of the molecule. If we were to include the stretching of the molecule we would have

$$E_{\rm rot}=\frac{\hbar^2}{2\mu_{AB}}\frac{L(L+1)}{R_0^2}\left[1-\frac{2\hbar^2L(L+1)}{k\mu_{AB}R_0^4}\right] \qquad (11.16)$$

We can get dipole transitions between the different rotational states if

$$|d|=Z_1er_1+Z_2er_2+|d_e|\neq 0 \qquad (11.17)$$

and $\Delta L=\pm 1$. We see that homonuclear diatomic molecules cannot emit dipole radiation due to changes in their rotational state. The energy of the radiation is given by

$$E_{L+1,L}=\frac{\hbar^2(L+1)}{\mu_{AB}R_0^2}\left[1-4\frac{\hbar^2(L+1)^2}{k\mu_{AB}R_0^4}\right] \qquad (11.18)$$

This energy is $\sim e^2/a_0(m/\mu_{AB})$ or $\sim 10^{-3}$ eV.

The transitions between vibrational states has a typical energy of $\sim e^2/a_0(m/\mu_{AB})^{1/2}$ or $\sim 10^{-1}$ eV. If one includes the centrifugal effects one finds that

$$E_v = \hbar\omega_L\left(v+\frac{1}{2}\right) \tag{11.19}$$

where

$$\omega_L \approx \mu_{AB}^{-1/2}\left[k+\frac{3\hbar^2 L(L+1)}{\mu_{AB}R_0^4}\right]^{1/2} \tag{11.20}$$

Morse found that the internuclear potential can often be well approximated by a function of the form

$$E_n(R) = E_{n,0} + B_n\left\{1-\exp\left[\beta_n\left(R-R_0\right)\right]\right\}^2 \tag{11.21}$$

The energy eigenvalues of this potential are

$$E_{nv} = \hbar\omega_{n0}\left(v+\frac{1}{2}\right) - \frac{\hbar^2\omega_0^2}{4B_n}\left(v+\frac{1}{2}\right)^2. \tag{11.22}$$

The vibrational levels get closer together as v increases and there are a finite number of vibrational levels

$$0 \leq v \leq \frac{(2\mu_{AB}B_n)^{1/2}}{\beta_n\hbar} - \frac{1}{2} \tag{11.23}$$

The selection rules for vibrational transtions are again $|d| \neq 0$ but also $d|d|/dR \neq 0$. We can change the vibrational level by $\Delta v = \pm 1$ and we must also have $\Delta L = L_{\rm lower} - L_{\rm upper} = +1$ (P branch) or $\Delta L = -1$ (R branch) or if there is an component of electronic orbital or spin angular momentum along the internuclear axis $\Delta L = 0$ (Q branch).

Fig. 11.3 shows the three branches for roto-vibrational transitions, if we neglect the stretching of the molecule. We see that for the R branch the transition energy decreases with increasing L. For the Q branch it is constant, and for the P branch the transition energy increases with increase L. The centrifugal stretching reduces the spacing of the angular momentum energy levels for large values of L (Eq. 11.18), but it stiffens the spring constant of the vibrational statres (Eq. 11.20). The latter dominates, so the stretching effect tends to make the transition energies increase with increasing L for large values of L. Fig. 11.4 depicts the roto-vibrational spectrum of CO (the most commonly observed molecule in astrophysics — it isn't the most common, what is the most common molecule and why isn't it commonly observed?) from samples of car exhaust. The CO molecule does not exhibit a Q-branch which would appear at about 2140 cm^{-1}.

The Fortrat diagram (Fig. 11.5) depicts the transition energies for various roto-vibrational transitions as a function of the rotational quantum number L. The figure shows the expected behaviour, especially for the "22" transitions for which all three branches are depicted. For small values of L the Q-branch has constant energy with L and then begins to increase with increasing L The energy of the P-branch transitions decreases with increasing L, and the opposite occurs for the R branch.

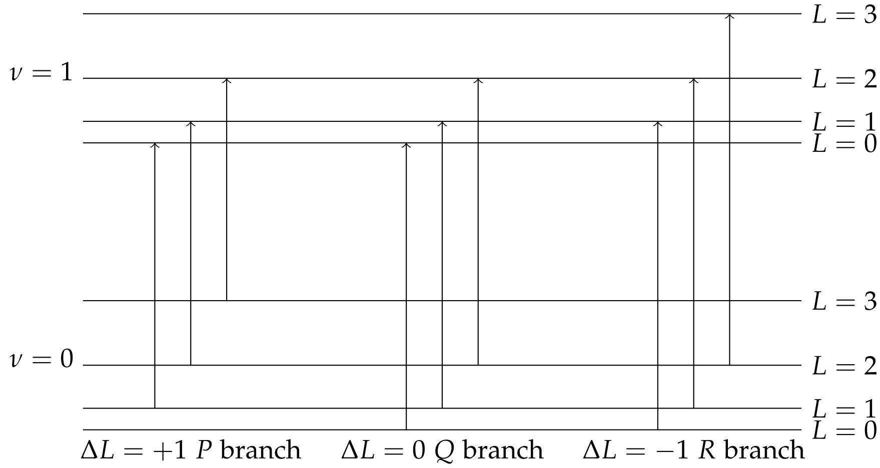


Figure 11.3: Roto-Vibrational Transitions Neglecting Stretching

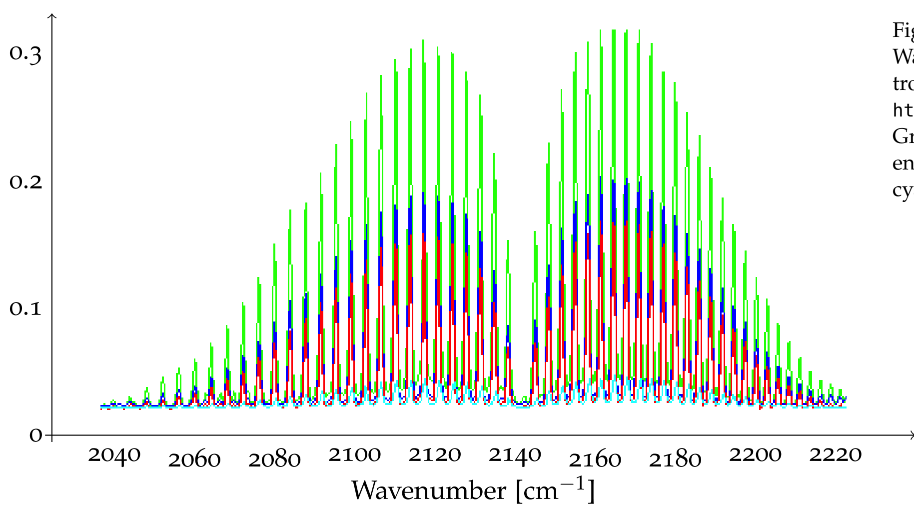


Figure 11.4: Absorption vs. Wavenumber for FTIR Spectroscopy of CO in car exhaust from `http://home.swipnet.se/~w-74877/ftir/ftir.htm`. Green is a cold engine, blue is a warm engine, red is a calibration reading and cyan is a 100ppm calibration.

In general each vibration transition includes a rotational transition as well so one gets group of transitions. The final wrinkle is that electronic transitions in molecules whose energy ~ 1 eV necessarily include changes in the rotational and vibrational state of the molecule. The general electronical-vibrational-rotational spectrum

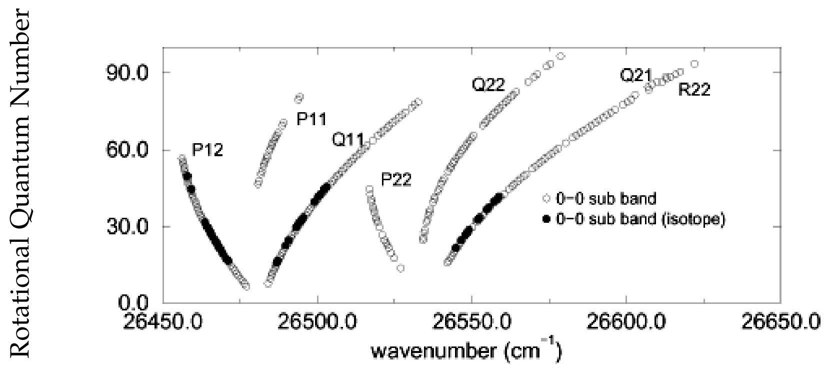


Figure 11.5: A Fortrat diagram for $^{24}Mg^{35}Cl$ and $^{24}Mg^{37}Cl$ (isotope) from Gutterres et al. (2003), Braz. J. Phys. vol. 33

takes the form of bands which can be resolved into individual lines if the broadening is weak.

Problems

1. **The Number of Levels**

   I fit a Morse function to the potential of H_2^+. The parameters were

   $$E_{n,0} = -0.065\frac{e^2}{a_0}, B_n = 0.07\frac{e^2}{a_0}, \beta_n = 0.7a_0^{-1}, R_0 = 2.5a_0 \qquad (11.24)$$

   How many vibrational levels does H_2^+ have? How many rotational levels does each vibrational level typically have?

2. **Nuclear Overlap**

   Consider two deuterons bound by a single electron as in question (1). What is the probability that the two deuterons lie on top of each other, *i.e.* that $R < 4$ fermi, the diameter of the deuteron? What is the probability if the two deuterons are bound by a single muon, $m_\mu \approx 207 m_e$? You can find the eigenfunctions of the Morse potential on Wikipedia.

   If you assume that whenever the deuterons overlap they fuse and that you get to "roll the dice" once each oscillation period, calculate the fusion rate in both cases.

3. **Stretching**

   Calculate the value of L for which the energy of the P branch transitions begins to increase.

4. **Temperature**

   Using the results depicted in Fig. 11.4, estimate the temperature of the hot and cold car exhaust and the relative concentration of CO in the two cases.

Part IV

Fluids

12
Fluid Mechanics

Phase-Space Density

The phase-space density of particles gives the number of particles in an infinitesimal region of phase space,

$$dN = f(x^\alpha, \mathbf{p}) d^3\mathbf{x} d^3\mathbf{p} \tag{12.1}$$

If there is no dissipation, the phase-space density along the trajectory of a particular particle is given by

$$\frac{df}{d\tau} = \mathcal{C} \tag{12.2}$$

where $\mathcal{C}$ accounts for two-body interactions between particles. This is known as the Boltzmann equation. If there are no collisions, $\mathcal{C} = 0$, so

$$\frac{df}{d\tau} = 0. \tag{12.3}$$

This is called Lioville's theorem or the collisionless Boltzmann equation. This limit applies in galactic dynamics. Here we are interested in particles in a gas that do collide so we expand out the derivative along the flow lines to get

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{p}} = \mathcal{C} \tag{12.4}$$

where $\mathbf{F}$ is a force that accelerates the particles. The collision term $\mathcal{C}$ must now be expressed in the lab frame of this equation that is no longer manifestly covariant. The requirement of no dissipation tells us that $\nabla_\mathbf{p} \cdot \mathbf{F} = 0$.

We would like to define some quantities that are integrals over momentum space that transform simply under Lorentz transformations. We derived earlier (§ 4) that

$$\frac{d^3\mathbf{p}}{p_t} = \frac{d^3\mathbf{p}'}{p'_t}. \tag{12.5}$$

We also know that $E_{\mathbf{p}} = p_t = (p^2c^2 + m^2c^4)^{1/2}$ so

$$\frac{d^3\mathbf{p}}{E_{\mathbf{p}}} = \text{Lorentz Invariant} \tag{12.6}$$

so we can define various integrals

$$\frac{n(x^\alpha)}{\bar{E}_{\text{har}}(x^\alpha)} \equiv \int \frac{d^3\mathbf{p}}{E_{\mathbf{p}}} f(x^\alpha, \mathbf{p}) \tag{12.7}$$

that transforms as a scalar where $n(x^\alpha)$ is the number density.. Unfortunately, there isn't much that one can do with it. One could use it as the source for a scalar theory of gravity, but it would violate the equivalence principle.

Particle Current

Next let's define

$$J^\mu(x^\alpha) \equiv c \int \frac{d^3\mathbf{p}}{E_{\mathbf{p}}} p^\mu f(x^\alpha, \mathbf{p}). \tag{12.8}$$

Because this is simply the sum of things that transform as a four-vector, J^μ also transforms as a four-vector. Let's look at it component by component

$$J^0(x^\alpha) = \int \frac{d^3\mathbf{p}}{E_{\mathbf{p}}} c p^0 f(x^\alpha, \mathbf{p}) = \int d^3\mathbf{p} f(x^\alpha, \mathbf{p}) = n(x^\alpha) \tag{12.9}$$

$$\mathbf{J}(x^\alpha) = \int \frac{d^3\mathbf{p}}{E_{\mathbf{p}}} c\mathbf{p} f(x^\alpha, \mathbf{p}) = \frac{1}{c} \int d^3\mathbf{p}\mathbf{v} f(x^\alpha, \mathbf{p}) = \frac{\langle \mathbf{v} \rangle}{c} n(x^\alpha) \tag{12.10}$$

If we assume that the scattering ($\mathcal{C}$) conserves energy, momentum and particles we have

$$J^\mu{}_{,\mu} = \frac{\partial J^\mu}{\partial x^\mu} = n \langle \nabla_{\mathbf{p}} \cdot \mathbf{F} \rangle. \tag{12.11}$$

We can prove this simply by integrating Eq. 12.4 over $d^3\mathbf{p}$. The first two terms yield the left-hand side of the equation above. The third term gives the right-hand side. This vanishes as long as

$$\nabla_{\mathbf{p}} \cdot \mathbf{F} = 0 \tag{12.12}$$

and the right-hand side vanishes if the scattering conserves energy and momentum.

Let's define $\mathbf{V} = \langle \mathbf{v} \rangle$ and write out Eq. 12.12 by components,

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{V}) = 0 \tag{12.13}$$

or

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{V}) = 0 \tag{12.14}$$

where $\rho = mn$ where m is the rest mass of the individual particles. This is the continuity equation. This must hold true regardless of the nature of the force, *i.e.* even if $\nabla_{\mathbf{p}} \cdot F \neq 0$. Because Eq. (12.11) is consistent with the Lioville equation (Eq. 12.3) and more generally with the Boltzmann equation (Eq. 12.2) and $J^{\mu}_{;\mu} = 0$ if particles are conserved, the Lioville and Boltzmann equations cannot hold if $\nabla_{\mathbf{p}} \cdot F \neq 0$ and particles are conserved.

Stress Tensor

Let's construct a tensor from the distribution function

$$T^{\mu\nu}(x^{\alpha}) \equiv c^2 \int \frac{d^3\mathbf{p}}{E_{\mathbf{p}}} p^{\mu} p^{\nu} f(x^{\alpha}, \mathbf{p}). \tag{12.15}$$

this is called the energy-momentum tensor or stress tensor of the system. Let's take one of the indices to be zero

$$T^{\mu 0}(x^{\alpha}) = c^2 \int \frac{d^3\mathbf{p}}{E_{\mathbf{p}}} p^{\mu} \frac{E_{\mathbf{p}}}{c} f(x^{\alpha}, \mathbf{p}) = c \int d^3\mathbf{p} p^{\mu} f(x^{\alpha}, \mathbf{p}) \tag{12.16}$$

which is the product of the total four momentum of the particle per unit volume with c. $T^{00}(x^{\alpha})$ gives the energy-density and the component $T^{0\mu}(x^{\alpha})$ gives the density of the $\mu-$component of the three-momentum.

We are free to fix a Lorentz frame that is moving with the material such that $J^{\mu} = 0$ for $\mu \neq 0$. If we are willing to neglect effects that depend on the gradient of the velocity (such as viscosity or heat conduction) we can define this frame globally. Furthermore, let's assume that the distribution function is isotropic in the momentum. Fluids for which this is possible are called *ideal fluids*. In this case we have

$$J^0 = 4\pi \int_0^{\infty} p^2 f(p) dp, J^{\mu} = 0 \tag{12.17}$$

and

$$\epsilon = T^{00} = 4\pi \int_0^{\infty} p^2 E(p) f(p) dp, T^{0\mu} = 0 \tag{12.18}$$

The space-space part of the energy-momentum tensor must be symmetric, isotropic and a three-dimensional tensor (a matrix). The only tensor that works is

$$T^i{}_j = P\delta^i{}_j \tag{12.19}$$

where

$$T^i{}_i = P\delta^i{}_i = 3P = c^2 \int \frac{d^3\mathbf{p}}{E_p} p^2 f(p) = 4\pi c^2 \int_0^{\infty} \frac{p^4}{E(p)} f(p) dp \tag{12.20}$$

so

$$P = \frac{4\pi}{3} c^2 \int_0^{\infty} \frac{p^4}{E(p)} f(p) dp. \tag{12.21}$$

Notice that the trace of the energy-momentum tensor $T^{\mu}{}_{\mu}$ is a scalar. In fact it is simply the product of m^2c^4 with the scalar density defined in Eq. 12.7.

Let's look at the non-relativistic limit of the energy-momentum tensor. Let's take $T^{00} = mc^2 n + \epsilon_{\rm nr}$ where

$$T^{00} = 4\pi \int_0^\infty p^2 \left(mc^2 + \frac{p^2}{2m} \right) f(p) dp = nmc^2 + \epsilon_{\rm nr} \tag{12.22}$$

where

$$\epsilon_{\rm nr} = 4\pi \int_0^\infty p^2 \frac{p^2}{2m} f(p) dp = \frac{2\pi}{m} \int_0^\infty p^4 f(p) dp. \tag{12.23}$$

Let's look at the non-relativistic limit of the pressure

$$P_{\rm nr} = \frac{4\pi}{3} c^2 \int_0^\infty \frac{p^4}{mc^2} f(p) = \frac{4\pi}{3m} \int_0^\infty p^4 f(p) dp \tag{12.24}$$

so $\epsilon_{\rm nr} = \frac{3}{2} P_{\rm nr}$. Now let's take the opposite limit

$$\epsilon_{\rm ur} = T^{00} = 4\pi \int_0^\infty p^2 (pc) f(p) dp, p_{\rm ur} = \frac{4\pi}{3} c^2 \int_0^\infty \frac{p^4}{pc} f(p) dp \tag{12.25}$$

so $P = \epsilon/3$ in the ultrarelativistic limit. We can transform from this special frame to a frame where the fluid moves and get

$$T^{\mu}{}_{\nu} = (P + \epsilon) U^\mu U_\nu - P \delta^{\mu}{}_{\nu} \tag{12.26}$$

and

$$J^\mu = n_{\rm prop} U^\mu \tag{12.27}$$

$n_{\rm prop}$ is the number density of the particles in a frame moving with the particles and U^μ is the bulk four-velocity of the fluid.

We can calculate the evolution of this tensor by integrating over Lioville's equation times p^μ to get

$$T^{\mu\nu}{}_{,\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\nu} = \begin{cases} n \langle \mathbf{v} \cdot \mathbf{F} \rangle, & \mu = 0 \\ cn \langle \mathbf{F} \rangle, & \mu > 0 \end{cases} . \tag{12.28}$$

We prove this by integrating Eq. 12.4 times p^μ over $d^3\mathbf{p}$. The zero-component is simply the work performed by the force on the particles in the volume. The other components account for the change in the momentum of the particles.

Non-relativistic limit Let's examine what this equation means in the non-relativistic limit,

$$T^{00} = c^2 \int d^3\mathbf{p} f(p) mc^2 \left(1 + \frac{1}{2} \frac{v^2}{c^2} \right) = \rho c^2 + \int d^3 p f \left(\frac{1}{2} m v^2 \right) = \rho c^2 + n \langle E \rangle \tag{12.29}$$

and

$$T^{0\nu} = c^2 \int d^3\mathbf{p} f mcv^\nu \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) = \left(\rho c\mathbf{V} + \frac{1}{c}\mathbf{q}\right)^\nu \tag{12.30}$$

where

$$\mathbf{q} = \int d^3\mathbf{p} \left(\frac{1}{2}mv^2\right)\mathbf{v}f \tag{12.31}$$

is the flux of kinetic energy. The first term is the flux of rest-mass energy. Finally for the space-space part we have

$$T^{\mu\nu} = \int d^3\mathbf{p} mv^\mu v^\nu f. \tag{12.32}$$

Let's look at the zero-component of Eq. 12.28

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\rho c^2 + n\langle E\rangle\right) + \nabla \cdot \left(\rho c\mathbf{V} + \frac{\mathbf{q}}{c}\right) = n\langle \mathbf{v}\cdot\mathbf{F}\rangle \tag{12.33}$$

We can subtract c times the continuity equation to get

$$\frac{\partial n\langle E\rangle}{\partial t} + \nabla\cdot\mathbf{q} = n\langle\mathbf{v}\cdot\mathbf{F}\rangle. \tag{12.34}$$

This ensures conservation of energy. We can divide the energy from the bulk flow from the random kinetic energy of the fluid

$$N\langle E\rangle = N\left\langle \frac{1}{2}m\left(\mathbf{V}+\mathbf{v}_r\right)^2\right\rangle = \frac{1}{2}\rho V^2 + \frac{1}{2}\rho\langle v_r^2\rangle = \frac{1}{2}\rho V^2 + \frac{3}{2}NT, \tag{12.35}$$

defining the temperature T of the fluid.

The spatial part of Eq. 12.29 gives

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\rho c\mathbf{V} + \frac{\mathbf{q}}{c}\right)^i + \frac{\partial T^{ik}}{\partial x^k} = n\langle\mathbf{F}\rangle. \tag{12.36}$$

The first term in the parentheses is larger by a factor of c^2 so to lowest order we have

$$\frac{\partial}{\partial t}\left(\rho V^i\right) + \frac{\partial T^{ik}}{\partial x^k} = n\langle\mathbf{F}\rangle. \tag{12.37}$$

This is equivalent to neglecting the momentum carried by the flow of energy.

Ideal Fluids

For an ideal fluid we found that the stress tensor took a particular form,

$$T^{\mu\nu} = (P+\epsilon)U^\mu U^\nu - Pg^{\mu\nu} \tag{12.38}$$

In the non-relativistic limit we find that space-space components are

$$T_{ik} = \rho V_i V_k + P\delta_{ik} \tag{12.39}$$

and

$$\mathbf{q} = \left[\frac{1}{2}V^2 + w\right]\rho\mathbf{V} \tag{12.40}$$

where $w = (\epsilon + P)/\rho$ is the heat function (enthalpy) per unit mass of the fluid. Notice that there is no energy flow without bulk motion. If we substitute this into the equations derived earlier we get

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{V}) = \frac{\partial\rho}{\partial t} + (\mathbf{V}\cdot\nabla)\,\rho + \rho\nabla\cdot\mathbf{V} = \frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{V} = 0 \tag{12.41}$$

and

$$\frac{\partial\mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\,\mathbf{V} = \frac{d\mathbf{V}}{dt} = -\frac{\nabla P}{\rho} + \frac{\langle\mathbf{F}\rangle}{m} \tag{12.42}$$

In the ideal fluid, no heat is tranferred between different parts of the fluid, so if we denote s as the entropy per unit rest mass we have

$$\frac{ds}{dt} = 0 \tag{12.43}$$

for a bunch of fluid; therefore, we also have a continuity equation for the entropy

$$\frac{\partial(\rho s)}{\partial t} + \nabla\cdot(\rho s\mathbf{v}) = 0. \tag{12.44}$$

We can use the continuity equation for particle number to simplify this further,

$$\rho\frac{\partial s}{\partial t} + \mathbf{v}\cdot\nabla(\rho s) = 0. \tag{12.45}$$

Isentropic flows

An important case of adiabatic flows is when the entropy s is initially constant. In such an isentropic flow, the entropy will remain constant and we can derive some additional useful forms of Eq. 12.42.

From the definition of the work function w and thermodynamics we know that

$$dw = Tds + \frac{1}{\rho}dp = \frac{dp}{\rho} \tag{12.46}$$

because the entropy is constant, $ds = 0$. We get

$$\frac{\partial\mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\,\mathbf{V} = \frac{d\mathbf{V}}{dt} = -\nabla w + \frac{\langle\mathbf{F}\rangle}{m} \tag{12.47}$$

From vector analysis we know

$$\frac{1}{2}\nabla v^2 = \mathbf{v}\times(\nabla\times\mathbf{v}) + (\mathbf{v}\cdot\nabla)\mathbf{v} \tag{12.48}$$

and we can get

$$\frac{\partial\mathbf{V}}{\partial t} + \frac{1}{2}\nabla v^2 - \mathbf{v}\times(\nabla\times\mathbf{v}) = -\nabla w + \frac{\langle\mathbf{F}\rangle}{m}. \tag{12.49}$$

If we take the curl of both sides we find that

$$\frac{\partial}{\partial t}\left(\nabla \times \mathbf{V}\right) = \nabla \times \left(\mathbf{V} \times \left(\nabla \times \mathbf{V}\right)\right) + \nabla \times \frac{\langle \mathbf{F} \rangle}{m} \tag{12.50}$$

$\omega = \nabla \times \mathbf{V}$ is called the vorticity.

If we assume that $\mathbf{F}/m = -\nabla \phi$ which is often the case, we find that if the flow in an isentropic, ideal fluid is initially irrotational it will remain irrotational.

We can go further than this. Let's define

$$\Gamma = \oint \mathbf{v} \cdot d\mathbf{l} \tag{12.51}$$

taken along some closed contour that moves with the fluid. Let's calculate

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\oint \mathbf{v} \cdot d\mathbf{l} = \frac{d}{dt}\oint \mathbf{v} \cdot \delta\mathbf{r} = \oint \frac{d\mathbf{v}}{dt} \cdot \delta\mathbf{r} + \oint \mathbf{v} \cdot \frac{d}{dt}\delta\mathbf{r}. \tag{12.52}$$

Because $\delta\mathbf{r}$ is the difference between two positions moving with the fluid we have

$$\mathbf{v} \cdot \frac{d}{dt}\delta\mathbf{r} = \mathbf{v} \cdot \delta\frac{d\mathbf{r}}{dt} = \mathbf{v} \cdot \delta\mathbf{v} = \delta v^2 \tag{12.53}$$

so

$$\frac{d}{dt}\oint \mathbf{v} \cdot d\mathbf{l} = \oint \frac{d\mathbf{v}}{dt} \cdot d\mathbf{l} = \oint \left(-\nabla w + \frac{\langle \mathbf{F} \rangle}{m}\right) d\mathbf{l} = 0 \tag{12.54}$$

if $\mathbf{F}/m = -\nabla\phi$, so the circulation around a contour moving with the fluid is constant if the flow is isentropic.

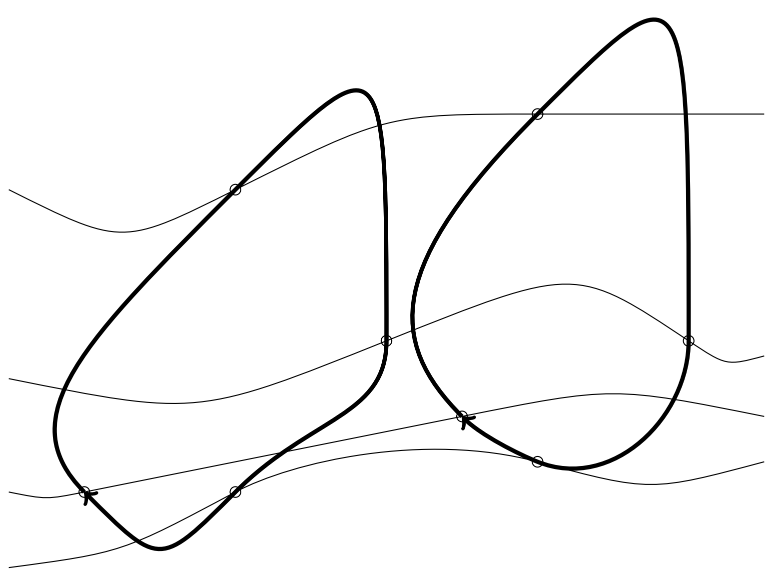


Figure 12.1: The circulation around a close contour (bold lines) that travels with the fluid along the streamlines (light lines) is conserved if the fluid is isentropic.

Hydrostatics

Let's assume that the fluid is not moving. If we look at Eq. 12.43 we get the equation of hydrostatic equilibrium. Let's further suppose that the force is derived from a potential, we obtain,

$$\frac{\nabla P}{\rho} = -\nabla\phi. \tag{12.55}$$

Let's take the divergence of both sides

$$\nabla \cdot \left(\frac{\nabla p}{\rho}\right) = -\nabla^2\phi = -4\pi G\rho \tag{12.56}$$

and in spherical coordinates we have

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho. \tag{12.57}$$

An interesting and important case is when the fluid is isentropic as well, then we have

$$\nabla w = -\nabla\phi. \tag{12.58}$$

and

$$\nabla^2 w = -4\pi G\rho \tag{12.59}$$

If we look at the Eq. 12.59 and imagine that the fluid is rotating we have an extra term,

$$\nabla w = -\nabla\phi + \Omega^2(\mathbf{r})\mathbf{R} \tag{12.60}$$

where **R** is a vector pointing from the rotation axis to the point in the fluid. Let's take the curl of both sides to get

$$0 = 0 + \nabla \times \left(\Omega^2(\mathbf{r})\mathbf{R}\right) \tag{12.61}$$

which tells us that

$$\Omega(\mathbf{r}) = \Omega(R) \tag{12.62}$$

or that isentropic stars must have constant angular velocity on cylindrical surfaces.

Really Little Sound Waves

The next order of complexity is to assume that the fluid is at rest with a small perturbation and to see what the perturbation does. We have

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{V}) = \frac{\partial\rho'}{\partial t} + \rho_0\nabla\cdot\mathbf{V}' = 0 \tag{12.63}$$

and

$$\frac{\partial\mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\,\mathbf{V} = \frac{d\mathbf{V}}{dt} + \frac{\nabla P}{\rho} = \frac{\partial\mathbf{V}'}{\partial t} + \frac{\nabla P'}{\rho_0} = 0. \tag{12.64}$$

We can write $P' = (\partial P/\partial\rho)_s \rho'$ and rewrite the continuity equation to get

$$\frac{\partial P'}{\partial t} + \rho_0 \left(\frac{\partial P}{\partial \rho}\right)_s \nabla \cdot \mathbf{V}' = 0 \tag{12.65}$$

Let's take the divergence of the Euler equation to get

$$\frac{\partial \nabla \cdot \mathbf{V}'}{\partial t} + \frac{\nabla^2 P'}{\rho_0} = 0. \tag{12.66}$$

and the time derivative of the continuity equation to get

$$\frac{\partial^2 P'}{\partial t^2} + \rho_0 \left(\frac{\partial P}{\partial \rho}\right)_s \nabla \cdot \frac{\partial \mathbf{V}'}{\partial t} = 0. \tag{12.67}$$

Finally we put the two together to get

$$\frac{\partial^2 P'}{\partial t^2} - \left(\frac{\partial P}{\partial \rho}\right)_s \nabla^2 P' = 0. \tag{12.68}$$

This is a wave equation with a sound speed of $c_s^2 = (\partial P/\partial\rho)_s$. Let's take a solution to this equation for the pressure,

$$P' = p' \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)] \tag{12.69}$$

with $k^2 c_s^2 = \omega^2$ and calculate the velocity of the fluid

$$\mathbf{V}' = \mathbf{v}' \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)] \tag{12.70}$$

From Eq. 12.65 we get

$$-\omega \mathbf{v}' + \frac{p'}{\rho_0}\mathbf{k} = 0 \tag{12.71}$$

and from Eq. 12.66 we get

$$-\omega p' + \rho_0 c_s^2 \mathbf{k} \cdot \mathbf{v}' = 0. \tag{12.72}$$

Combining these results gives

$$-\omega \mathbf{v}' + \frac{c_s^2 \mathbf{k} \cdot \mathbf{v}'}{\omega}\mathbf{k} = 0 \tag{12.73}$$

and rearranging

$$\mathbf{v}' = \frac{c_s^2 \mathbf{k} \cdot \mathbf{v}'}{\omega^2}\mathbf{k} = v' \frac{\mathbf{k}}{k}. \tag{12.74}$$

Therefore, the fluid is displaced in the direction of the propagation of the wave; it is a longitudinal wave.

Steady Supersonic Flow

Many disturbances travel through a fluid at a finite speed (changes in the entropy or vorticity move with the fluid). If the fluid itself travels faster than the speed of sound, a disturbance starting at particular point only can travel downstream so the upsteam flow cannot know about it. A flow can become supersonic abruptly as in a shock or continuously. We will examine this latter case here.

Let's imagine that a fluid is flowing through a pipe of variable cross section $A(x)$ and that the flow is steady so that all partial time derivatives vanish. We can write the continuity equation as $\rho v A =$constant. The Euler equation becomes

$$v\frac{dv}{dt} = -\frac{1}{\rho}\frac{dp}{dt} = -\frac{c_s^2(\rho)}{\rho}\frac{d\rho}{dt} \tag{12.75}$$

where we have assumed that the fluid flows in the $x-$direction. From the continuity equation we know

$$-\frac{1}{A}\frac{dA}{dt} = \frac{1}{\rho v}\frac{d(\rho v)}{dt} = \frac{1}{\rho v}\left(v\frac{d\rho}{dt} + \rho\frac{dv}{dt}\right). \tag{12.76}$$

We can combine the two equations to get

$$\frac{d\ln A}{dt} = \frac{c_s^2}{v^2}\left(1 - \frac{v^2}{c_s^2}\right)\frac{d\ln\rho}{dt} = \frac{p}{\rho v^2}\left(1 - \frac{v^2}{c_s^2}\right)\frac{d\ln p}{dt} = -\left(1 - \frac{v^2}{c_s^2}\right)\frac{d\ln v}{dt}. \tag{12.77}$$

If $v < c_s$ we have the following situation,

- If the area of the pipe decreases (nozzle) in the direction of the flow, the velocity increases and the pressure and density decrease.
- If the area of the pipe increases (diffuser) in the direction of the flow, the velocity decreases and the pressure and density increase.

On the other hand if the flow is supersonic ($v > c_s$) we have

- If the area of the pipe decreases (nozzle) in the direction of the flow, the velocity decreases and the pressure and density increase.
- If the area of the pipe increases (diffuser) in the direction of the flow, the velocity increases and the pressure and density decrease.

If we have a tube in which the flow is initially subsonic and the area of the tube decreases, the flow will accelerate. If the area of the tube increases again the flow will decelerate and you're back where you started. On the other hand let's imagine that the area of the tube decreases sufficiently that the velocity of the flow reaches the speed of sound at the cinch point of the tube, the fluid will exit the cinch point supersonically and accelerate as the tube inreases in cross-section. Now you know why a rocket engine is shaped like it is (this is called a de Laval nozzle).

Flow through a Channel

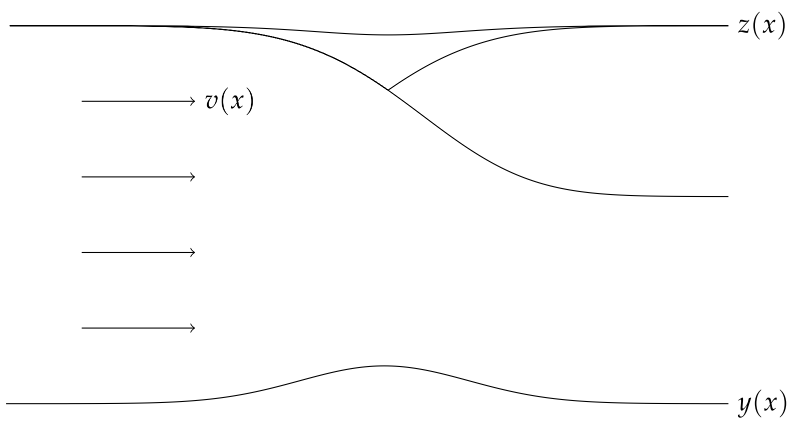


Figure 12.2: A channel of variable depth

We can see many of the features of the supersonic flow through a de Laval nozzle in the flow through a channel. Much of the intuition developed in hydraulics carries over to other fluid systems (even astrophysical ones); furthermore, water running through a channel is something with which most are familiar.

Let's look at a channel of constant width but varying depth $y(x)$. Let's take y equals to zero well before the bump. We would like to know how the height of the surface of the flow $z(x)$ changes as it passes through the channel (as shown in Fig. 12.2).

Let's write out the Bernoulli equation (divided by g as customary in hydraulics) for the fluid moving along the surface,

$$\frac{v^2}{2g} + z + \frac{p_a}{g} = \frac{v_0^2}{2g} + z_0 + \frac{p_a}{g} = h_0 \tag{12.78}$$

where p_a is the pressure of the atmosphere. The quantity h_0 is a constant along the surface streamline, and it is so important in hydraulics that it has a special name, *specific head*. From continuity we have

$$v\left(z - y\right) = v_0 z_0 = q_0 \tag{12.79}$$

where q_0 is the flux. Notice that we have neglected the vertical velocity of the flow. This is a common assumption in hydraulics. Combining the equations yields

$$\frac{v_0^2}{2}\left[\left(\frac{z_0}{z-y}\right)^2 - 1\right] + g\left(z - z_0\right) = 0. \tag{12.80}$$

Taking the derivative with respect to x yields

$$\frac{dy}{dx} = \frac{dz}{dx}\left[1 - \frac{g\left(z-y\right)^3}{v_0^2 z_0^2}\right] = \frac{dz}{dx}\left[1 - \frac{g\left(z-y\right)}{v^2}\right] = \frac{dz}{dx}\left[1 - \mathrm{Fr}^{-2}\right]. \tag{12.81}$$

where Fr is the Froude number, the ratio of the speed of flow to the speed of small wavelength gravity waves (see § 15). We can rearrange this a bit to yield

$$\frac{dz}{dx} = -\frac{dy}{dx}\frac{\mathrm{Fr}^2}{1-\mathrm{Fr}^2}, \tag{12.82}$$

so if the fluid is *subcritical* or streaming ("subsonic") over the bump, the surface will dip, and if the fluid is *supercritical* or shooting ("supersonic") the surface will bulge. For the equation to make sense, if the flow becomes supercritical, it must do so at the top of the bump.

Fig. 12.2 shows the various possibilities. Essentially for given values of g, v_0 and z_0 and for a small enough values of $y(x)$, there are two solutions for $z(x)$. One is a small deviation in the level of the surface that corresponds to a subcritical flow. The second has a large deviation (supercritical flow). When the flow is critical these two solutions coincide. The upper curves show the surface for various values of v_0. The uppermost curve is always well in the subcritical regime. The middle curve nearly reaches the critical point at the top of the bump, and the lower curve reaches the critical point and is supercritical to the right of the bump.

When the flow is critical at the top of the bump, we see that the smooth solution is the one that goes supercritical after the bump. At first glance the whole setup seems quite sensitive. In particular what if the bump is so big that the flow goes critical before reaching the top of the bump? For a particular set of initial conditions we can calculate the height of the bump where the fluid goes critical to be

$$y_c = z_0 + \frac{v_0^2}{2g} - \frac{3\left(v_0 z_0\right)^{2/3}}{2g^{1/3}} = h_0 - \frac{3q_0^{2/3}}{2g^{1/3}} \tag{12.83}$$

We can actually do much better than this. If we look at Eq. 12.80 and make the substitution $z = y + 1/u$ we get

$$\frac{v_0^2 z_0^2}{2}u^3 - \left[\frac{v_0^2}{2} + g\left(z_0 - y\right)\right]u + g = 0. \tag{12.84}$$

This equation has the form

$$Au^3 - Bu + C = 0. \tag{12.85}$$

Let us substitute $u = \sqrt{4B/3A}\cos t$ to give after some manipulation

$$\cos 3t = -\frac{3C}{2B}\sqrt{\frac{3A}{B}} \tag{12.86}$$

which gives three real solutions for u as long as the absolute value of the right-hand side does not exceed unity. We can also use this result to solve for v_0 in terms of y_c in Eq. 12.83. Both this and the solutions to Eq. 12.84 are left for the exercises.

Real Sound Waves

Let's take a closer look at sound waves. As before we shall assume that the background is static so before we perturb the medium the entropy is constant throughout. Let's perturb the fluid in a particular wave so that s remains constant. In this case, we can express the pressure in terms of the density alone.

Furthermore, let's assume that the velocity of fluid at any point depends on the density alone and look at the continuity and Euler equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0, \quad \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0 \tag{12.87}$$

where we have assumed that the wave is a plane wave travelling in the $x-$direction. Using the relationships between the pressure, velocity and density we can obtain,

$$\frac{\partial \rho}{\partial t} + \frac{d(\rho v)}{d\rho}\frac{\partial \rho}{\partial x} = 0, \quad \frac{\partial v}{\partial t} + \left(v + \frac{1}{\rho}\frac{dP}{dv}\right)\frac{\partial v}{\partial x} = 0. \tag{12.88}$$

We can define the speed of the wave as the rate at which regions of the same density or velocity move forward along the $x-$direction,

$$\left(\frac{\partial x}{\partial t}\right)_\rho = -\left[\frac{\frac{\partial \rho}{\partial t}}{\frac{\partial \rho}{\partial x}}\right] = \frac{d(\rho v)}{d\rho} = v + \rho\frac{dv}{d\rho} \tag{12.89}$$

and

$$\left(\frac{\partial x}{\partial t}\right)_v = -\left[\frac{\frac{\partial v}{\partial t}}{\frac{\partial v}{\partial x}}\right] = v + \frac{1}{\rho}\frac{dP}{dv}. \tag{12.90}$$

These two velocities must be equal so we get

$$\rho\frac{dv}{d\rho} = \frac{1}{\rho}\frac{dP}{dv} = \frac{c_s^2}{\rho}\frac{d\rho}{dv} \tag{12.91}$$

so

$$v = \pm\int\frac{c_s}{\rho}d\rho = \pm\int\frac{dP}{\rho c_s}. \tag{12.92}$$

We can find how fast a portion of the wave travels by substituting into Eq. 12.91 to get

$$\left(\frac{\partial x}{\partial t}\right)_v = v \pm c_s(v) \tag{12.93}$$

so we find that

$$x = t[v \pm c_s(v)] + f(v) \tag{12.94}$$

where $f(v)$ determines the initial shape of the wave. Let's derive an expresion for the sound speed as a function of the density,

$$c_s = \left[\frac{dP}{d\rho}\right]^{1/2} = \left[\gamma K\rho^{\gamma-1}\right]^{1/2} \tag{12.95}$$

so

$$\rho = \rho_0 \left(\frac{c_s}{c_0} \right)^{2/(\gamma-1)}. \tag{12.96}$$

If we substitute this into Eq. 12.95 we get

$$c_s = c_0 \pm \frac{1}{2}(\gamma - 1)v. \tag{12.97}$$

We can use this result to express the density and pressure in terms of the fluid velocity

$$\rho = \rho_0 \left[1 \pm \frac{1}{2}(\gamma-1)\frac{v}{c_0} \right]^{2/(\gamma-1)}, \quad P = P_0 \left[1 \pm \frac{1}{2}(\gamma-1)\frac{v}{c_0} \right]^{2\gamma/(\gamma-1)}. \tag{12.98}$$

Putting things together we find that

$$x = t \left[\pm c_0 + \frac{1}{2}(\gamma+1)v \right] + f(v) \tag{12.99}$$

or rearranging

$$v = F \left\{ x - \left[\pm c_0 + \frac{1}{2}(\gamma+1)v \right] t \right\} \tag{12.100}$$

If we look at Eq. 12.100 we see that we can have a situation where the same value of x has more than one value of v. This isn't physical. This first occurs at a time t when

$$\left(\frac{\partial x}{\partial v} \right)_t = 0, \left(\frac{\partial^2 x}{\partial v^2} \right)_t = 0, \tag{12.101}$$

or at

$$t = -\frac{2f'(v)}{\gamma+1}, f''(v) = 0. \tag{12.102}$$

The first expression tells us when the shock forms, and the second tells us that the shock forms at a point of inflection in the wave.

Another important situation is when the discontinuity forms at a boundary between gas that is moving at gas that is stationary ($v = 0$). In this case we have

$$t = -\frac{2f'(0)}{\gamma+1} \tag{12.103}$$

As an example let's assume that we have a pipe closed at one end by a piston and we start to move the piston according to $v_{\text{pist}} = at$. Because the gas at the edge of the piston must move with the piston we have $v = at$ at $x = \frac{1}{2}at^2$, so we can write down an expression for

$$f(v) = f(at) = \frac{1}{2}at^2 - -c_0 t - \frac{1}{2}(\gamma+1)at^2 = -c_0 tb - \frac{1}{2}\gamma at^2 = -\left(\frac{c_0}{a}\right) v - \frac{1}{2}\frac{\gamma}{a}v^2. \tag{12.104}$$

Using the expression for x we get

$$x - \left[c_0 + \frac{1}{2}(\gamma+1)v \right] t = f(v) = -\left(\frac{c_0}{a}\right) v - \frac{1}{2}\frac{\gamma}{a}v^2. \tag{12.105}$$

Solving for v gives

$$v = \frac{1}{\gamma}\left(\left\{\left[c_0 - \frac{1}{2}(\gamma+1)at\right]^2 + 2a\gamma(c_0 t - x)\right\}^{1/2} - \left[c_s - \frac{1}{2}(\gamma+1)at\right]\right\}. \tag{12.106}$$

If $a < 0$ a rarefaction wave travels through the gas. On the other hand if $a > 0$ we get a shock at a time,

$$t = -\frac{2f'(0)}{\gamma+1} = \frac{2c_0}{a(\gamma+1)}. \tag{12.107}$$

We leave the exploration of shocks to the next chapter; suffice it to say for now that shocks happen.

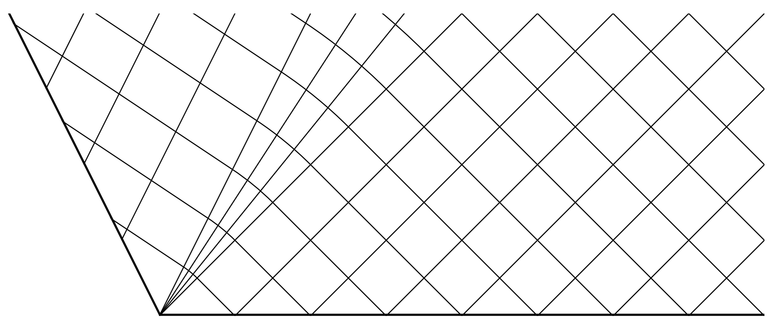


Figure 12.3: The Characteristic Structure for a Rarefaction Wave, $a < 0$. Notice the smooth transition between the regions.

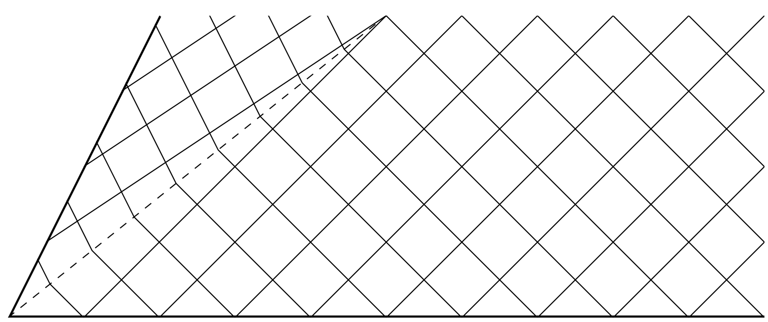


Figure 12.4: The Characteristic Structure for a Compression Wave, $a > 0$. Notice the abrupt transition between the regions.

Problems

1. **Maximum Flux**

   Calculate from the Euler equation and the continuity equation, at what velocity does the flux (ρV) reach its maximum for fluid flowing through a tube of variable cross-sectional area? At which velocities does the flux vanish? You can consider the flow to be adiabatic.

2. **Stream Bed**

   Fig. 12.2 shows how the level of the surface changes for a flow passing over an obstacle. For an initial depth of $z_0 = 1$ and $g = 10$ and a bump height of $y(x) = 0.1e^{-x^2}$, find the solutions to Bernoulli's equation (Eq. 12.80) for z as a function of x and the initial velocity v_0. You may find several solutions for a given x. Also you should only worry about the positive real solutions for z. What are the values of the critical velocities v_0?

3. **Sound Velocity**

   Show that for a linear sound wave *i.e.* one in which $\delta\rho \ll \rho$ that the velocity v of fluid motion is much less than c_s. Estimate the maximum longitudinal fluid velocity in the case of a sound wave in air at STP in the case of a disturbance which sets up pressure fluctuations of order 0.1%.

13
Shock Waves

Non-relativistic Shocks

Discontinuities signal a failure of fluid mechanics as we have formulated it. Fluid mechanics assumes that the material is continuous so quantities cannot change discontinuously. In practice viscosity and thermal conduction save the day, so although the wave may get really steep, a discontinuity doesn't actually form. To understand the structure of a shock, one needs to include viscosity, but one can understand the behaviour of shocks without including viscosity as we shall see.

Let's stand in the frame of the shock. The fluid approaches the shock supersonically on the left side and exits subsonically on the right side. Let P_1, ρ_1 and v_1 denote the physical quantities on the left-hand side (the pre-shock fluid) and P_2, ρ_2 and v_2 in the post-shock fluid.

First we have

$$\rho_1 v_1 = \rho_2 v_2 = j. \tag{13.1}$$

What goes into the shock must come out of the shock. If you remember the energy flux for an ideal fluid is

$$\mathbf{q} = \left[\frac{1}{2}V^2 + w\right]\rho\mathbf{V}. \tag{13.2}$$

This flux must be the same on each side (unless the shock is radiative) so we have

$$\left(\frac{1}{2}v_1^2 + w_1\right)\rho_1 v_1 = \left(\frac{1}{2}v_2^2 + w_2\right)\rho_2 v_2. \tag{13.3}$$

Because of conservation of mass we can simplify this to

$$\frac{1}{2}v_1^2 + w_1 = \frac{1}{2}v_2^2 + w_2. \tag{13.4}$$

This states that the sum of the kinetic and internal energy per unit mass is conserved across the shock. Third we have to conserve the momentum flux

$$P_1 + \rho_1 v_1^2 = P_2 + \rho_2 v_2^2 \tag{13.5}$$

We can use the defintion of the enthalpy to eliminate it from the equations

$$w = \epsilon + \frac{P}{\rho} = \frac{\gamma}{\gamma - 1}\frac{P}{\rho} = \frac{\gamma}{\gamma - 1} k_B T. \tag{13.6}$$

Let's define the Mach number of the incoming flow as $M_1 = v_1/c_s$ and rewrite Eq. 13.5 as

$$\frac{P_2}{P_1} = 1 + \frac{\rho_1}{P_1}v_1^2 - \frac{\rho_2}{P_1}v_2^2 = 1 + \gamma M_1^2 \left(1 - \frac{\rho_1}{\rho_2}\right). \tag{13.7}$$

We can also rewrite Eq. 13.4 to yield

$$\frac{P_2}{P_1} = \frac{\rho_2}{\rho_1} + \frac{1}{2} M_1^2 (\gamma - 1) \left(\frac{\rho_2}{\rho_1} - \frac{\rho_1}{\rho_2}\right) \tag{13.8}$$

and we can equate these two expressions to solve for ρ_2/ρ_1. From inspection we see that one solution is $\rho_2 = \rho_1$, which means that there is not discontinuity. The other solution yields

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{(\gamma+1) + (\gamma+1)(M_1^2 - 1)}{(\gamma+1) + (\gamma-1)(M_1^2-1)} = \frac{(\gamma+1)M_1^2}{2 + M_1^2(\gamma-1)} \tag{13.9}$$

$$\frac{P_2}{P_1} = \frac{(\gamma+1) + 2\gamma(M_1^2-1)}{(\gamma+1)} = \frac{1 - \gamma + 2M_1^2\gamma}{(\gamma+1)}, \tag{13.10}$$

$$\frac{T_2}{T_1} = \frac{(1 - \gamma + 2M_1^2\gamma)[2 + M_1^2(\gamma-1)]}{(\gamma+1)^2 M_1^2} \tag{13.11}$$

$$M_2^2 = \frac{(\gamma+1) + (\gamma-1)(M_1^2-1)}{(\gamma+1) + 2\gamma(M_1^2-1)} = \frac{2 + M_1^2(\gamma-1)}{1 - \gamma + 2M_1^2\gamma} \tag{13.12}$$

where intermediate expressions are given to show that if $M_1 > 1$, then $\rho_2 > \rho_1$, $P_2 > P_1$ and $M_2 < 1$. The fluid enters the shock supersonically and leaves the shock subsonically. The post-shock fluid has higher pressure and density. It is not obvious from the expression but the post-shock temperature always exceeds the pre-shock value.

As we take the limit of a stong shock $M_1 \to \infty$ we find that the compresssion ratio and square of the downstream Mach number approach

$$\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1} \text{ and } M_2^2 = \frac{1}{2} - \frac{1}{2\gamma}. \tag{13.13}$$

For $\gamma = 5/3$ the compression ratio is 4 and the downstream Mach number is $1/\sqrt{5}$. For a diatomic gas ($\gamma = 7/5$) the maximum compression ratio is larger at 6 and the square of the downstream Mach number is $1/\sqrt{7}$ — in fact the compression ratio ρ_2/ρ_1 always equals $1/M_2^2 - 1$ for any value of M_1.

Although the solution outlined above gives the ratio of pressures, densities and other quatities as a function of the incoming Mach number M_1, there is an alternative approach that is somewhat more illustrative. First let us define the specific volume of the fluid $V = 1/\rho$, so we can write $v_1 = jV_1$ and $v_2 = jV_2$ from Eq. 13.1. Let's substitute this in Eq. 13.5 to give

$$p_1 + j^2 V_1 = p_2 + j^2 V_2 \tag{13.14}$$

so

$$j^2 = \frac{p_2 - p_1}{V_1 - V_2} = -\frac{\Delta p}{\Delta V}. \tag{13.15}$$

Let's use this value of j^2 to determine the velocity difference

$$v_1 - v_2 = j\left(V_1 - V_2\right) \text{ so } \left(v_1 - v_2\right)^2 = \left(p_2 - p_1\right)\left(V_1 - V_2\right) = -\Delta p \Delta V. \tag{13.16}$$

Both Eq. 13.15 and 13.16

Now let's use the same values of v_1 and v_2 in the energy equation

$$w_1 + \frac{1}{2} j^2 V_1^2 = w_2 + \frac{1}{2} j^2 V_2^2 \tag{13.17}$$

and

$$w_1 - w_2 + \frac{1}{2}\left(V_1 + V_2\right)\left(p_2 - p_1\right) = 0. \tag{13.18}$$

Because the specific enthalpy w is a function of P and V, Eq. 13.18 defines a curve. Let's specialize for an ideal gas, for which $w = \gamma/(\gamma - 1)pV$, so

$$p_2 = p_1 \frac{\frac{\gamma}{\gamma-1} V_1 - \frac{1}{2}\left(V_1 + V_2\right)}{\frac{\gamma}{\gamma-1} V_2 - \frac{1}{2}\left(V_1 + V_2\right)}. \tag{13.19}$$

The denominator vanishes for

$$\frac{V_1}{V_2} = \frac{\rho_2}{\rho_1} = \frac{\gamma + 1}{\gamma - 1}, \tag{13.20}$$

the same value as Eq. 13.13 Fig. 13.1 depicts the shock or Hugoniot adiabat for a shock with a preshock pressure p_1 and specific volume V_1. Any point along the curve a to the left of (p_1, V_1) is a possible postshock condition. A particular postshock condition (p_2, V_2) is highlighted. The minimum flux passing through the shock is given by the negative of the slope at (p_1, V_1), and it increases as the shock gets stronger. The velocity difference vanishes for small shocks and grows as the area of the box as the shock grows.

The curve b to the left of (p_2, V_2) shows the possible postshock conditions if the preshock condition is (p_2, V_2). Notice that it also intersects the curve a at (p_1, V_1). There are two (or no) shock adiabats that connect any two points in the $p - V-$plane. One corresponds

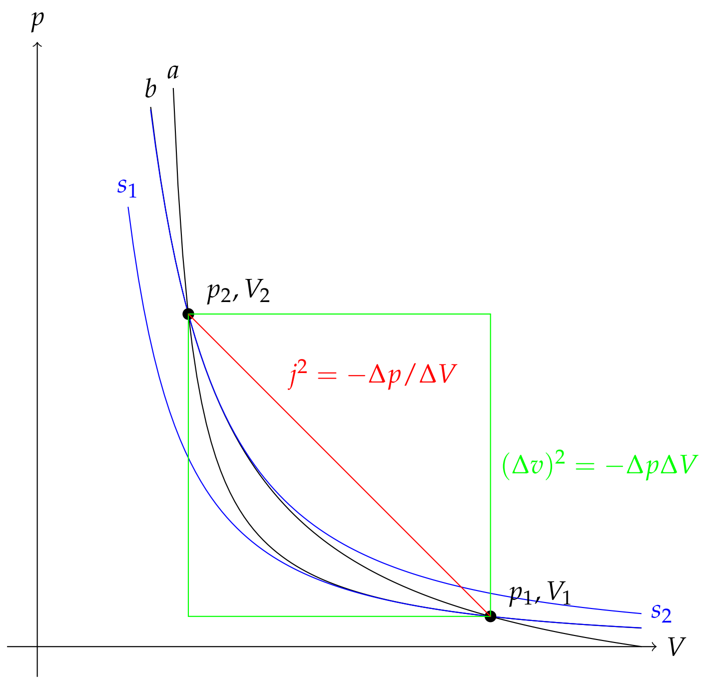


Figure 13.1: Shock (Hugoniot) Adiabats (in black) and Standard (Poisson) Adiabats (in blue)

to pressure and density increasing through the shock (curve a), and one corresponds to pressure and density decreasing through the shock (curve b). Earlier it was stated that these quantities must increase through the shock, but no reason was given. Fig. 13.1 shows the curves of constant entropy or standard (Poisson) adiabats on the $p - V-$plane corresponding to the values of the entropy at (p_1, V_1), s_1, and at (p_2, V_2), s_2. We know that if the pressure is higher at a particular density or specific volume than the entropy is larger so $s_2 > s_1$. From the second law of thermodynamics we know that entropy cannot decrease in an isolated system, so the initial state of the flow must be (p_1, V_1) and the final state is (p_2, V_2). Furthermore, we can see that the curves of constant entropy not only pass through the correponding plots in the plane (this is by design) but they are also tangent to and have the same radius of curvature as the shock adiabats. This means that both the first and second derivatives coincide for these two sets of curves and that the increase in entropy is third order in the size of the shock.

Although the curve b that travels from (p_2, V_2) to (p_1, V_1) is an unphysical solution for a shock because entropy decreases along the path, it does provide some great insights. What is the velocity of the flow on either side of the shock? We have

$$v_1^2 = j^2 V_1^2 = -\frac{\Delta p}{\Delta V} V_1^2 \text{ and } v_2^2 = -\frac{\Delta p}{\Delta V} V_2^2. \tag{13.21}$$

What is the sound speed on either side of the shock? We have

$$c_{s,1}^2 = \left.\frac{\partial P}{\partial \rho}\right|_{s_1} = -V_1^2 \left.\frac{\partial P}{\partial V}\right|_{s_1} = -V_1^2 \left.\frac{\partial P}{\partial V}\right|_a \text{ and } c_{s,2}^2 = -V_1^2 \left.\frac{\partial P}{\partial V}\right|_b \tag{13.22}$$

so the Mach numbers on each side of the shock are given by the ratio of the slope of the secant to the slope of the tangent. That is,

$$M_1^2 = \frac{\Delta p}{\Delta V} \Big/ \left.\frac{\partial P}{\partial V}\right|_a \text{ and } M_2^2 = \frac{\Delta p}{\Delta V} \Big/ \left.\frac{\partial P}{\partial V}\right|_b. \tag{13.23}$$

Because all of the adiabats are concave up in the $p - V-$plane, the slope of the secant must be larger than that of the tangent at (p_1, V_1), so the flow enters the shock supersonically. Conversely at (p_2, V_2) the slope of the secant must be smaller than that of the tangent, so the flow exits the shock subsonically. As the shock decreases in intensity, the figure demonstrates that both Mach numbers approach unity.

A Spherical Shock - The Sedov Solution

Let's imagine that we dump a really large amount of energy into a small region. The energy is initially carried by a small mass (m). Initially, as long as

$$m \gg \frac{4}{3}\pi r^3 \rho. \tag{13.24}$$

where the right-hand side is the mass swept up. The ejecta will freely expand. Let's imagine sometime later when the mass of the ejecta is negligible but that the energy of the explosion is still large compared to the enthapy of the swept-up material

$$E \gg \frac{4}{3}\pi r^3 \rho w. \tag{13.25}$$

This is equivalent to $p_2 \gg p_1$, so we have a strong shock, so $\rho_2/\rho_1 = (\gamma+1)/(\gamma-1)$

In this situation, we only have four numbers of interest,

- r, the distance from the centre of the explosion
- t, the time since the explosion and
- E, the energy of the explosion.
- ρ_1, the density of the undisturbed gas

By combining E, t and ρ we can find only one expression with the dimension of length, so let us take the radius of the shock to be

$$R(t) = R_0 \left(\frac{Et^2}{\rho_1}\right)^{1/5}. \tag{13.26}$$

where R_0 is a dimensionless constant that we will determine from the solution. The velocity of the shock wave with respect to the undisturbed gas is

$$v_s = -v_{1,s} = \frac{dR}{dt} = \frac{2R}{5t} = \frac{2}{5} R_0 E^{1/5} \rho_1^{-1/5} t^{-3/5} \tag{13.27}$$

We would like to know the speed of the gas relative to the undisturbed gas after the shock has passed,

$$v_{2,s} - v_{1,s} = v_{2,u} - v_{1,u} = v_{2,u} \tag{13.28}$$

For a strong shock we have

$$v_{2,u} = \frac{\gamma - 1}{\gamma + 1} v_{1,s} - v_{1,s} = \frac{2}{\gamma + 1} v_s \tag{13.29}$$

and

$$\rho_2 = \rho_1 \frac{\gamma + 1}{\gamma - 1} \text{ and } P_2 = \frac{2}{\gamma + 1} \rho_1 u_1^2. \tag{13.30}$$

Behind the shock, the fluid is ideal so we can use the continuity, Euler and energy equations

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0 \tag{13.31}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial r} + \frac{2\rho v}{r} = 0 \tag{13.32}$$

$$\left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial r} \right) \ln \left(\frac{p}{\rho^\gamma} \right) = 0. \tag{13.33}$$

The final equation says that the entropy per unit mass does not change. The trick to solve these equations is to assume that all of the variables depend on the similarity variable $\xi = [r/R(t)]$ with the right scaling. For example we have

$$v = \frac{2r}{5t} V(\xi), \rho = \rho_1 G(\xi), c_s^2 = \gamma \frac{P}{\rho} = \frac{4r^2}{25t^2} Z(\xi). \tag{13.34}$$

The shock jump conditions give the boundary conditions for the solution,

$$V(1) = \frac{2}{\gamma + 1}, G(1) = \frac{\gamma + 1}{\gamma - 1}, Z(1) = \frac{2\gamma(\gamma - 1)}{(\gamma + 1)^2}. \tag{13.35}$$

Padmanabhan gives a general solution to these equations in closed form, but let's look at the results for $\gamma = 5/3$ depicted in Fig. 13.2. In general the value of V ranges between $1/\gamma$ at $\xi = 0$ and $2/(\gamma + 1)$ at $\xi = 1$; it doesn't change much with the similarity variable; most of the change in the velocity from the center to the edge comes from the scaling in Eq. 13.34 and the values of $G(\xi)$ and $Z(\xi)$ can be written in terms of $V(\xi)$. We can find one of these relationships by examining the conservation of energy in the flow. The total energy of the flow is simply E because no energy leaves the flow and the material swept up by the shock is assumed to have no energy. Furthermore, the flow is self-similar so the energy contained within a spherical shell labelled by the similarity variable ξ must be constant with time. Let's look at a the flow at particular value of ξ. The total energy flowing outward through a spherical surface over a time dt is

$$dE = 4\pi r^2 \left(w + \frac{v^2}{2} \right) \rho v dt. \tag{13.36}$$

On the other hand the volume swept out as the flow expands self-similarly is

$$dV = 4\pi r^2 \left. \frac{\partial r}{\partial t} \right|_\xi dt. \tag{13.37}$$

We can combine this with the energy density to yield the energy in the region

$$dE = 4\pi r^2 \left. \frac{\partial r}{\partial t} \right|_\xi \left(\epsilon + \frac{v^2}{2} \right) \rho dt. \tag{13.38}$$

We can equate these two energies and solve for $c_s^2 = \gamma P/\rho$ to give

$$Z(\xi) = \frac{\gamma(\gamma-1)V(\xi)^2 \left[1 - V(\xi)\right]}{2\left[\gamma V(\xi) - 1\right]} \tag{13.39}$$

The final step in completing the solution is to find the relationship between the energy of the explosion and the value of R_0. We have

$$E = \int_0^R 4\pi r^2 dr \rho \left[\frac{v^2}{2} + \epsilon \right] = \int_0^R 4\pi r^2 dr \rho \left[\frac{v^2}{2} + \frac{c_s^2}{\gamma(\gamma-1)} \right] \tag{13.40}$$

and change variables to ξ with $r = R(t)\xi$ to yield

$$E = R(t)^5 \rho_1 \frac{16\pi}{25 t^2} \int_0^1 \xi^4 G(\xi) \left[\frac{V(\xi)^2}{2} + \frac{Z(\xi)}{\gamma(\gamma-1)} \right] \tag{13.41}$$

and

$$1 = R_0^5 \frac{16\pi}{25} \int_0^1 \xi^4 G(\xi) \left[\frac{V(\xi)^2}{2} + \frac{Z(\xi)}{\gamma(\gamma-1)} \right], \tag{13.42}$$

showing explicitly that the value of R_0 is a dimensionless number that only depends on the value of γ. The value R_0 is approximately 1.033 for $\gamma = 7/5$ and 1.152 for $\gamma = 5/3$. It ranges from 0.783 to 1.232 as γ goes from 1.1 to 1.9. Amazingly without knowing γ well one can get an estimate of the energy of the blast (within a factor of three) simply from measuring the value of the radius of the shock at a particular time and taking $R_0 = 1$.

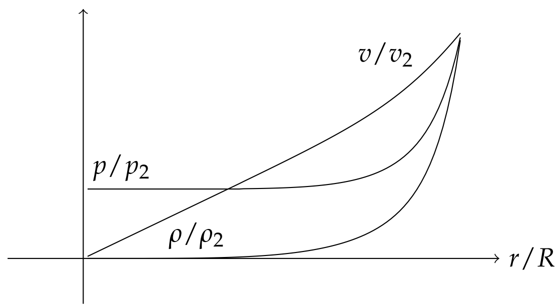


Figure 13.2: Variation of the density, velocity and pressure behind the shock for the Sedov-Taylor blast wave

Detonation Waves

In a detonation as the material passes through the shock energy is released either through chemical changes or nuclear burning. One could have either a release of energy through the shock (like combustion) or a consumption of energy (like ionization). The jump conditions are with the exception of the energy equation

$$\frac{1}{2}v_1^2 + w_1 = \frac{1}{2}v_2^2 + w_2 \tag{13.43}$$

where w_1 and w_2 are now different functions of p and V to reflect the different chemical or nuclear composition of the gas before and after the reactions. Fig. 13.3 depicts the situation graphically. The lower curve a is the shock adiabat without the chemical changes and a' is the detonation abiabat which uses the functional form of the enthalpy in the burnt gas. We still have the relationships between the flux, velocities and slopes and areas on the $p - V-$plane, because these originated from the conservation of momentum and mass not from the energy equation that has changed.

The various points outline how the gas changes as it passes through the shock and burns. An example of the general case, the gas enters the shock supersonically at A and is compressed to point B and leaves the shock subsonically and at the same time or after the gas burns and moves along the chord to point E. Because the flux of the flow is conserved through the transition, the state of the gas must remain on the chord AB. There is a minimum flux that can pass through the detonation front, $j_{\rm min}$, and this flux also corresponds to the minimum velocity jump through the front where the final state is O or the Jouguet point. In the case of a shock without a chemical change there is no minimum velocity jump.

Although the minimal value of the flux appears to be a special case, it is actually occurs in nature often. A detonation that proceeds from A to D and then to O minimizes the entropy incease in the front. Furthermore, for final states above O along a' the gas leaves the front subsonically. If the final state lies at O, the gas leaves the detonation front right at the speed of sound in the downstream flow. This special situation often arises when the combustion itself creates the shock. Let's take a specific example of a detonation front that starts near the closed end of a tube. The front must be followed by a rarefaction wave that travels up the tube at the speed of sound through the postshock gas. If the postshock gas is traveling subsonically relatively to the shock then the rarefaction wave will eventually catch up to the back of the shock reducing the flux through the shock by reducing the postshock pressure and shock velocity relative to the preshock gas until the minimum flux is achieved. At this point the postshock gas leaves the front at the sound speed so the rarefaction wave no longer overtakes the shock and the combined detonation front and rarefaction wave achieves a steady state. The detonation adiabat below the Jouguet point O cannot be reached if the combustion begins after the gas is compressed. The point E for example has a lower entropy than point C so the gas cannot pass from C to E either immediately after the shock or through a subsequent shock.

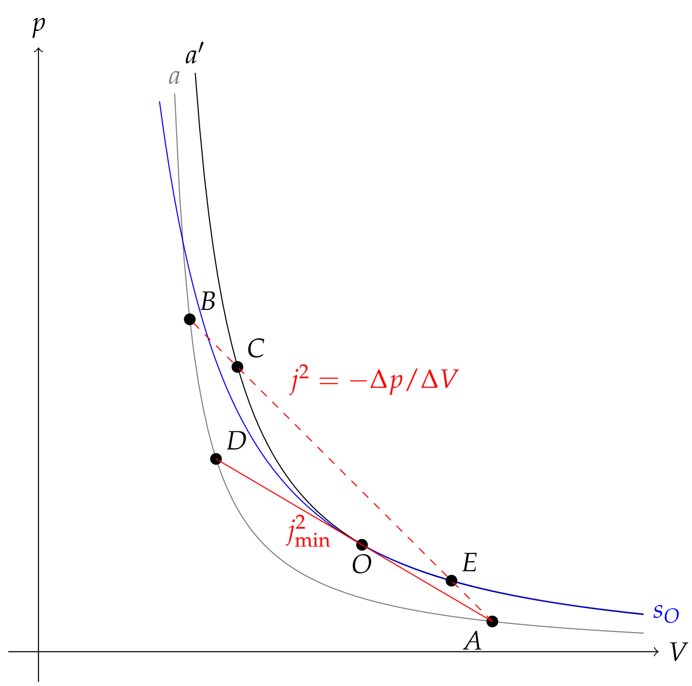


Figure 13.3: Shock (Hugoniot) Adiabat (in gray), Detonation Adiabat (in black) and Standard (Poisson) Adiabat (in blue)

Radiative Shocks

So far we have assumed that the energy in a fluid element is conserved through the shock, that no energy leaves the flow or is radiated. The opposite extreme is that the shock heats the gas sufficiently that radiative losses are important near the shock and the gas rapidly cools. In this case we must abandon the conservation of energy flux through the shock (Eq. 13.4) and find another criterion to understand how the gas changes through the shock. Astrophysically the rate that gas cools can depend very sensitively on the temperature of the gas. In particular gas above about 10^4 K radiates much more effectively than cooler gas. Imagine if the gas before the shock was just below the critical temperature at which cooling set in. As it passes through the shock, it goes above this temperature and then rapidly begins to cool and rapidly returns to its initial temperature. The additional condition that we seek is that final temperature equals the initial temperature.

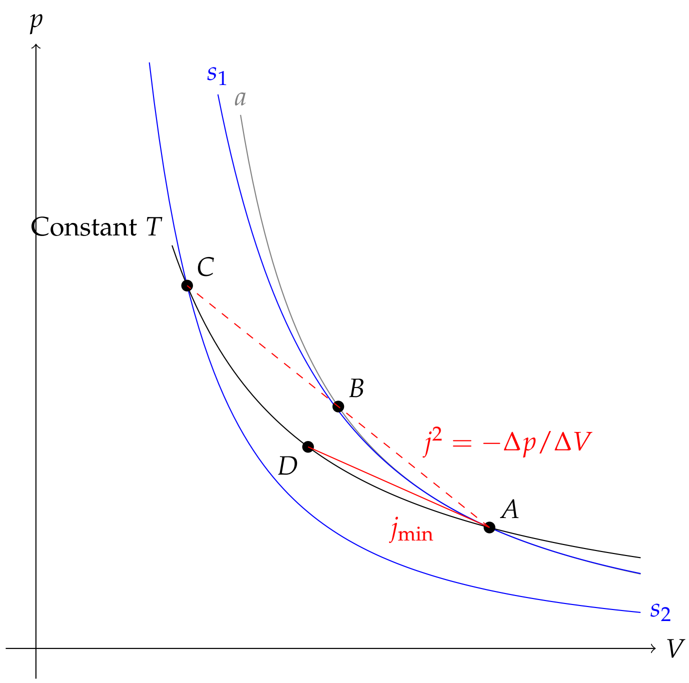


Figure 13.4: Isotherm (in black), Shock (Hugoniot) Adiabat (in gray), Standard (Poisson) Adiabats (in blue)

Again we still have the relationships between the flux, velocities, slopes and areas on the $p - V-$plane that result from the conservation of momentum and mass, but the shock adiabat is replaced with an isotherm as shown in Fig. 13.4. From the diagram it is apparent that the entropy of the gas decreases through an isothermal shock; as a gas is compressed at constant temperature, its entropy decreases. The radiation carries away both energy and entropy. Because the standard adiabats are generally steeper than the isotherms, the gas always leaves the shock subsonically. Again because the momentum flux is conserved, the gas must remain on the chord AC throughout. As it passed through the shock it is heated from A to point B and then as it cools it travels from B to C. As for the case of a detonation, we find that there is a minimum flux that can pass through an isothermal shock and a minimal velocity change. Just above the flux $j_{\rm min}$ the flow enters the shock slightly supersonically and leaves subsonically.

The initial and final Mach numbers and densities are related through

$$M_2 = \frac{1}{\gamma M_1}, \frac{\rho_2}{\rho_1} = \gamma M_1^2, \frac{P_2}{P_1} = \gamma M_1^2. \tag{13.44}$$

The ratio of the energy flux entering the radiative shock to that leaving is given by

$$\frac{q_1}{q_2} = \gamma^2 M_1^2 \frac{(\gamma-1)M_1^2 + 2}{2\gamma^2 M_1^2 + \gamma - 1}. \tag{13.45}$$

For large values of M_1 the initial energy flux is much larger than the final energy flux. At the other end the minimum value of M_1 is of course unity. This yields a minimum energy ratio for the isothermal shock of

$$\left.\frac{q_1}{q_2}\right|_{\rm min} = \frac{\gamma^2}{2\gamma - 1}. \tag{13.46}$$

Even this weakest of isothermal shocks results in a compression ratio $\rho_2/\rho_1 = \gamma$.

Sometimes the temperature of the gas is held constant through the interaction with an external radiation field, so that even slight departures from isothermality disappear on a short timescale. In this case it makes sense to take $\gamma = 1$. If we substitute $\gamma = 1$ in Eq. 13.9 through 13.12 we obtain Eq. 13.44. However, the enthalpy of an isothermal gas is given by

$$w = c_s^2 \ln\left(\frac{\rho}{\rho_0}\right), \tag{13.47}$$

so if we take the reference density $\rho_0 = \rho_1$ we find

$$\frac{q_1}{q_2} = \frac{M_1^4}{1 + \ln M_1^4} \tag{13.48}$$

Relativistic Shocks

We will look at relativistic shocks as an example of relativistic hydrodynamics. In particular we will look at the relativistic jump conditions across the shock. The particle flux must be conserved across the shock (Eq. 12.27)

$$J_{x,1} = J_{x,2}, \frac{U_1}{V_1} = \frac{U_2}{V_2} \tag{13.49}$$

where $V_1 = 1/n_{\text{prop},1}$ and $U_1 = \gamma_1 v_1/c$ is the spatial component of four-velocity of the flow before the shock and $\gamma_1 = (1 - v_1^2/c^2)^{-1/2}$. It is most clear to use the rest-mass energy density for n_{prop}. The components of the stress-energy tensor must also be conserved (Eq. 12.38)

$$T_{x0,1} = T_{x0,2}, w_1 U_1 \gamma_1 = w_2 U_2 \gamma_2 \tag{13.50}$$

and

$$T_{xx,1} = T_{xx,2}, w_1 U_1^2 + p_1 = w_2 U_2^2 + p_2 \tag{13.51}$$

where $w = \epsilon + p$ and ϵ includes the rest-mass energy of the particles. Here w is the enthalpy per unit volume whereas in previous sections it denoted the enthalpy per unit mass, $w_{\text{mass}} = w_{\text{volume}} V$.

By combining the particle flux, Eq.13.49, and the momentum equations, Eq. 13.51, we obtain the results

$$j^2 = \frac{p_2 - p_1}{w_1 V_1^2 - w_2 V_2^2} = \frac{p_2 - p_1}{V_{w,1} - V_{w,2}}, \tag{13.52}$$

$$(U_1 - U_2)^2 = j^2 (V_1 - V_2)^2 = \frac{V_1 - V_2}{V_{w,1} - V_{w,2}} (p_2 - p_1)(V_1 - V_2) \tag{13.53}$$

where $V_w = wV^2 = (p + \epsilon)V^2$. These are analogous to Eq. 13.15 and 13.16. Finally we can derive the equation of the shock adiabat, using the identity $\gamma^2 = 1 + U^2$, to yield

$$w_1 V_{w,1} - w_2 V_{w.2} + (V_{w,1} + V_{w,2})(p_2 - p_1) = 0 \tag{13.54}$$

which is very close in form to Eq. 13.18. In the non-relativistic limit for the second term we can take $V_w = V$, but we must look at the first terms more closely because the result depends on the difference of two quantities that are equal to lowest order in the non-relativistic limit. In particular,

$$w_1 V_{w,1} = w_1^2 V_1^2 = \left(\rho_1 c^2 + w_{NR,1}\right)^2 V_1^2 = 1 + 2 w_{NR,1} V_1 + w_{NR,1}^2 V_1^2 \tag{13.55}$$

where we have used $\rho_1 c^2 V_1 = 1$. We can drop the last term. The first term cancels in Eq. 13.54, leaving the middle term which equals twice the enthalpy per unit mass and results in twice Eq. 13.18.

Hydraulic Jump

Let's revist the dynamics of water travelling down a shallow channel. We neglect the vertical motion of the fluid and assume that all dimensions are large compared with the depth of the fluid — this is the *hydraulic* approximation. If the flow only depends on the position x along the channel and time t, the continuity and momentum equation are

$$\frac{\partial h}{\partial t} + \frac{\partial (vh)}{\partial x} = 0, \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = -g \frac{\partial h}{\partial x}. \tag{13.56}$$

where the depth h is assumed to be constant across the channel. We can define a surface density $\bar{\rho} = \rho h$ and a mean pressure $\bar{p} = \rho g h^2/2$ and recast the equations as

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial (v\bar{\rho}}{\partial x} = 0, \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x}. \tag{13.57}$$

These equations are identical to the equations for the adiabatic flow of a gas with $\bar{p} \propto \bar{\rho}^2$. We can apply the results from gas dynamics to hydraulics as along as the flow is abiabatic — no shocks.

These equations do not include the conservation of energy equation because we assume that the flow does not have any internal energy or entropy. In practice the energy in the flow can be transferred to small scale motion of the fluid which is quickly dissipated. Let us examine discontinuities in the fluid height and velocity by using the conditions of continuity on the particle and momentum flux. Such discontinuities are known as *hydraulic jumps*. The mass flux density is simply $j = \rho v h$ and the momentum flux is

$$\int_0^h \left(p + \rho v^2\right) dz = \frac{1}{2}\rho g h^2 + \rho v^2 h. \tag{13.58}$$

The jump conditions are

$$v_1 h_1 = v_2 h_2, v_1^2 h_1 + \frac{1}{2} g h_1^2 = v_2^2 h_2 + \frac{1}{2} g h_2^2. \tag{13.59}$$

We can express any two quantities in terms of the others in particular we have the velocities in terms of the heights

$$v_1^2 = \frac{1}{2} g \frac{h_2}{h_1}\left(h_1 + h_2\right), v_2^2 = \frac{1}{2} g \frac{h_1}{h_2}\left(h_1 + h_2\right). \tag{13.60}$$

If we look at the energy flux in the channel we have

$$q = \int_0^h \left(\frac{p}{\rho} + \frac{1}{2}v^2\right)\rho v dz = \frac{1}{2} j \left(gh + v^2\right) \tag{13.61}$$

and the difference in energy flux is

$$q_1 - q_2 = gj\frac{\left(h_1^2 + h_2^2\right)\left(h_2 - h_1\right)}{4 h_1 h_2}. \tag{13.62}$$

Because the energy flux of the flow must decrease through the jump $h_2 > h_1$ — the height of the fluid must increase downstream of the jump. Substituting $h_2 < h_1$ into Eq. 13.60 shows that $v_1 . \sqrt{g h_1}$ and $v_2 > \sqrt{g h_2}$. The flow enters the jump supercritically and leaves the jump subcritically.

Problems

1. **Shock Entropy** Show that the entropy of the fluid increases as it passes through a shock. Hint: the equation of state of an isentropic fluid is $P = K\rho^\gamma$ where the value of K increases with increasing entropy.

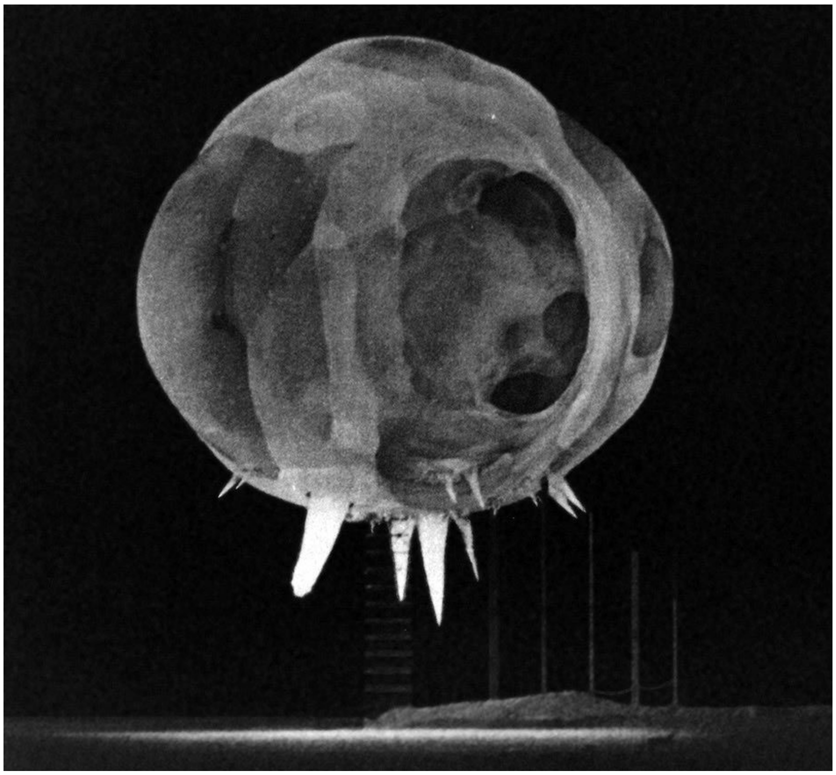


Figure 13.5: The explosion of nuclear device in 1952 about 2 ms after detonation.

2. **Bomb Yield**

   Fig. 13.5 shows shocked air heated to incandescence about two milliseconds after the detonation of a nuclear bomb. The height of the device was 90 meters. What was the approximate yield of the device?

3. **Relativistic Shock**

   Find the incoming and outgoing velocity of a relativistic shock in terms of the energy density and pressure on either side of the shock.

4. **Relativistic Bernoulli**

   Find the relativistic generalisation of Bernoulli's equation for a streamline (you can neglect gravitiy).

5. **Bathtub Physics**

   When water flows into a bathtub, a circular hydraulic jump forms around the incoming stream of water. If you assume that the flow rate is constant and the flow is initially vertical, calculate the height of the water downstream of the jump as a function of the radius of the jump and the flow rate. You may neglect friction and assume that the velocity upstream of the jump is constant. If the bathtub is large compared to the radius of the jump and the walls are vertical, how does the radius of the jump change with time?

14
Accretion and Winds

We continue looking at steady flows with two specific applications: matter flowing onto an object (accretion) and matter flowing away from an object (winds).

Spherical Accretion

We can apply what we learned about the de Laval nozzle to accretion onto an astrophysical object. We will assume that the accretion is steady at a rate $\dot{M}$ and that the pressure $P \propto \rho^\gamma$ with $1 < \gamma < 5/3$. First let's write the continuity equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho v\right) = 0 \tag{14.1}$$

so $\rho v r^2$ =constant. Let's write down the Euler equation

$$v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{GM}{r^2} = 0 \tag{14.2}$$

Let's use the equation of state to eliminate p from the Euler equation and use the continuity equation to eliminate ρ,

$$\frac{1}{\rho}\frac{\partial \rho}{\partial r} = -\frac{1}{vr^2}\frac{\partial}{\partial r}(vr^2) \text{ and } \frac{\partial p}{\partial r} = \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial r} = c_s^2 \frac{\partial \rho}{\partial r}, \tag{14.3}$$

we get

$$v\frac{\partial v}{\partial r} - \frac{c_s^2}{vr^2}\frac{\partial}{\partial r}(vr^2) + \frac{GM}{r^2} = 0. \tag{14.4}$$

We can rearrange this

$$\frac{1}{2}\left(1 - \frac{c_s^2}{v^2}\right)\frac{\partial v^2}{\partial r} = -\frac{GM}{r^2}\left(1 - \frac{2c_s^2 r}{GM}\right). \tag{14.5}$$

Far away from the star, the sound speed c_s approaches some constant value, so the right hand side of the equation will be positive at large r. We would like the gas to accelerate toward the star, so $\frac{\partial v^2}{\partial r} < 0$ at large distances. For this to be the case $v < c_s$ far from the star. As the gas falls toward the star, it accelerates until it reaches the critical radius.

$$\frac{2c_s^2 r_c}{GM} = 1 \tag{14.6}$$

If we use $c_s^2 = \gamma p/\rho = \gamma kT$ we get

$$r_c = \frac{GM}{2c_s^2(r_c)} \approx 7.5 \times 10^{13} \left(\frac{T}{10^4 K}\right)^{-1} \left(\frac{M}{\mathrm{M}_\odot}\right) \mathrm{cm} \tag{14.7}$$

that is much larger that most stars, so we have to worry about what happens with r_c. If the flow is subsonic when it gets to r_c it will decelerate within r_c and the accretion stagnates. If the flow becomes supersonic before reaching r_c, then $\frac{\partial v^2}{\partial r}$ diverges as the flow becomes supersonic. This is unphysical because you get two values of v^2 at a single value of r.

The only viable accretion mode is for the flow to become supersonic precisely at r_c. It then accelerates for $r < r_c$ as well. We can work further to determine the flow by integrating Eq. 14.2 to get a Bernoulli equation

$$\frac{v^2}{2} + \frac{c_s^2}{\gamma - 1} - \frac{GM}{r} = \text{constant} \tag{14.8}$$

We know that as $r \to \infty$, $v^2 \to 0$ so we have

$$\frac{v^2}{2} + \frac{c_s^2 - c_s^2(\infty)}{\gamma - 1} - \frac{GM}{r} = 0. \tag{14.9}$$

At the critical radius we have $v^2 = c_s^2$ and $GM/r_c = 2c_s^2$, so

$$\frac{c_s^2(r_c)}{2} + \frac{c_s^2(r_c) - c_s^2(\infty)}{\gamma - 1} - 2c_s^2(r_c) = 0. \tag{14.10}$$

We can find that

$$c_s^2(r_c) = c_s^2(\infty) \left[\frac{2}{5 - 3\gamma}\right] \tag{14.11}$$

so

$$r_c = \frac{GM}{c_s^2(\infty)} \frac{5 - 3\gamma}{4} \tag{14.12}$$

and

$$\rho(r_c) = \rho(\infty) \left[\frac{2}{5 - 3\gamma}\right]^{1/(\gamma-1)} \tag{14.13}$$

We can determine

$$\dot{M} = 4\pi r_c^2 \rho(r_c) c_s(r_c) = \pi G^2 M^2 \frac{\rho(\infty)}{c_s^3(\infty)} \left(\frac{2}{5 - 3\gamma}\right)^{(5-3\gamma)/2(\gamma-1)} \tag{14.14}$$

$$= 1.4 \times 10^{11} \mathrm{g\ s}^{-1} \left(\frac{M}{\mathrm{M}_\odot}\right)^2 \left[\frac{\rho(\infty)}{10^{-24} \mathrm{g\ cm}^{-3}}\right] \left[\frac{c_s(\infty)}{10 \mathrm{km\ s}^{-1}}\right]^{-3} \tag{14.15}$$

for $\gamma = 1.4$. The gamma-dependent factor ranges from $e^{3/2}$ for $\gamma = 1$, to $5/2$ at $\gamma = 7/5$ and 1 at $\gamma = 5/3$.

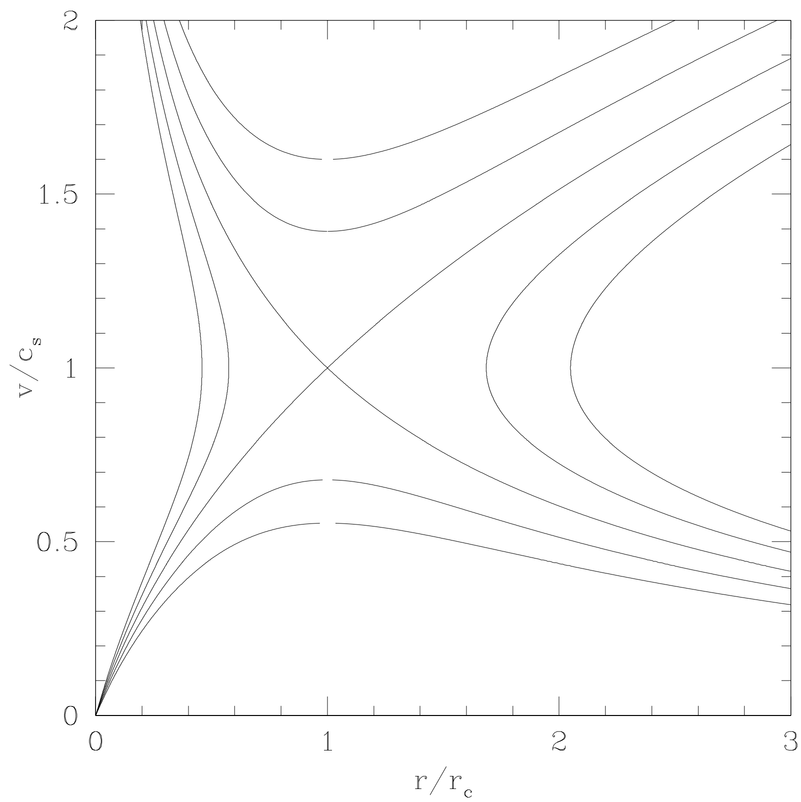


Figure 14.1: The solution to Bondi accretion with $\gamma = 7/5$ (an adiabatic, diatomic gas).

Accretion Disks

The preceding section ignores an important aspect of accretion: the angular momentum of the accreta. If the material starts with some net angular momentum it can only collapse so far before its angular velocity will be sufficient to halt further collapse. For the accetion to proceed, angular momentum must be transferred outwards through the accreted material or removed from the central regions with ejecta. Understanding the production of ejecta is beyond our scope, but examining the transport of angular momentum through a rotating disk of material is not once we add an additional ingredient to our analysis, viscosity.

First let's see why angular momentum can play a crucial role in accretion. If we assume that the material had some initial velocity v relative to the star, and that without gravity it would come within a distance b of the star (the impact parameter). The initial specific angular momentum is vb. If the material conserves angular momentum we can compare the centripetal acceleration with gravitational acceleration to give

$$\frac{(vb)^2}{r^3} = \frac{GM}{r^2}, \tag{14.16}$$

so the accretion will stall at

$$r = \frac{(vb)^2}{GM} = 10^{-3}\text{AU}\left(\frac{v}{1\text{km/s}}\frac{b}{1\text{AU}}\right)^2 \frac{\text{M}_\odot}{M}. \tag{14.17}$$

Around this radius, the accretion flow must make a transition between a spherical inflow and a disk. Without viscosity the accretion will cease, so the crucial ingredient to move further is a prescription for the viscosity. Unfortunately, natural estimates for the microscopic viscosity of astrophysical gas are too small by many orders of magnitude to account for the structure of accretion disks.

It is likely that accretion disks are turbulent magnifying the effects of small-scale viscosity to larger scales. However, without simulating the turbulence directly, it is difficult to estimate the effective viscosity. Instead let's assume there is some viscosity that we don't know exact and look at the angular momentum transport needed to maintain accretion.

Angular Momentum Transport

The specific angular momentum of material in circular orbit is given by the orbital velocity times the square of the radius,

$$l = \Omega r^2 = (GMr)^{1/2}. \tag{14.18}$$

Because matter is falling toward the centre the angular momentum flows inward

$$\dot{L}^{+} = \dot{M} \left(GMr\right)^{1/2}. \tag{14.19}$$

Also some angular momentum ends up on the central object

$$\dot{L}^{-} = \beta \dot{M} \left(GMr_I\right)^{1/2} \tag{14.20}$$

where r_I is the inner radius of the disk. Therefore, there is a torque acting in the disk

$$\tau = f_\phi \left(2\pi r\right) \left(2h\right) \left(r\right) = \dot{L}^{+} - \dot{L}^{-} = \dot{M} \left[\left(GMr\right)^{1/2} - \beta \left(GMr_I\right)^{1/2}\right] \tag{14.21}$$

The viscous torque is the product of the viscous stress in the tangential direction, the area upon which the stress acts (the half-height of the disk is h) and the radius. The viscous stress is proportional to the viscosity and the angular velocity gradient,

$$f_\phi = -\eta \frac{d\Omega}{d\ln r} = -\eta r \frac{d}{dr} \left(\sqrt{GM} r^{-3/2}\right) = \frac{3}{2} \eta \Omega. \tag{14.22}$$

Both Eq. 14.21 and 14.22 give the stress. We can combine these two equations to yield the value of the coefficient of dynamical viscosity, η,

$$\eta = \frac{\dot{M}}{6\pi r^2 h \Omega} \left[\left(GMr\right)^{1/2} - \beta \left(GMr_I\right)^{1/2}\right]. \tag{14.23}$$

For example we can now determine the energy generated per unit area of the disk

$$2hQ \approx 2h \frac{\left(f_\phi\right)^2}{\eta} = \frac{9}{2} \Omega^2 h \eta = \frac{3\dot{M}}{4\pi r^2} \frac{GM}{r} \left[1 - \beta \left(\frac{r_I}{r}\right)^{1/2}\right]. \tag{14.24}$$

Emission

If we assume that the energy is radiated through the surface we find that the flux per unit area is half this value (two surfaces) and that the total luminosity of the disk is

$$L = \int_{r_I}^{\infty} Q 2\pi r dr = \left(\frac{3}{2} - \beta\right) \frac{GM\dot{M}}{r_I}. \tag{14.25}$$

If one assumes that the disk radiates locally as a blackbody, the spectrum is simply the sum of the various blackbodies (the so-called multi-temperature disk model).

Vertical Structure

We have assumed that the disk is thin. Well, how thin is it? The pressure gradient in the disk must resist the vertical component of gravity. Since the disk is not self-gravitating, this force comes from the central object so we have

$$\frac{1}{\rho}\frac{dP}{dz} = -\frac{GM}{r^2}\frac{z}{r}, \frac{P_c}{\rho_c h} \approx \frac{GM}{r^3}h, h \approx \left(\frac{P}{\rho}\right)^{1/2}\left(\frac{r^3}{GM}\right)^{1/2} \approx \frac{c_s}{\Omega}. \quad (14.26)$$

Let us assume that the disk is thin, we have

$$\frac{h}{r} \ll 1, \left(\frac{P}{\rho}\right)^{1/2}\left(\frac{r}{GM}\right)^{1/2} \ll 1, \frac{2kT}{m_p}\frac{r}{GM} \ll 1, kT \ll \frac{1}{2}\frac{GMm_p}{r}, \quad (14.27)$$

so for the disk to remain thin, most of the gravitational energy that is released as the material spirals down must be emitted. To determine how the thickness varies with radius we can use the various scalings in Eq. 14.26 and assume that the temperature is given by the effective temperature of the surface. This is essentially assuming that the disk is isothermal vertically. We know that Ω increases inward as $r^{-3/2}$ and $T_{\text{eff}} \propto r^{-3/4}$, so

$$\frac{h}{r} \propto \frac{r^{-3/8}}{rr^{-3/2}} = r^{1/8}. \quad (14.28)$$

The relative thickness of the disk remains nearly constant with radius if only internal heating is important in a vertically isothermal disk. Because the gas in the central plane of the disk can only be hotter than at the surface, the thickness estimated in this manner is a lower limit. Furthermore, close to the central object radiation from the central object itself may heat the disk further, thickening the inner regions.

We can do a bit better than this by calculating the temperature gradient through the disk. We have (Eq. 1.120)

$$F(z) = -\frac{16\sigma T^3}{3\kappa_R}\frac{\partial T}{\partial \Sigma} = -\frac{4}{3\kappa_R}\frac{\partial \sigma T^4}{\partial \Sigma} = -\frac{c}{\kappa_R}\frac{\partial P_{\text{rad}}}{\partial \Sigma} \quad (14.29)$$

$$= hQ \approx \frac{4\sigma T_c^4}{3\kappa_R(T_c, \rho_c)\rho_c h} \quad (14.30)$$

To go further we need an estimate of the density of the disk. We know the accretion rate but the disk could be of relatively low mass with material spiralling in quickly or of higher mass with material slowly spiralling in.

Modelling the Stress

Looking back at Eq. 14.21, we find that stress has units of angular momentum per unit time per unit volume or erg cm^{-3} in cgs units; therefore, it is quite natural to assume that the stress is proportional to the pressure $f_\phi = \alpha P$. Shakura and Sunyaev argued that the viscosity is produced by turbulent eddies so its natural value is

$$\eta \approx \rho v_{\text{turb}} l_{\text{turb}} < \rho c_s h \quad (14.31)$$

where the inequality holds because the turbulent velocity is limited by the sound speed, and the size of the eddies is limited by the thickness of the disk. We know that the stress is given by

$$f_\phi = \frac{3}{2}\eta\Omega < \frac{3}{2}\rho_c c_s h\Omega \approx \rho c_s^2 \approx P \tag{14.32}$$

so the value of α must be less than or equal to unity.

We can combine the α-stress with the angular momentum transport equation to give

$$\alpha P \left(4\pi r^2 h\right) = \dot{M}\left[(GMr)^{1/2} - \beta\left(GMr_I\right)^{1/2}\right] \tag{14.33}$$

and substituting what we know about the vertical structure (*i.e.* $P \approx \rho h^2\Omega^2$ from Eq. 14.26) to get

$$\alpha h^2\Omega^2\rho\left(4\pi r^2 h\right) = \dot{M}\left[(GMr)^{1/2} - \beta\left(GMr_I\right)^{1/2}\right]. \tag{14.34}$$

After some rearrangement we get

$$h^3 = \frac{1}{\alpha}\frac{\dot{M}}{4\pi\rho\Omega}\left[1 - \beta\left(\frac{r_I}{r}\right)^{1/2}\right], \frac{h^2}{r^2} = \frac{1}{2\alpha}\frac{v_r}{r\Omega}\left[1 - \beta\left(\frac{r_I}{r}\right)^{1/2}\right]. \tag{14.35}$$

The disk gets thinner as the value of α increases and gets fatter as the infall velocity approaches the orbital velocity.

We can combine the α-prescription with vertical radiative transfer (Eq. 14.29) to obtain an estimate of the central density and temperature of the disk. First we shall assume that the photons domiante the pressure and electron scattering dominates the opacity (*i.e.* the equation of state and opacity at the midplane of the disk), so

$$P_c \approx \frac{1}{3}aT^4 = \frac{4}{3}\frac{\sigma T_c^4}{c} \approx h^2\Omega^2\rho_c \tag{14.36}$$

and

$$\frac{4h\Omega^2 c}{\kappa_{\rm es}} = \frac{3\dot{M}}{8\pi r^2}\frac{GM}{r}\left[1 - \beta\left(\frac{r_I}{r}\right)^{1/2}\right] \tag{14.37}$$

Now combining Eq. 14.34 with Eq. 14.37, we obtain

$$\rho = \frac{128\pi^2}{27}\frac{c^3}{\alpha\Omega\dot{M}^2\kappa_{\rm es}^3}\left[1 - \beta\left(\frac{r_I}{r}\right)^{1/2}\right]^{-2} \tag{14.38}$$

and

$$h = \frac{3}{8\pi}\frac{\dot{M}\kappa_{\rm es}}{c}\left[1 - \beta\left(\frac{r_I}{r}\right)^{1/2}\right] \tag{14.39}$$

if we assume that electron scattering dominates the opacity and radiation pressure dominates (appropriate for high temperatures). Essentially, the thickness of the disk in this case is constant except near the inner edge where it becomes thinner. The thickness increases with the accretion rate and decreases rapidly with α.

We can combine Eq. 14.38 and 14.39 to obtain an estimate of the pressure in the midplane of the disk

$$P \approx \rho_c h^2 \Omega^2 = \frac{2c\Omega}{3\kappa_{\rm es}\alpha} \qquad (14.40)$$

and the ratio of the radial motion to the azimuthal motion

$$\frac{v_r}{\Omega r} = \frac{\dot{M}}{4\pi r \rho h \Omega r} = \frac{9}{64\pi^2}\frac{\alpha \dot{M}^2 \kappa_{\rm es}^2}{r^2 c^2}\left[1-\beta\left(\frac{r_I}{r}\right)^{1/2}\right] \qquad (14.41)$$

$$= \frac{9}{4}\alpha\left(\frac{L}{L_{\rm Edd}}\right)^2 \frac{r_I^2}{r^2}\left(\frac{3}{2}-\beta\right)^{-2}\left[1-\beta\left(\frac{r_I}{r}\right)^{1/2}\right] \qquad (14.42)$$

Winds

We have already discussed material flowing away from an object in the context of the Sedov solution that applies for explosions. Here we are interested in the situation where the flow is more or less steady; that is, it lasts for many dynamical times.

In our treatment of spherical accretion, the direction of the radial velocity did not enter. The solutions looked the same whether the matter flowed inward or outward, so the solution for spherical accretion may be useful here if the effects of angular momentum may be neglected. Typically the radiation from the star drives the material outward, so only the initial angular momentum of the gas is important. At the surface of the star we know that the centripetal acceleration must be less than the gravitational acceleration, so

$$\frac{\left(\Omega_* r_*^2\right)^2}{r^3} < \frac{GM}{r^2} \qquad (14.43)$$

at the surface of the star. The ratio of the centripetal acceleration to the gravitational acceleration decreases as r^{-1} as the material flows outward, so within a few stellar radii the angular momentum is no longer important to the dynamics of the flow. On the other hand, the flow can carry away a significant amount of angular momentum from the star, accounting for why stellar rotation decreases with age.

Because angular momentum is only important near the star, we can use the results of § 14 to understand winds as well. The crucial difference is that the boundary conditions for a wind differ from those for accretion. Let's start with

$$v\frac{\partial v}{\partial r} - \frac{c_s^2}{v r^2}\frac{\partial}{\partial r}(v r^2) + \frac{GM}{r^2} = 0. \qquad (14.44)$$

One can assume that the stellar wind is approximately isothermal ($\gamma = 1$) — if one assumes otherwise one gets Eq. 14.8. We can integrate this equation to yield

$$\frac{v^2}{2} - c_s^2 \ln\left(v r^2\right) - \frac{GM}{r} = \text{Constant}. \qquad (14.45)$$

and after some rearrangement

$$\frac{v^2}{c_s^2} - \ln \frac{v^2}{c_s^2} - 4 \ln \frac{r}{r_c} - \frac{2GM}{rc_s^2} = \text{Constant} \tag{14.46}$$

where $v = c_s$ at $r = r_c \equiv GM/(2c_s^2)$ for the critical solution from Eq. 14.5 to yield

$$\mathcal{M}^2 - \ln \mathcal{M}^2 = 4 \ln \frac{r}{r_c} + \frac{4r_c}{r} - 3 \tag{14.47}$$

for the critical, transonic solution where $\mathcal{M} = v/c_s$.

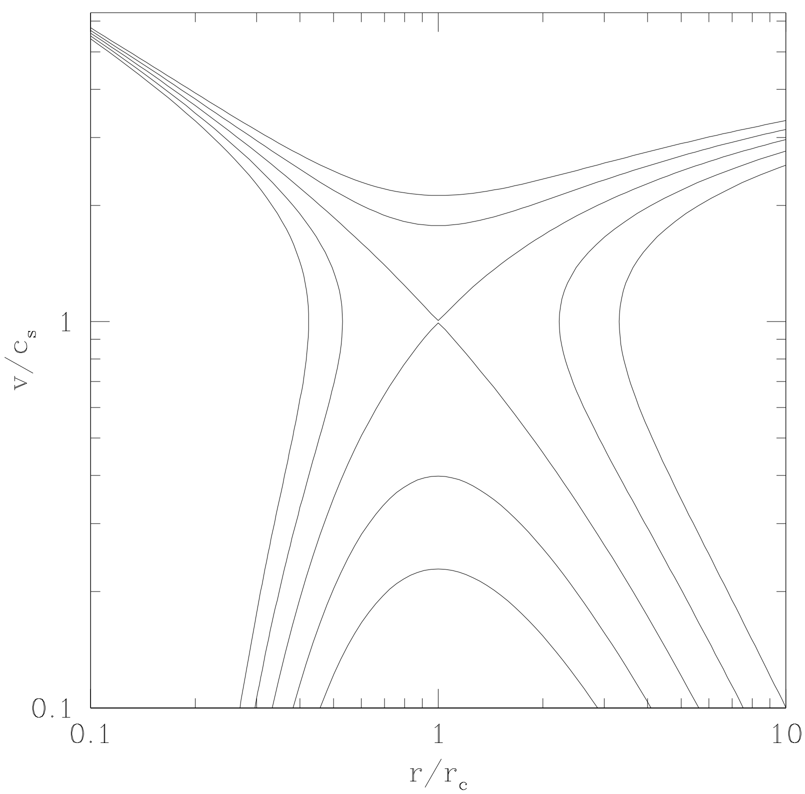


Figure 14.2: The velocity structure for an isothermal wind, neglecting angular momentum and magnetic fields.

Problems

1. **Exact Solutions**

   For which values of γ can the Bernoulli equation (Eq. 14.9) be solved using elementary methods (linear, quadratic and cubic equations of the form in Eq. 12.85). There are many, however only a few have $1 < \gamma < 5/3$.

2. **Monoatomic Gas**

   For a monoatomic gas, the value of γ is 5/3. According Eq. 14.13, the density at the critical radius is infinite, and the critical radius itself goes to zero. Explain how accretion from atomic gas would proceed.

3. **Accretion Disk**

   Calculate the surface temperature of the accretion disk as a function of radius, central mass and accretion rate. You may assume that all of the energy generated by viscous stresses is radiated locally as blackbody emission. Calculate the cumulative amount of flux as a function of temperature. Does most of the radiation emerge from regions at high temperature, at low temperatures or somewhere in between?

4. **Bondi Solution**

   Generate a picture like Fig. 14.1 for the Bondi solution to spherical accretion. Use $\gamma = 9/7$.

5. **Bondi Solution — Harder**

   Generate a Fig. 14.1 for the Bondi solution to spherical accretion. Use $\gamma = 7/5$.

6. **Accretion Energetics**

   (a) Let's use Newtonian gravity for simplicity here. How much kinetic energy does a gram of material have if it falls freely from infinity to the surface of a star of mass M and radius R?

   (b) How much energy is released if a gram of material falls from a circular orbit just above the stellar surface onto the stellar surface? To put it another way, what is the kinetic energy of the material in the circular orbit?

   (c) Hydrogen burning releases about 6×10^{18} erg/g. How does accretion of hydrogen onto a neutron star ($R = 10$ km, $M = 1.4\mathrm{M}_\odot$) differ from accretion onto a white dwarf ($R = 10000$ km, $M = 0.6\mathrm{M}_\odot$)?

   (d) What is the total about of energy released per gram of material as it falls from infinity to the surface of a neutron star? How many grams of material would have to fall each second on the neutron star to generate an Eddington luminosity through accretion? This is called the Eddington accretion rate.

7. **A Simplified Accretion Disk** This is a simplified model for an accretion disk. It is simpler than the model outlined in the chapter

but it will give the right order of magnitude for things. We are also using Newtonian gravity.

(a) Let's divide the accretion disk into a series of rings each of mass dm. What is the total energy of a ring at a distance r from the central black hole of mass M?

(b) Let's say that the ring shrinks by a distance dr. What is the change in the energy of the ring (dE/dr)? As the ring shrinks mass is moving toward the black hole. Divide both sides the answer to (b) by dt to get an equation for the energy loss rate per radial interval.

(c) What is the energy loss rate per unit area?

(d) Let's assume that this energy is radiated at the radius where it is liberated. Using the blackbody formula what is the temperature of the surface of the disk?

(e) Let's assume that the disk extends from an outer radius r_A to an inner radius r_0. What is the total luminosity of the disk if the accretion rate is dm/dt? What and where is the peak temperature of the disk? What and where is the minimum temperature of the disk?

(f) Sketch the spectrum from the accretion disk on a log-log plot. You can use temperature units for the energy axis (i.e. $kT_{\max}$ and $kT_{\min}$). To do this you will have to think about the peak flux from a blackbody at a particular temperature and the size of the disk that radiates at $T_{\max}$ and $T_{\min}$.

(g) The accretion rate is determined by the evolution of the orbit of the black hole with its companion, so it doesn't know about the Eddington limit of the black hole. What do you suppose happens if the rate that matter falls onto the disk exceeds the Eddington limit?

(h) What major bit of physics has been left out of this analysis?

15
Fluid Instabilities

Gravity Waves and Rayleigh-Taylor Instability

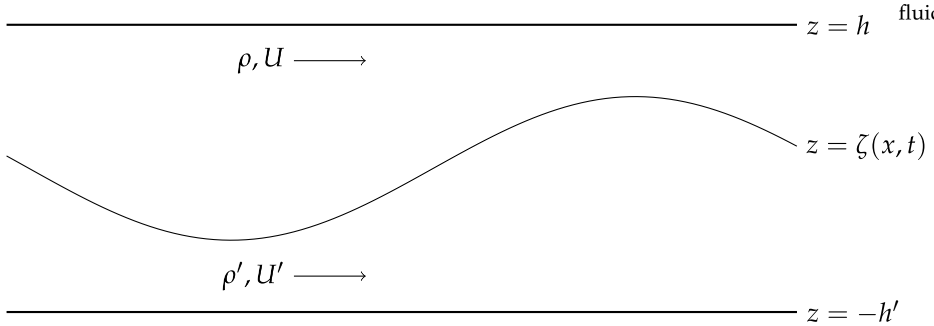


Figure 15.1: Shearing flow in a stratified fluid

Let's imagine a different type of wave on a fluid. Let's imagine we have two fluids in a gravitational field. The lower fluid has density ρ', velocity U' and thickness h' and the upper fluid has density ρ, velocity U and thickness h. Let's have $\zeta(x, t)$ denote the displacement of the interface in the $z-$direction. Let's assume that both fluids are incompressible and the flow is irrotational, so we can define

$$\mathbf{v} = -\nabla\Phi \text{ and } \mathbf{v}' = -\nabla\Phi' \tag{15.1}$$

where

$$\Phi = -Ux + \phi \text{ and } \Phi' = -U'x + \phi'. \tag{15.2}$$

To make further progress let us assume that the displacement of the interface has the form

$$\zeta(x, t) = A\cos(kx - \omega t) \tag{15.3}$$

and furthermore let velocity potentials also have a similar dependence

$$\phi = C\sin(kx - \omega t) f(z) \text{ and } \phi' = C'\sin(kx - \omega t) f'(z). \tag{15.4}$$

Because the fluids are assumed to be incompressible, we have $\nabla^2\phi = 0$ and the boundary condition gives $\partial\phi/\partial z = 0$ at $z = h$ and similarly for lower fluid, so we have

$$\phi = C\sin(kx-\omega t)\cosh\left[k(z-h)\right] \tag{15.5}$$
$$\phi' = C'\sin(kx-\omega t)\cosh\left[k(z+h')\right]. \tag{15.6}$$

The Lagrangian derivative of the displacement of the interface $D\zeta(x,t)/Dt$ gives the vertical velocity of the fluid at the interface, so

$$-\frac{\partial\phi}{\partial z} = \frac{\partial\zeta}{\partial t} + U\frac{\partial\zeta}{\partial x} \text{ and } -\frac{\partial\phi'}{\partial z} = \frac{\partial\zeta}{\partial t} + U'\frac{\partial\zeta}{\partial x}. \tag{15.7}$$

This yields a relationship between the constants A, C and C', the wavenumber and frequency,

$$A\left(kU-\omega\right) = -kC\sinh kh, \tag{15.8}$$
$$A\left(kU'-\omega\right) = kC'\sinh kh'. \tag{15.9}$$

We seek a relationship between ω and k, so we require an additional equation to eliminate the unknowns A, C and C'. Specifically, the pressure on each side of the interface must be equal. To examine the pressure let's look at Euler's equation for the ideal fluid (Eq. 12.42) and substitute $\mathbf{V} = -\nabla\Phi$ to yield

$$-\frac{\partial\nabla\Phi}{\partial t} + \frac{1}{2}\nabla V^2 = -\frac{\nabla P}{\rho} - g\hat{z}. \tag{15.10}$$

Since the fluid is incompressible we can write

$$\nabla\left[-\frac{\partial\Phi}{\partial t} + \frac{V^2}{2} + \frac{P}{\rho} + gh\right] = 0, \tag{15.11}$$

so for each fluid we have

$$-\rho\frac{\partial\Phi}{\partial t} + \rho\frac{v^2}{2} + gz\rho + p = B(t), \tag{15.12}$$
$$-\rho\frac{\partial\Phi'}{\partial t} + \rho'\frac{v'^2}{2} + g\rho' z + p = B'(t). \tag{15.13}$$

At the upper and lower surface the velocity of the perturbation vanishes and $p(z=-h') - p(z=h)$ must equal $g\rho h + g\rho' h'$, so

$$B(t) - B'(t) = \frac{\rho U^2}{2} - \frac{\rho' U'^2}{2}. \tag{15.14}$$

Let's take the difference of the two Bernoulli equations and evaluate it at $z = \zeta(x,t)$ to yield

$$\rho\left(-\frac{\partial\phi}{\partial t} - U\frac{\partial\phi}{\partial x} + g\zeta\right) = \rho'\left(-\frac{\partial\phi'}{\partial t} - U'\frac{\partial\phi'}{\partial x} + g\zeta\right). \tag{15.15}$$

to first order in the small quantities ϕ and ϕ'. Substituting the expressions for ϕ, ϕ' and ζ yields

$$\rho \left[C \cosh kh \left(\omega - kU \right) + gA \right] = \rho' \left[C' \cosh kh' \left(\omega - kU' \right) + gA \right] . \tag{15.16}$$

Combining this result with Eq. 15.8 and 15.9 yields the equation,

$$\rho \left(\omega - kU \right)^2 \coth kh + \rho' \left(\omega - kU' \right)^2 \coth kh' = kg \left(\rho' - \rho \right) , \tag{15.17}$$

and the dispersion relation,

$$\begin{aligned} \frac{\omega}{k} &= \frac{\rho U \coth kh + \rho' U' \coth kh'}{\rho \coth kh + \rho' \coth kh'} \pm \\ & \left[\frac{g}{k} \frac{\rho' - \rho}{\rho \coth kh + \rho' \coth kh'} - \frac{\rho\rho' \coth kh \coth kh' \left(U - U' \right)^2}{\left(\rho \coth kh + \rho' \coth kh' \right)^2} \right]^{1/2} \end{aligned} \tag{15.18}$$

The first interesting limit is where $U = U' = 0$ which yields the simpler expression

$$\frac{\omega^2}{k^2} = \frac{g}{k} \frac{\rho' - \rho}{\rho \coth kh + \rho' \coth kh'} \tag{15.19}$$

If $\rho' > \rho$, then $\omega^2 > 0$ and we have a stable wave. There are several interesting limits to this result.

- If $\rho = 0$, then $\omega^2 = gk \tanh kh$.
- If $\rho = 0$ and $kh \gg 1$, then $\omega^2 = gk$ (deep-water waves).
- If $\rho = 0$ and $kh \ll 1$, then $\omega^2 = ghk^2$ (shallow-water waves).
- If $\rho \neq 0$, $kh' \gg 1$ and $kh \gg 1$, then (both liquids very deep)

$$\omega^2 = \frac{kg(\rho' - \rho)}{\rho + \rho'} \tag{15.20}$$

- If $\rho \neq 0$, $kh' \ll 1$ and $kh \ll 1$, then (long waves)

$$\omega^2 = k^2 \frac{g(\rho' - \rho) hh'}{\rho h' + \rho' h} \tag{15.21}$$

On the other hand if $\rho' < \rho$, then $\omega^2 < 0$ and the perturbation simply grows (it does not oscillate). This is the Rayleigh-Taylor instability. This instability occurs whenever a low density gas underlies a higher density gas, for example in a supernova explosion. The gravitational acceleration g can be due to gravity (as in a supernova) or due to a deceleration of the fluid, if a low-density fluid plows into a high-density fluid. According to Eq. 15.21 the smallest scales have the highest growth rates. This is countered by viscosity and surface tension, so a particular scale dominates the growth at least initially.

Kelvin-Helmholtz or Shearing Instability

If we look at the term in the brackets in Eq. 15.18 for $U \neq U'$ we see that if $g = 0$ waves with all values of k are unstable and if $g \neq 0$ for sufficiently large values of k (small wavelengths), waves are unstable even if $\rho' > \rho$. The critical value of k is

$$k_{\rm crit} = \frac{g}{(U-U')^2} \frac{\rho'-\rho}{\rho\rho'} \frac{\rho \coth kh + \rho' \coth kh'}{\coth kh \coth kh'} \tag{15.22}$$

and for $k > k_{\rm crit}$ the growth rate increases monotonically. In reality for really small wavelengths other effects come into play, such as surface tension and viscosity; therefore, unless the velocity difference is sufficiently large, waves will not grow, and furthermore a particular wavelength grow the fastest.

We have also assumed that the velocity change is abrupt. It turns out that even if the velocity changes gradually with position, the flow is unstable, so we would like to get a heuristic understanding of the Kelvin-Helmholtz instability. We have two fluids moving in opposite directions along their shared interface which may be thick. We do not include gravity.

Let's assume that the flow is initially steady and irrotational that so we have Euler's equation

$$(\mathbf{V} \cdot \nabla)\mathbf{V} + \frac{\nabla P}{\rho} = 0 \tag{15.23}$$

We know that

$$\frac{1}{2}\nabla v^2 = \mathbf{V} \times (\nabla \times \mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{v} \tag{15.24}$$

which yields

$$\frac{1}{2}\nabla V^2 + \frac{\nabla P}{\rho} = 0 \tag{15.25}$$

and

$$\frac{1}{2}V^2 + \frac{P}{\rho} = \text{constant} \tag{15.26}$$

for the flow. Therefore, regions where V^2 is large have lower pressure. In the figure we have chosen a reference frame where the fluids are moving with equal and opposite velocities. We will also assume that the depths of both fluids are really large and the densities are equal. Therefore, the picture of what goes on in one fluid is mirrored in the other. If we focus on the wrinkle in the interface on the right hand side, the upper fluid must travel a bit farther to get around the wrinkle than the lower fluid, so it must travel faster and according to Eq. 15.26, its pressure must drop more than the fluid below the interface. The pressure on the inside of the curve is greater than on the outside. These pressure gradient causes the wrinkle to grow. We could even imagine a rubber sheet or less dramatically a layer of fluid moving at intermediate velocities lying along the interface and the forces would still be the same, and the instability remains.

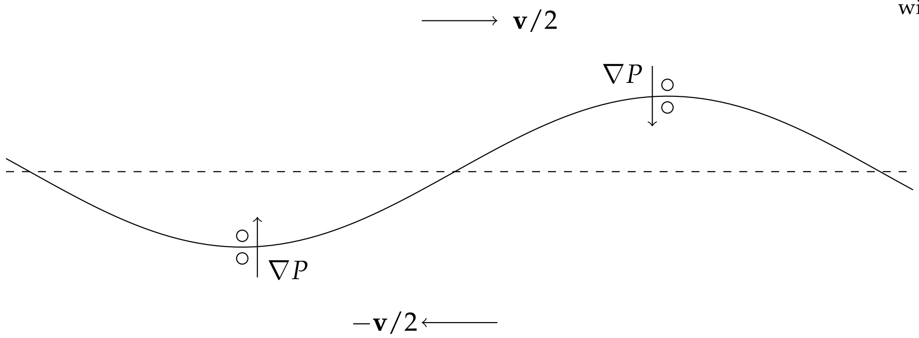


Figure 15.2: Illustration of shearing flow with pressure gradients

Gravitational Instability

Let's revisit our small sound waves but this time we will include the effects of self-gravity. have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = \frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{V}' = 0 \tag{15.27}$$

and

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\, \mathbf{V} + \frac{\nabla P}{\rho} = \frac{d\mathbf{V}}{dt} + \frac{\nabla P}{\rho} = \frac{\partial \mathbf{V}'}{\partial t} + \frac{\nabla P'}{\rho_0} = -\nabla \phi' \tag{15.28}$$

We can write $P' = (\partial P/\partial \rho)_s \rho'$ and rewrite the continuity equation to get

$$\frac{\partial P'}{\partial t} + \rho_0 \left(\frac{\partial P}{\partial \rho}\right)_s \nabla \cdot \mathbf{V}' = 0 \tag{15.29}$$

Let's take the divergence of the Euler equation to get

$$\frac{\partial \nabla \cdot \mathbf{V}'}{\partial t} + \frac{\nabla^2 P'}{\rho_0} = -\nabla^2 \phi' \tag{15.30}$$

and the time derivative of the continuity equation to get

$$\frac{\partial^2 P'}{\partial t^2} + \rho_0 \left(\frac{\partial P}{\partial \rho}\right)_s \nabla \cdot \frac{\partial \mathbf{V}'}{\partial t} = 0. \tag{15.31}$$

Finally we put the two together to get

$$\frac{\partial^2 P'}{\partial t^2} - \left(\frac{\partial P}{\partial \rho}\right)_s \left(\nabla^2 P' + \rho_0 \nabla^2 \phi'\right) = 0. \tag{15.32}$$

This is a wave equation with a sound speed of $c_s^2 = (\partial P/\partial\rho)_s$ but there is an extra term.

$$\nabla^2\phi' = 4\pi G\rho' \tag{15.33}$$

We can write $P' = c_s^2\rho'$. This eliminates density from the equation to get

$$\frac{\partial^2 P'}{\partial t^2} - c_s^2\nabla^2 P' - 4\pi G\rho_0 P' = 0. \tag{15.34}$$

Let's try a trial plane wave to find a solution to this equation

$$P' = p'\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)] \tag{15.35}$$

We get

$$-\omega^2 + c_s^2 k^2 - 4\pi G\rho_0 = 0 \tag{15.36}$$

so

$$\omega^2 = c_s^2 k^2 - 4\pi G\rho_0. \tag{15.37}$$

If $k^2 > 4\pi G\rho_0/c_s^2$, then $\omega^2 > 0$ and the wave is stable. On the other hand, if $k^2 < 4\pi G\rho_0/c_s^2$, the perturbation will grow. We can define a Jeans length

$$l_{\rm Jeans} = \frac{2\pi}{k_{\rm crit}} = \sqrt{\frac{\pi}{G\rho_0}}c_s \tag{15.38}$$

and a Jeans mass

$$M_{\rm Jeans} = \frac{4}{3}\pi l_{\rm Jeans}^3\rho_0 = \frac{4}{3}\sqrt{\frac{\pi^5}{G^3\rho_0}}c_s^3 \tag{15.39}$$

$$= 1.4\times 10^{42}{\rm g}\left[\frac{\rho_0}{10^{-24}{\rm g~cm}^{-3}}\right]^{-1/2}\left[\frac{c_s}{10{\rm km~s}^{-1}}\right]^3 \tag{15.40}$$

Thermal Instability

So far we have examined instabilities where energy does not leave or enter the fluid. In general hot gas emits and absorbs radiation; it may release energy through nuclear or chamical reactions as well. If the power absorbed and generated within the gas equals the power emitted by the gas, the temperature of the gas will remain constant and equilibrium is achieved. The question remains whether this equilibrium is stable. Heuristically we can see that if the cooling rate increases faster with temperature than the heating rate, then a slight increase in temperature will result in the gas cooling faster and the temperture returning to its equilibruum value. On the other hand, if the heating rate increases with temperature faster than the cooling rate, the slight temperature increase will be compounded with a further increase in temperature.

Problems

1. **X-ray Bursts:**

   We will try to model Type-I X-ray bursts using a simple model for the instability. We will calculate how much material will accumulate on a neutron star before it bursts.

   (a) Let us assume that the star accretes pure helium, that the temperature of the degenerate layer is constant down to the core (T_c), how much luminosity emerges from the surface of the star?

   (b) Let us assume that the helium layer has a mass, dM, and that the enregy generation rate for helium burning is given by

   $$\epsilon_{3\alpha} = 3.5 \times 10^{20} T_9^{-3} \exp(-4.32/T_9) \text{erg} \text{s}^{-1} \text{g}^{-1}$$

   where $T_9 = T/10^9$K. The energy generation rate is a function of density too, but let's forget about that to keep things simple. How much power does the helium layer generate as a function of dM?

   (c) Equate your answer to (a) to the answer to (b) and solve for dM. This is the thickness of a layer in thermal equilibrium.

   (d) Let's assume that the potential burst starts by the temperature in the accreted layer jiggling up by a wee bit. If the surface luminosity increases faster with temperature than the helium burning rate, then the layer is stable. Calculate dL_{surface}/dT and dP_{helium}/dT.

   (e) Calculate the value of dM for which dP_{helium}/dT exceeds dL_{surface}/dT and the layer bursts.

   (f) Equate your value of dM in (c) and (e) and solve for T. What is dM? How long will it take for such a layer to accumulate if the star is accreting at one-tenth of the Eddington accretion rate?

Part V

Appendicies

A
Mathematical Appendix

The Integral of $x^3/(e^x-1)$

The integral can be evaluated using a Taylor series

$$\int_0^\infty \frac{x^3}{e^x-1}dx = \int_0^\infty \frac{x^3 e^{-x}}{1-e^{-x}}dx = \int_0^\infty x^3 \sum_{n=1}^\infty e^{-nx}dx \qquad \text{(A.1)}$$

Let's look at each term in the sum (we can do this because each term in the sum is a convergent integral)

$$\int_0^\infty x^3 e^{-nx}dx = \frac{1}{n^4}\int_0^\infty u^3 e^{-u}du. \qquad \text{(A.2)}$$

The integral

$$\int_0^\infty u^3 e^{-u}du = -u^3e^{-u}\Big|_0^\infty + 3\int_0^\infty u^2 e^{-u}du \qquad \text{(A.3)}$$

and

$$\int_0^\infty u^3 e^{-u}du = 3\left(-u^2e^{-u}\Big|_0^\infty + 2\int_0^\infty ue^{-u}du\right) \qquad \text{(A.4)}$$

and

$$\int_0^\infty u^3 e^{-u}du = 3\times 2\times\left(-ue^{-u}\big|_0^\infty + \int_0^\infty e^{-u}du\right) = 6 \qquad \text{(A.5)}$$

to yield

$$\int_0^\infty \frac{x^3}{e^x-1}dx = \sum_{n=1}^\infty \frac{6}{n^4}. \qquad \text{(A.6)}$$

This result can be generalized to yield

$$\int_0^\infty \frac{x^\alpha}{e^x-1}dx = \sum_{n=1}^\infty \frac{\Gamma(\alpha+1)}{n^\alpha} = \Gamma(\alpha+1)\zeta(\alpha+1). \qquad \text{(A.7)}$$

For odd positive values of α the summation can be solved with contour integration. Let's start with

$$\sum_{n=1}^\infty \frac{1}{n^4}$$

to evaluate. We will use a basic result from complex analysis that the integral of an analytic function around a closed contour vanishes if the contour contains no poles. Let's examine the function

$$f(z) = \frac{\pi \cot(\pi z)}{z^4} dz \tag{A.8}$$

that has poles at $z = \ldots, -2, -1, 0, 1, 2 \ldots$ and for large values of z $f(z)$ quickly approaches zero, so the integral

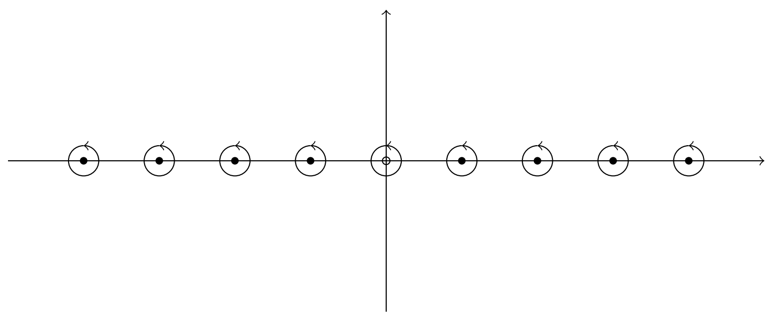


Figure A.1: The poles of $f(z)$ in the complex plane

$$\lim_{R \to \infty} \oint_{C_R} f(z) dz = 0 \tag{A.9}$$

where C_R is a circle of radius R. The sum of the integrals about all of the poles must vanish. Fig. A.1 shows all of the poles. At the poles (solid points in the figure) other than at the origin, the function is given by

$$f(z) \approx \frac{1}{n^4} \frac{1}{z-n} \tag{A.10}$$

that we can integrate along the loops in the figure by substituting $z = n + Re^{i\theta}$ so $dz = iRe^{i\theta} d\theta$ and

$$\lim_{R \to 0} \oint_{C_R} f(z) dz = \lim_{R \to 0} \int_0^{2\pi} \frac{1}{n^4} \frac{1}{Re^{i\theta}} iRe^{i\theta} d\theta = \frac{i}{n^4} \int_0^{2\pi} d\theta = 2\pi i \frac{1}{n^4} \tag{A.11}$$

where C_R is a circle of radius R centered on the pole. Let's combine this result with the integral around the large loop (Eq. A.9) to give

$$0 = 4\pi i \sum_{n=1}^{\infty} \frac{1}{n^4} + \lim_{R \to 0} \oint_{C_R} f(z) dz \tag{A.12}$$

where the first term is the sum we seek and the second term is an integral is over a circle surrounding the origin. The leading term in the integral about the origin is proportional to z^{-5} and $\oint z^{-n} dz = 0$ if $n \neq 1$, so we have to look at higher order terms, specifically

$$f(z) = \frac{1}{z^5} - \frac{\pi^2}{3z^3} - \frac{\pi^4}{45z} + \cdots \tag{A.13}$$

so we have

$$0 = 4\pi i \sum_{n=1}^{\infty} \frac{1}{n^4} + 2\pi i \left(-\frac{\pi^4}{45} \right) \tag{A.14}$$

and

$$\sum_{n=1}^{\infty} \frac{6}{n^4} = 6 \frac{\pi^4}{45 \times 2} = \frac{\pi^4}{15}. \tag{A.15}$$

Parseval's Theorem

We have stated a rather useful result,

$$\int_{-\infty}^{\infty} |E(t)|^2 dt = 2\pi \int_{-\infty}^{\infty} |\hat{E}(\omega)|^2 d\omega. \tag{A.16}$$

We now have the tools to prove it quickly,

$$\begin{aligned} \int_{-\infty}^{\infty} |E(t)|^2 dt &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \hat{E}(\omega') e^{-i\omega' t} d\omega' . \int_{-\infty}^{\infty} \hat{E}^*(\omega) e^{i\omega t} d\omega & \text{(A.17)} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt d\omega' d\omega \hat{E}(\omega') \hat{E}^*(\omega) e^{-i\omega' t} e^{i\omega t} & \text{(A.18)} \end{aligned}$$

The integral over time is simply Fourier transform of $2\pi e^{-i\omega' t}$ which we know,

$$\begin{aligned} \int_{-\infty}^{\infty} |E(t)|^2 dt &= 2\pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega' d\omega \hat{E}(\omega') \hat{E}^*(\omega) \delta(\omega - \omega') & \text{(A.19)} \\ &= 2\pi \int_{-\infty}^{\infty} d\omega \hat{E}(\omega) \hat{E}^*(\omega) = 2\pi \int_{-\infty}^{\infty} |\hat{E}(\omega)|^2 d\omega & \text{(A.20)} \end{aligned}$$

B

Selected Solutions

Chapter 1

1. **Hot Cloud**

X-ray photons are produced in a cloud of radius R at the uniform rate (photons per unit volume per unit times). the cloud is a distance d away. Assume that the cloud is optically thin. A detector at Earth has an angular acceptance beam of half-angle $\Delta\theta$ and an effective area A.

(a) If the cloud is fully resolved by the detector, what is the observed intensity of the radiation as a function of position?

(b) If the cloud is fully unresolved, what is the average intensity when the source is in the detector?

Answer:

(a) If the source is resolved, we can discern different parts of the cloud, so the observed intensity is the integral of the emission coefficient through the cloud,

$$I = \int j ds = \int_{-\sqrt{R^2-b^2}}^{\sqrt{R^2-b^2}} \frac{\Gamma}{4\pi} ds = \frac{\Gamma}{2\pi}\sqrt{R^2-b^2} = \frac{\Gamma}{2\pi} R \sqrt{1-\frac{b^2}{R^2}} \tag{B.1}$$

where b/R is the relative distance between our line of sight and the centre of the cloud.

(b) If the source is not resolved, the observed intensity is given by the flux from the source divided by the solid angle of acceptance of the detector.

$$I = \frac{F}{\pi\Delta\theta^2} = \frac{\frac{4}{3}\pi R^3\Gamma}{4\pi d^2}\frac{1}{\pi\Delta\theta^2} = \frac{\Gamma R^3}{3\pi d^2 \Delta\theta^2} \tag{B.2}$$

Clearly, this is the minimum value of the actual intensity of the source because it may actually subtend a smaller region of the sky that $\Delta\theta$ but we have no way of know because our detector cannot resolve below this scale.

3. **Blackbody**

Only one or no neutrinos can occupy a single state. Calculate the spectrum of the neutrino field in thermal equilibrium (neglect the mass of the neutrino). Neutrinos like photons have two polarization states. What is the ratio of the Stefan-Boltzmann constant for neutrinos to that of photons?

Answer:

The main difference between the neutrinos and the photons is the partition function. The mean energy of the neutrinos with a certain value of ν is

$$\bar{E} = \frac{\sum_{i=0}^{1} nh\nu e^{-nh\nu/kT}}{\sum_{i=0}^{1} e^{-nh\nu/kT}}. \qquad \text{(B.3)}$$

For photons the sum is from 0 to infinity. So we have

$$B_\nu(T) = \frac{2h}{c^2} \frac{\nu^3}{\exp(h\nu/kT) + 1}. \qquad \text{(B.4)}$$

for neutrinos. The ratio of the Stefan-Boltzmann constants is

$$R = \frac{\int_0^\infty x^3 (e^x + 1)^{-1}}{\int_0^\infty x^3 (e^x - 1)^{-1}} = \frac{7\pi^4/120}{\pi^4/15} = \frac{7}{8} \qquad \text{(B.5)}$$

4. **Surface Emission from the Crab Pulsar:** The neutron star that powers the Crab Pulsar can be assumed to have a mass of $1.4\mathrm{M}_\odot$ and a radius of 10 km with constant internal density and an effective temperature of 10^6 K. The frequency of the Crab Pulsar is 30 Hz and its period increases by 38 ns each day. Compare the power from the surface emission to the power lost as the neutron star spins down. The total power of the Crab Nebulae is about 75,000 times that of the Sun. What is the likely source of this power?

   Answer:

   The blackbody flux from the surface of the star is given by

$$F = 4\pi R^2 \sigma T^4 = 7 \times 10^{32} \text{ erg/s} = 7 \times 10^{25} \text{ W} = 0.17\mathrm{L}_\odot. \qquad \text{(B.6)}$$

   As the neutron star spins down it loses kinetic energy at a rate

$$\frac{dE}{dt} = -I\Omega\dot{\Omega} = -4\pi^2\nu^3 I\dot{P} = -5 \times 10^{38} \text{ erg/s} = -5 \times 10^{31} \text{ W} = 10^5\mathrm{L}_\odot \qquad \text{(B.7)}$$

   where $I \approx \frac{2}{5}MR^2 \approx 10^{45}\text{gcm}^2$. The spin-down power is approximately the power needed to power the nebula so it is a possible source of energy.

5. **Power-Law Atmosphere**

   Assume the following

   - The Rosseland mean opacity is related to the density and temperature of the gas through a power-law relationship,

$$\kappa_R = \kappa_0 \rho^\alpha T^\beta; \qquad \text{(B.8)}$$

   - The pressure of the gas is given by the ideal gas law;
   - The gas is in hydrostatic equilibrium so $p = g\Sigma$ where g is the surface gravity; and

- The gas is in radiative equilibrium with the radiation field so the flux is constant with respect to z or Σ.

Calculate the temperature of the gas as a function of Σ.

Answer:

First we take the equation of radiative transfer

$$F(z) = -\frac{16\sigma T^3}{3\kappa_R}\frac{\partial T}{\partial \Sigma} = \frac{16\sigma T^3}{3\kappa_0 \rho^\alpha T^\beta}\frac{\partial T}{\partial \Sigma} \tag{B.9}$$

We eliminate the variable ρ using the ideal gas law and the equation of hydrostatic equilibrium,

$$g_s \Sigma = \frac{1}{\mu m_p}\rho k T \tag{B.10}$$

so we have

$$\frac{\partial T}{\partial \Sigma} = \frac{3\kappa_0}{16\sigma F}\Sigma^\alpha T^{\beta-\alpha-3}\left(\frac{\mu m_p}{g_s k}\right)^\alpha \tag{B.11}$$

which can be integrated by the separation of variables to yield

$$\frac{T^{4+\alpha-\beta}}{4+\alpha-\beta} = \frac{3\kappa_0}{16\sigma F}\frac{\Sigma^{\alpha+1}}{\alpha+1}\left(\frac{\mu m_p}{g_s k}\right)^\alpha \tag{B.12}$$

7. **Goggles**

Calculate from thermodynamic principles how much objects are magnified or demagnified while viewed through goggles underwater. N.B. The wavenumber of a photon of a given frequency is proportional to the index of refraction.

Answer:

If we have a blackbody underwater and a blackbody in air at equal temperatures, the underwater blackbody will emit

$$F_{water} = n^2 F_{air} \tag{B.13}$$

energy per unit area per unit time. You can see this from the definition of the density of states

$$\rho_s = 4\pi k^2 dk = 4\pi \left(\frac{n\nu}{c}\right)^2 d\left(\frac{n\nu}{c}\right) \tag{B.14}$$

which is larger by a factor of n^3, so the energy density within the water of the blackbody radiation is larger by a factor of n^3 than in air. However, flux is related to the intensity which is energy density times the velocity so the flux is only larger by a factor of n^2.

For the underwater blackbody to absorb as much as radiation from the blackbody in air as the blackbody in air receives from it, the solid angle subtended by the underwater BB must be larger by n^2 so it is magnified linearly by a factor of$n \approx 1.33$.

Chapter 2

1. **Coulomb's Law**

   Derive Coulomb's law from Maxwell's Equations

   Answer:

   The first of Maxwell's equations is

$$\nabla \cdot \vec{E} = 4\pi\rho \tag{B.15}$$

   Let's assume that there is a single charge q located at r=0 and integrate over a spherical region centered on the origin we get

$$\int_V dV \nabla \cdot \vec{E} = \int dV 4\pi\rho = 4\pi q \tag{B.16}$$

   However the integral of the left-hand side is a integral of a divergence over a volume so we have

$$\int_{\partial V} dV \nabla \cdot \vec{E} = \int \vec{E} \cdot dA = |\vec{E}| 4\pi R^2 = 4\pi q \tag{B.17}$$

   so

$$\vec{E} = \frac{q}{R^2}\hat{r} \tag{B.18}$$

2. **Ohm's Law**

   In certain cases the process of aborption of radiation can be treated by means of the macroscopic Maxwell equations. For example, suppose we have a conducting medium so that the current density j is related to the electric field E by Ohm's law: $\vec{j} = \sigma\vec{E}$ where σ is the conductivity (cgs unit = sec^{-1}</sup>). Investigate the propagation of electromagnetic waves in such a medium and show that:

   (a) The wave vector $\vec{k}$ is complex $\vec{k}^2 = \frac{\omega^2 m^2}{c^2}$ where m is the complex index of refraction with

$$m^2 = \mu\epsilon\left(1 + \frac{4\pi i\sigma}{\omega\epsilon}\right) \tag{B.19}$$

   (b) The waves are attenuated as they propagate, corresponding to an absorption coefficient.

$$\alpha = \frac{2\omega}{c}\Im(m) \tag{B.20}$$

   Answer:

   Let's take the third and fourth of Maxwell's equations

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{B.21}$$

and

$$\nabla \times \vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t} \tag{B.22}$$

Let's substitute

$$\mu \vec{H} = \vec{B}, \vec{D} = \epsilon \vec{E} \tag{B.23}$$

and

$$\vec{J} = \sigma \vec{E} \tag{B.24}$$

to get

$$\nabla \times \vec{B} = \mu\sigma \frac{4\pi}{c}\vec{E} + \frac{1}{c}\mu\epsilon \frac{\partial \vec{E}}{\partial t} \tag{B.25}$$

Let's take the curl of this equation to get

$$-\nabla^2 \vec{B} = \mu\sigma \frac{4\pi}{c}\nabla \times \vec{E} + \frac{1}{c}\mu\epsilon \frac{\partial \nabla \times \vec{E}}{\partial t} \tag{B.26}$$

and substitute in the other Maxwell's equation to get

$$-\nabla^2 \vec{B} = -\mu\sigma \frac{4\pi}{c^2}\frac{\partial \vec{B}}{\partial t} - \frac{1}{c^2}\mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2} \tag{B.27}$$

Let's substitute

$$\vec{B} = \vec{B}_0 \exp\left[i\left(\vec{k}\cdot\vec{x} - \omega t\right)\right] \tag{B.28}$$

to get

$$k^2 = i\mu\sigma\frac{4\pi}{c^2}\omega + \frac{1}{c^2}\mu\epsilon\omega^2 = \frac{\omega^2 m^2}{c^2} \tag{B.29}$$

with

$$m^2 = \mu\epsilon\left(1 + i\frac{4\pi\sigma}{\omega\epsilon}\right) \tag{B.30}$$

If we substitute this into the formula for the wave we find

$$\vec{B} = \vec{B}_0 \exp\left[i\left(\vec{k}\cdot\vec{x} - \omega t\right)\right] = \vec{B}_0 \exp\left[i\left(\Re\vec{k}\cdot\vec{x} - \omega t\right)\right]\exp\left[-\Im\vec{k}\cdot\vec{x}\right] \tag{B.31}$$

so the magnetic field decreases with a mean-free path $\frac{c}{\omega \Im m}$. The energy is proportional to B^2 so the absorption coefficient is $\frac{2\omega}{c}\Im m$

3. **The Edge of the Crab**

Fig. 2.3 shows the x-ray emission of the Crab pulsar wind nebula at a distance of 2 kpc. The x-ray emitting gas is contained by magnetic fields causing the x-ray emission regions to end sharply. We can relate the frequency of the emission to the energy of the electrons and the strength of the magnetic field by

$$\omega = \left(\frac{E}{m_e c^2}\right)^2 \frac{eB}{m_e c} \tag{B.32}$$

and assume that the electrons are relativistic so their inertial mass is E/c^2. Use the sharpness of the emission regions to determine the energy of the electrons and the strength of the magnetic field.

Answer:

Let's assume that the particles are doing cyclotron motion so

$$F = \frac{mv^2}{r} = \frac{evB}{c}, v \approx c, m \approx \frac{E}{c^2} \tag{B.33}$$

so

$$\frac{E}{r} = eB, E = eBr$$

where $r < 2$ kpc $\times$ 1 arcsecond $= 2000$ AU and we also know that ω lies in the X-rays, so $\hbar\omega \sim 1$ keV and

$$\omega = E^2 B \frac{e}{m_e^3 c^5} = E^2 \frac{E}{er} \frac{e}{m_e^3 c^5}$$

so

$$E^3 = m_e^3 c^5 \omega r, B^3 = \frac{m_e^3 c^5 \omega}{e^3 r^2}.$$

The fact the the edge is unresolved yields an upper limit on the energy and a lower limit on the magnetic field strength. If we use $\hbar\omega = 1$ keV, we obtain

$$E < 6 \times 10^{13} \text{ eV}, B > 7 \times 10^{-11} \text{ G}$$

4. **Momenta** This problem is meant to deduce the momentum and angular momentum properties of radiation and does not recesarily represent any real physical system of interest. Consider a charge Q in a viscous medium where the viscous force is proportional to velocity:

$$F_{visc} = -\beta v \tag{B.34}$$

Suppose a circular polarized wave passes through the medium. The equation of motion of the charge is

$$m\frac{dv}{dt} = F_{visc} + F_{Lorentz} \tag{B.35}$$

We assume that the terms on the right dominate the inertial term on the left, so that approximately

$$0 = F_{visc} + F_{Lorentz} \tag{B.36}$$

Let the frequency of the wave be ω and the strength of the electric field be E.

(a) Show that to lowest order (neglecting the magnetic force) the charge moves on a circle in a plane normal to the direction of propagation of the wave with speed QE/β and with radius $QE/(\beta\omega)$.

(b) Show that the power transmitted to the fliud by the wave is Q^2E^2/β

(c) . By considering the small magnetic force acting on the particle show that the momentum per unit time (force) given to the fluid by the wave is in the direction of propagation and has the magnitude $Q^2E^2/(\beta c)$.

(d) Show that the angular momentum per unit time (torque) given to the fluid by the wave is in the direction of propagation and has magnitude $\pm Q^2E^2/(\beta\omega)$ where the $+$ is for left and $-$ is for right circular polarization.

(e) Show that the absorption cross section of the charge is $4\pi Q^2/(\beta c)$.

(f) If we regard the radiation to be composed of circular polarized photons of energy $E_\gamma = h\nu$, show that these results imply that the photon has momemtum $p = h/\lambda = E_\gamma/c$ and has angular momemtum $J = \pm\hbar$ along the direction of propagation.

(g) Repeat this problem for a linearly polarized wave

Answer:

(a) We have $\vec{v} = \frac{Q}{\beta}\vec{E}$. The electric field traces a circle so the particle traces a circle with a speed $\frac{QE}{\beta}$. The angular velocity of the particle is ω of the wave, so $\omega r = \frac{QE}{\beta}$ so $r = \frac{QE}{/}\beta\omega$.

(b) Power is $Q\vec{v}\cdot\vec{E} = \frac{Q^2E^2}{\beta}$.

(c) The magnetic force is in the direction $\vec{v}\times\vec{B}$ but the velocity points in the direction of the electric field so the force is in the direction $\vec{E}\times\vec{B}$, the direction of propagation. The magnitude of magnetic field equals that of the electric field so we have $F = \frac{Q^2E^2}{\beta c}$

(d) Torque is $\vec{r}\times\vec{F} = \frac{Q^2E^2}{\beta\omega}$

(e) The cross section is power absorbed divided by the Poynting vector

$$\sigma = \frac{Q^2E^2}{\beta}\left[\frac{c}{4\pi}E^2\right]^{-1} = \frac{4\pi Q^2}{\beta c} \tag{B.37}$$

(f) If the wave comes in energy units of $h\nu$. The ratio of the momentum unit to the energy unit must be the ratio of the force (momentum per unit time) to the power (energy per unit time), so we get

$$h\nu\frac{Q^2E^2}{\beta c}\frac{\beta}{Q^2E^2} = \frac{h\nu}{c} \tag{B.38}$$

The ratio of the angular momentum unit to the energy unit must be the ratio of the torque (angular momentum per unit time) to the power (energy per unit time), so we get

$$h\nu \frac{Q^2 E^2}{\beta\omega} \frac{\beta}{Q^2 E^2} = \frac{h\nu}{\omega} = \hbar \tag{B.39}$$

(g) For the linearly polarized wave, the particle moves up and down sinusoidally. The size of the up and down path is twice the value of R above (the circle is squished along one axis to be a line). The velocity varies sinusoidally, the power magnetic force vary as $\sin^2 \omega t$. The torque vanishes. The cross section is the same as is the momentum of a photon. The angular momentum vanishes (because the torque vanishes).

5. **Maxwell before Maxwell**

Show that Maxwell's equations before Maxwell, that is, without the "displacement current" term, $c^{-1}\frac{\partial D}{\partial t}$, unacceptably constrained the sources of the field and also did not permit the existence of waves.

Answer:

Let's take the divergence of the Maxwell's equation

$$\nabla \times \vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t} \tag{B.40}$$

to get

$$0 = \frac{4\pi}{c}\nabla \cdot \vec{J} \tag{B.41}$$

where we have left the displacement current out. This states that the divergence of the current must vanish, which means that either charge is not conserved or that the charge density is constant (neither is good).

Let's take the curl of the Maxwell's equation

$$\nabla^2 \vec{B} = 0 \tag{B.42}$$

and we would get the same thing for the electric field. This is not a wave equation.

6. **Coulomb gauge** Derive the equations describing the dynamics of the electric and vector potentials in the Coulomb gauge

$$\nabla \cdot \mathbf{A} = 0$$

Look at the equation for the electric potential. What is the solution to the electric potential given the charge density ρ? Why is this called the Coulomb gauge?

How does the expression for the scalar potential in the Coulomb gauge differ from that in the Lorenz gauge? What is strange about it? Is it physical?

Now look at the equation for the vector potential. Show that the LHS can be arranged to be the same as in the Lorenz gauge but the RHS is not just the current but the current plus something else.

Show that the RHS can be expressed as

$$\frac{4\pi}{c}\left(\mathbf{J}-\mathbf{J}_{\text{long}}\right)$$

where

$$\mathbf{J}_{\text{long}} = -\frac{1}{4\pi}\nabla\int\frac{\nabla'\cdot\mathbf{J}}{|\mathbf{x}-\mathbf{x}'|}d^3x$$

Answer: In the Coulomb gauge the scalar potential follows Coulomb's law

$$\nabla^2\phi = -4\pi\rho.$$

That is why it is called the Coulomb gauge. The potential everywhere right now depends on the charge here right now, so it is acausal (strange); however, because we cannot actually measure the scalar potential the acausality has no physical consequence.

Now for the vector potential

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right) = -\frac{4\pi}{c}\mathbf{J}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\left(\mathbf{J}-\mathbf{J}_{\text{long}}\right)$$

where

$$\begin{aligned}\mathbf{J}_{\text{long}} &= \frac{1}{4\pi}\nabla\left(\frac{\partial\phi}{\partial t}\right) = \frac{1}{4\pi}\nabla\left(\int\frac{\partial\rho}{\partial t}\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3x\right)\\ &= -\frac{1}{4\pi}\nabla\left(\int\frac{\nabla\cdot\mathbf{J}}{|\mathbf{x}-\mathbf{x}'|}d^3x\right) = -\frac{1}{4\pi}\nabla\left(\int\frac{\nabla'\cdot\mathbf{J}}{|\mathbf{x}-\mathbf{x}'|}d^3x\right)\end{aligned}$$

What remains of the current after subtracting the longitudinal current is the transverse current which is given by the expression

$$\mathbf{J}_{\text{trans}} = \frac{1}{4\pi}\nabla\times\nabla\left(\int\frac{\mathbf{J}}{|\mathbf{x}-\mathbf{x}'|}d^3x\right)$$

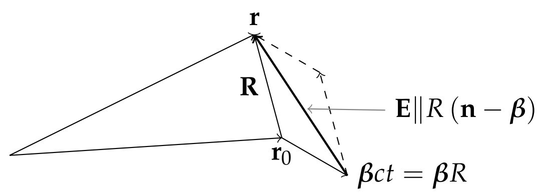
so the source for the wave equation for $\mathbf{A}$ is given by the transverse current alone.

Chapter 3

1. **Constant Velocity Charge**

Show that if charge is not accelerating, the electric field vector points to the current (not the retarded) position of the charge.

Answer:



4. **Synchrotron Cooling:**

A particle of mass m, charge q, moves in a plane perpendicular to a uniform, static, magnetic field B.

(a) Calculate the total energy radiated per unit time, expressing it in terms of the constants already defined and the ratio $\gamma = 1/\sqrt{1-\beta^2}$ of the particle's total energy to its rest energy. You can assume that the particle is ultrarelativistic.

(b) If at time $t = 0$ the particle has a total energy $E_0 = \gamma_0 mc^2$, show that it will have energy $E = \gamma mc^2 < E_0$ at a time t, where

$$t \approx \frac{3m^3c^5}{2q^4B^2}\left(\frac{1}{\gamma} - \frac{1}{\gamma_0}\right).$$

Answer:

The synchrotron power is given by the power emitted by a particle performing circular motion

$$P_\perp = \frac{2}{3}\frac{q^2}{m^2c^3}\gamma^2\left(\frac{d\mathbf{p}}{dt}\right)^2$$

where for an ultrarelativistic charged particle in a magnetic field we have

$$\left|\frac{d\mathbf{p}}{dt}\right| = qB$$

so

$$P_\perp = \frac{2}{3}\frac{q^5}{m^2c^3}\gamma^2B^2 = -\frac{dE}{dt} = -mc^2\frac{d\gamma}{dt}$$

and

$$\frac{d\gamma}{dt} = -\frac{2}{3}\frac{q^5}{m^3c^5}B^2\gamma^2.$$

Separating the variables and integrating yields the required answer.

5. **Classical HI:** A particle of mass m and charge q moves in a circle due to a force $\mathbf{F} = -\hat{\mathbf{r}}\frac{q^2}{r^2}$. You may assume that the particle always moves non-relativistically.

   (a) What is the acceleration of the particle as a function of r?

   (b) What is the total energy of the particle as a function of r? The potential energy is given by $-q^2/r$.

   (c) What is the power radiated as a function of r?

   (d) Using the fact the $P = -dE/dt$ and the answer to (2), find dr/dt.

   (e) Assuming that the particle starts with $r = r_i$ at $t = 0$, find the value of t where $r = 0$.

   (f) Let's assume that $q = e$, the charge of the electron, and $m = m_e$, the mass of the electron. Write your answer in (4) in terms of r_i, r_0 (the classical electron radius) and c.

   (g) What is the time if $r_i = 0.5$Å(for an hydrogen)?

   (h) Compare this to the lifetime of a hydrogen atom.

Answer:

(a)

$$\dot{\mathbf{u}} = -\hat{r}\frac{q^2}{r^2 m}$$

(b)

$$E = -\frac{q^2}{r} + \frac{1}{2}mv^2 = -\frac{q^2}{r} + \frac{1}{2}\left(\frac{q^2}{r}\right) = -\frac{1}{2}\frac{q^2}{r}$$

where I used

$$\frac{mv^2}{r} = \frac{q^2}{r^2}$$

for circular motion.

(c)

$$P = \frac{2q^2\dot{u}^2}{3c^3} = \frac{2q^2}{3c^3}\left(\frac{q^2}{r^2 m}\right)^2$$

(d)

$$\frac{dE}{dt} = \frac{d}{dt}\left(-\frac{1}{2}\frac{q^2}{r}\right) = \frac{1}{2}\frac{q^2}{r^2}\frac{dr}{dt}$$

$$\frac{dE}{dt} = -P = -\frac{2q^6}{3m^2c^3}\frac{1}{r^4} = \frac{1}{2}\frac{q^2}{r^2}\frac{dr}{dt}$$

$$\frac{dr}{dt} = -\frac{4q^4}{3m^2c^3}\frac{1}{r^2}$$

(e)

$$t = \int_{r_i}^{0} \frac{dt}{dr} dr = -\frac{3m^2c^3}{4q^4} \int_{r_i}^{0} r^2 dr = \frac{r_i^3 m^2 c^3}{4q^4}$$

(f)

$$t = \frac{r_i^3 m^2 c^3}{4e^4} = \frac{1}{4c} r_i \left(\frac{r_i}{r_0}\right)^2$$

(g)

$$r_0 = 2.82 \times 10^{-13} \text{ cm}, 1\text{Å} = 10^{-8} \text{ cm}$$

$$t = \frac{1}{12 \times 10^{10} \text{cm/s}} 0.5 \times 10^{-8} \text{ cm} (17000)^2 = 1.2 \times 10^{-11} \text{ s}$$

(h) It is much smaller than the lifetime of a hydrogen atom.

6. **The Eddington Luminosity:**

There is a natural limit to the luminosity a gravitationally bound object can emit. At this limit the inward gravitational force on a piece of material is balanced by the outgoing radiation pressure. Although this limiting luminosity, the Eddington luminosity, can be evaded in various ways, it can provide a useful (if not truly firm) estimate of the minimum mass of a particular source of radiation.

(a) Consider ionized hydrogen gas. Each electron-proton pair has a mass more or less equal to the mass of the proton (m_p) and a cross section to radiation equal to the Thompson cross-section (σ_T).

(b) The radiation pressure is given by outgoing radiation flux over the speed of light.

(c) Equate the outgoing force due to radiation on the pair with the inward force of gravity on the pair.

(d) Solve for the luminosity as a function of mass.

The mass of the sun is 2×10^{33}g. What is the Eddington luminosity of the sun?

Answer:

(a) OK

(b) $P = \frac{F}{c} = \frac{L}{4\pi r^2 c}$

(c) $F_{\text{out}} = P\sigma_T = \frac{L\sigma_T}{4\pi r^2 c}$, $F_{\text{in}} = \frac{GMm_p}{r^2}$

(d) $F_{\text{in}} = F_{\text{out}}$ for $L = L_{\text{Edd}}$ so $L_{\text{Edd}} = \frac{4\pi c G M m_p}{\sigma_T}$

(e) $L_{\text{Edd}} = 1.26 \times 10^{38} \text{erg/s} \left(\frac{M}{M_\odot}\right) = 3.2 \times 10^4 \left(\frac{M}{M_\odot}\right) L_\odot$.

Chapter 4

1. **The Ladder and the Barn: A Spacetime Diagram:**

This problem will work best if you have a sheet of graph paper. In a spacetime diagram one draws a particular coordinate (in our case x) along the horizontal direction and the time coordinate vertically. People also generally draw the path of a light ray at 45°. This sets the relative units of the two axes.

(a) Draw a spacetime diagram and label the axes x and t. The t-axis is the path of Emma through the spacetime.

(b) Draw the world line of someone travelling at $\frac{3}{5}$ of the speed of light. The world line should intersect with the origin of the spacetime diagram. Label this world line t'. The t'-axis is the path of Kara through the spacetime.

(c) Draw the x' axis on the graph. Here's a hint about where it should go. The light ray bisects the angle between the x and t axes. Kara who is travelling along t' will find that the speed of light is the same for her, so the light ray must also bisect the angle between x' and t'.

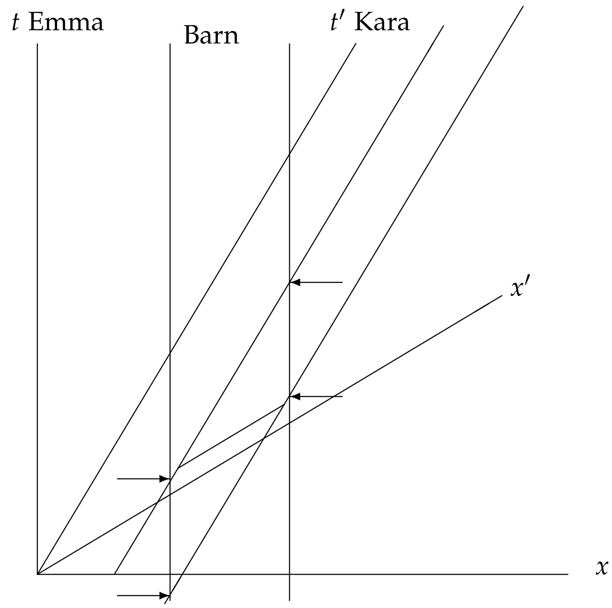
(d) Parallel to Emma's time axis draw the walls of the barn in pencil. The barn is 4.5 meters wide in Emma's frame.

(e) Draw Kara's ladder along Kara's x-axis. The ladder is 5 meters long in Kara's frame. How long is it in Emma's frame.

(f) Draw the world lines of the ends of Kara's ladder. These lines are parallel to Kara's time axis.

(g) Erase a portion of the barn walls to allow Kara's ladder to fit through.

(h) Using the diagram, explain how Kara and Emma can understand how the too-long ladder fits in the too-small barn.



Erase the sections between the arrows. Emma sees the ladder inside the barn with the two doors closed at the same time. Kara sees the forward door open before the back door has shut.

2. **The Fermi Process:**

One model to understand how cosmic rays are accelerated is through shocks. The main idea is that a charge particle can cross a shock and turned around by the tangled mangetic field and recross the shock. Each time the charge does this it gains energy.

To understand this let's use a simplified model in which two mirrors are travelling toward each other at some velocity v. When a particle hits the mirror, its energy in the frame of the mirror remains unchanged but its velocity and therefore the spacelike components of the four-momentum change sign.

(a) Draw a diagram with the two mirrors.

(b) For argument's sake, let's first focus on the mirror on the left and consider that the mirror on the right is moving. What is the four-velocity in this frame of the mirror on the left (U_l^μ)? What is the four-velocity in this frame of the mirror on the right (U_r^μ)?

$$U_l^\mu = \begin{bmatrix} c \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad U_r^\mu = \begin{bmatrix} \gamma c \\ -\gamma v \\ 0 \\ 0 \end{bmatrix}$$

(c) Now let's focus on the mirror on the right and consider that the mirror on the left is moving. What is the four-velocity in this frame of the mirror on the left ($U_l'^\mu$)? What is the four-velocity in this frame of the mirror on the right ($U_r'^\mu$)?

$$U_l^\mu = \begin{bmatrix} \gamma c \\ \gamma v \\ 0 \\ 0 \end{bmatrix} \qquad U_r^\mu = \begin{bmatrix} c \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(d) To start let's assume that the particle of mass m approaches the mirror on the left at the velocity of the mirror on the right. What is the four-momemtum of the particle (p^μ) in the frame of the mirror on the left?

$$p^\mu = \begin{bmatrix} m\gamma c \\ -m\gamma v \\ 0 \\ 0 \end{bmatrix}$$

(e) The particle bounces off of the mirror. What is its four-momentum now?

$$p^\mu = \begin{bmatrix} m\gamma c \\ m\gamma v \\ 0 \\ 0 \end{bmatrix}$$

(f) Now the particle is approaching the mirror on the right. What is the zeroth component of the four-momentum of the particle in the frame of the right-hand mirror? One could do a Lorentz transformation but it is easier to use $U_r^\mu p_\mu$ to determine the energy of the particle in the primed frame.

$$U_r^\mu p_\mu = \begin{bmatrix} \gamma c \\ -\gamma v \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} m\gamma c & -m\gamma v & 0 & 0 \end{bmatrix} = m\gamma^2 \left(c^2 + v^2\right)$$

(g) Using the answer to 6, construct the four-momentum of the particle in the frame of the right-hand mirror (p'_μ).

$$p^\mu = \begin{bmatrix} mc\frac{1+\beta^2}{1-\beta^2} \\ m\frac{2v}{1-\beta^2} \\ 0 \\ 0 \end{bmatrix}$$

(h) The particle bounces off of the mirror. What is its four-momentum now?

$$p^\mu = \begin{bmatrix} mc\frac{1+\beta^2}{1-\beta^2} \\ -mc\frac{2\beta}{1-\beta^2} \\ 0 \\ 0 \end{bmatrix}$$

(i) Now the particle is approaching the mirror on the left. What is the zeroth component of the four-momentum of the particle in the frame of the left-hand mirror? Again one could do a Lorentz transformation but it is easier to use $U_l'^\mu p_\mu'$ to determine the energy of the particle in the unprimed frame.

$$U_l^\mu p_\mu = \begin{bmatrix} \gamma c \\ \gamma v \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} mc\gamma^2(1+\beta^2) & 2\beta\gamma^2 mc & 0 & 0 \end{bmatrix} = mc^2\gamma^3\left(1+3\beta^2\right)$$

(j) Compare the energy of the particle in step (d) to the energy of the particle in step (i). Has the energy of the particle increased? Let's let the relative velocity of the mirrors approach the speed of light.

$$\beta \approx 1 - \frac{1}{2\gamma^2}$$

By what factor does the energy of the particle increase each time it goes back and forth.

The energy has increased by a factor of

$$\gamma^2\left(1+3\beta^2\right) \approx 4\gamma^2$$

(k) The final element is the fact that only a tiny fraction of the particles bounce back and forth. Let's take that fraction to be 10^{-5} and $\gamma = 100$. What can you say about the final distribution of particle energies?

The final distribution will be a power-law with slope given by

$$s = \ln 10^{-5} / \ln(4\gamma^2) \approx -1.1$$

3. **Boosting** We are going to figure out how times and energies measured by someone in motion differ from what we might measure.

   (a) Use special relativity (the Minkowski metric) to figure this out. I measure a photon to have an energy E. What is the four-momentum of the photon?

   (b) My pal is travelling toward me in the opposite direction of the photon at a velocity βc. What is his four-velocity? Use the definition $\gamma = \left(1-\beta^2\right)^{-1/2}$ to simplify the expression. What energy would he measure for the photon? What does the expression look like as γ gets much larger than one?

   (c) If my pal observes the photon to have an energy of 100 MeV while I say its energy is less than 500 keV, what is the minimal value of γ for my pal (take $\beta \approx 1$ to make life easier)?

   (d) My pal is still coming toward me at a velocity βc. When he is a distance r away from me (at a time t_0) he emits a photon toward me. How long does it take this photon to reach me?

   (e) From his point of view a short time Δt later he emits another photon toward me. How long is Δt in my frame and when do I receive the second photon? What is the difference in time between when I receive the first and second photons? What does the expression look like as γ gets much larger than one? Compare it with you answer to (b).

Answer:

(a)

$$p^\mu = \frac{E}{c}\begin{bmatrix} 1 \\ \mathbf{n} \end{bmatrix} \text{ Take } p^\mu = \begin{bmatrix} \frac{E}{c} \\ \frac{E}{c} \\ 0 \\ 0 \end{bmatrix} \tag{B.43}$$

(b)

$$u^\mu = \begin{bmatrix} \gamma c \\ -\beta\gamma c \\ 0 \\ 0 \end{bmatrix} \text{ and } E' = -u_\mu p^\mu = \gamma E + \beta\gamma E \approx 2\gamma E \tag{B.44}$$

(c) $E = 500$ keV and $E' = 100$ Mev $= 2\gamma(500$ keV$)$ so $\gamma_{\text{min}} = 100$.

(d) $t_{\text{Arrival}} = t_0 + \frac{r}{c}$

(e)

$$\Delta t_{\text{me}} = \gamma \Delta t_{\text{him}} \tag{B.45}$$

$$t_{\text{Arrival,2}} = t_0 + \gamma\Delta t_{\text{him}} + \frac{1}{c}\left(r - \beta c \gamma \Delta t_{\text{him}}\right) \tag{B.46}$$

$$= t_0 + \frac{r}{c} + \gamma \Delta t_{\text{him}} (1 - \beta) \quad \text{(B.47)}$$

$$\Delta t_{\text{Arrival}} = \Delta t_{\text{him}} \gamma (1 - \beta) \quad \text{(B.48)}$$

$$\Delta t_{\text{Arrival}} = \Delta t_{\text{him}} \frac{1}{\gamma (1 + \beta)} \approx \Delta t_{\text{him}} \frac{1}{2\gamma}. \quad \text{(B.49)}$$

where to get the penultimate result, one uses the identity $(1 - \beta)(1 + \beta) = \gamma^{-2}$ and in general we have

$$\frac{\Delta t_{\text{Arrival}}}{\Delta t_{\text{him}}} = \frac{E}{E'}. \quad \text{(B.50)}$$

Chapter 5

1. **Circular Orbit** The equation for a geodesic (an orbit) is given by

$$\frac{du^\mu}{ds} + \Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta = 0$$

where u^μ is the four-velocity. When an index in an expression is repeated you are supposed to sum over the index. The indices run through t, r, θ and ϕ.

(a) Let's suppose that the particle at one moment is just going around the center of the black hole so the velocities in the r and θ directions vanish and we'll take $\theta = \pi/2$ (the equatorial plane).

In this situation u^t and u^ϕ are the only components of the four velocity that don't vanish and $\Gamma^r{}_{tt}$ and $\Gamma^r{}_{\phi\phi}$ are the only Christoffel symbols that don't have vanishing coefficients. Write out the geodesic equations in terms of the Christoffel symbols (don't calculate the Christoffel symbols).

(b) We would like for the velocity to be constant around the circular orbit so we would like the first term in the geodesic equation to vanish. Solve for $\Omega = u^\phi / u^t$ in terms of the Christoffel symbols.

(c) The two Christoffel symbols that play a role are

$$\Gamma^r{}_{tt} = \frac{(r-2M)M}{r^3} \text{ and } \Gamma^r{}_{\phi\phi} = (2M - r)\sin^2\theta.$$

What is Ω in terms of M and r?

(d) Substitute your value of Ω into the Schwarzschild metric and calculate ds^2 along the circular orbit. Over what range of radii can a material object (a toaster, UBC undergrad etc.) travel in a circular orbit around a Schwarzschild black hole.

Answer:

(a) The geodesic equation gives

$$\frac{du^t}{d\tau} = \frac{du^\theta}{d\tau} = \frac{du^\phi}{d\tau} = 0$$

and

$$\frac{du^r}{d\tau} + \Gamma^r{}_{tt} u^t u^t + \Gamma^r{}_{\phi\phi} u^\phi u^\phi = 0$$

(b) Because the first term vanishes, the other terms cancel each other; this gives

$$\Omega^2 = \frac{u^\phi u^\phi}{u^t u^t} = -\frac{\Gamma^r{}_{tt}}{\Gamma^r{}_{\phi\phi}}$$

(c) $\Omega^2 = M/r^3$. Kepler would be pleased.

(d) $ds^2 = (1 - 3M/r)dt^2$. If $r > 3M$, then ds^2 is positive which is what you need for a teaster to orbit the black hole.

2. **Photon Orbit**

We are going to find a radius at which a light will orbit a black hole.

(a) Start with the Schwarzschild metric. We want a circular orbit so we will set $dr = 0, d\theta = 0$ and $\theta = \pi/2$. What is ds^2 for a photon (a photon travels along a null geodesic)? Solve for $(d\phi/dt)^2$.

(b) $d\phi/dt$ is simply Ω for the photon orbit. Kepler's third law works in the Schwarzschild spacetime for circular orbits. Solve for R.

Answer:

(a) $ds^2 = 0$ for a photon so we get

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{1 - \frac{2M}{r}}{r^2}$$

(b) $\Omega^2 = M/r^3$, so $M = r - 2M$ and $r = 3M$

3. **Thermodynamics and General Relativity:**

In general relativity if two bodies are in thermodynamic equilibrium,

$$\frac{T_1}{1 + z_1} = \frac{T_2}{1 + z_2}$$

We can exploit this relationship along with Kirchoff's law to derive some interesting facts about how light travels from a neutron star to our telescopes.

(a) Because everything is in thermodynamic equilibrium, we can safely assume that the neutron star of mass M and radius R emits as a blackbody at a temperature T. Calculate the total power emitted from the neutron star surface in the frame of the neutron star surface.

(b) Calculate the total power received at infinity. Let the redshift of the surface be z.

(c) Let the space surrounding the star be filled with blackbody radiation in thermal equilibrium with the surface of the neutron star. You can imagine that the neutron star is in a gigantic thermos bottle. Let T_∞ be the temperature of this blackbody radiation measured at infinity (i.e. $z = 0$). What is T_∞?

(d) Now here comes Kirchoff's law: in thermodynamic equilibrium a body emits as much as it receives. How much power does the neutron star absorb from the blackbody at infinity? This is the product of the surface area of the neutron star with the flux per unit area of the blackbody radiation.

(e) A conundrum: compare the answer to (b) with the answer to (f). They differ. Does the neutron star cool down because (b) is greater than (d)?

(f) The neutron star can't cool down because it is already in equilibrium, so one of our assumptions must be wrong. It turns out that the most innocuous sounding assumption is incorrect. The power that the neutron star absorbs is the product of its apparent surface area with the flux per unit area of the blackbody radiation. Let the apparent radius be R_∞ and recalculate the answer to (d).

(g) Equate (b) and (e) and solve for R_∞.

$R_\infty \neq R$ because in the vicinity of a neutron star light does not travel in a straight line. One can also derive the value of R_∞ by solving for a null geodesic that is tangent to the surface of the neutron star. What is the minimum value of R_∞ for a constant value of M? You will need to know that

$$1 + z = \frac{1}{\sqrt{1 - \frac{2GM}{Rc^2}}}$$

What is the value of R? Call this radius R_γ.

(h) Prove the size of the image of the neutron star must decrease or remain the same as the radius of the neutron star decreases. Use the fact that rays that we ultimately see remain outgoing throughout their journey to us (otherwise by symmetry they would hit the surface a second time).

(i) What happens to the size of the image of the star if the radius of the star is less that R_γ?

(j) The calculation of the apparent radius of the star from thermodynamics hinges on the assumption that the outgoing flux from the surface reaches infinity. For $R < R_\gamma$, the size of the image no longer increases while thermodynamics says it should, so we must conclude that for radii less than R_γ initially outgoing photons can become incoming photons.

From these arguments and spherical symmetry speculate what might happen to a photon emitted precisely at R_γ tangentially, i.e. neither ingoing or outgoing.

(k) Calculate how much radiation a star whose radius is less than R_γ will absorb.

The answer to (k) falls short of (b) again. We know that the star can't heat up, so an assumption must be wrong. Within R_γ not every photon emitted can escape to infinity, many photons return and hit the surface.

(l) Using the answers to (b) and (k), calculate the fraction of the outgoing photon flux emitted from the surface that manages to escape.

(m) Use the fact that a blackbody emits isotropically to determine the opening angle of the cone into which the escaping photons are emitted. This region is symmetric around the radial direction.

(n) Pat yourself on the back. You have derived many of the quirky things about the Schwarschild metric (the metric that surrounds a spherically symmetric mass distribution). List the key assumptions that you have made to make this derivation work.

Chapter 6

1. **Bremsstrahlung:**

   Consider a sphere of ionized hydrogen plamsa that is undergoing spherical gravitational collapse. The sphere is held at uniform temperature, T_0, uniform density and constant mass M_0 during the collapse and has decreasing radius R_0. The sphere cools by emission of bremsstrahlung radiation in its interior. At $t = t_0$ the sphere is optically thin.

   (a) What is the total luminosity of the sphere as a function of $M_0, R(t)$ and T_0 while the sphere is optically thin?

   (b) What is the luminosity of the sphere as a function of time after it becomes optically thick in terms of $M_0, R(t)$ and T_0?

   (c) Give an implicit relation in terms of $R(t)$ for the time t_1 when the sphere becomes optically thick.

   (d) Draw a curve of the luminosity as a function of time.

   Answer:

   (a) $L = \epsilon^{ff} \frac{4}{3}\pi R^3 = \frac{2^5 \pi e^6}{3hmc^3} \left(\frac{2\pi kT}{3m}\right)^{1/2} \left(\frac{M}{m_p \frac{4}{3}\pi R^3}\right)^2 \bar{g}_B \frac{4}{3}\pi R^3$, so $L \propto R^{-3}$.

   (b) $L = \sigma_{SB} T^4 4\pi R^2$

   (c) $\sigma_{SB} T^4 4\pi R^2 = \frac{2^5 \pi e^6}{3hmc^3} \left(\frac{2\pi kT}{3m}\right)^{1/2} \left(\frac{M}{m_p}\right)^2 \frac{3}{4\pi R^3} \bar{g}_B$

   (d) Draw your graph with luminosity increasing with time as $R(t)^{-3}$ and then decreasing after a certain time as $R(t)^2$.

Chapter 7

1. **Synchrotron Radiation:**

   An ultrarelativistic electron emits synchrotron radiation. Show that its energy decreases with time according to

$$\gamma = \gamma_0 \left(1 + A\gamma_0 t\right)^{-1}, A = \frac{2e^4 B_\perp^2}{3m^3 c^5}. \qquad \text{(B.51)}$$

   Here γ_0 is the initial value of γ and $B_\perp = B \sin \alpha$. Show that the time for the electron to lose half its energy is

$$t_{1/2} = \left(A\gamma_0\right)^{-1} \qquad \text{(B.52)}$$

   How do you reconcile the decrease of γ with the result of constant γ for motion in a magnetic field?

   Answer:

$$P = -\frac{dE}{dt} = -m_e c^2 \frac{d\gamma}{dt} = \frac{2}{3} r_0^2 c \beta^2 \gamma^2 B_\perp^2$$

   so

$$\frac{d\gamma}{dt} = -\frac{2}{3} \frac{e^4}{m_e^3 c^5} B_\perp^2 \gamma^2$$

   where we have taken $\beta \approx 1$. If the Lorentz factor $\gamma = \gamma_0$ at $t = 0$, integrating this yields

$$\frac{1}{\gamma} - \frac{1}{\gamma_0} = \frac{2}{3} \frac{e^4}{m_e^3 c^5} B_\perp^2 t,$$

   and rearranging yields the answers above.

2. **Synchrotron Cooling More Precisely:**

   Derive the evolution of the energy of the electron (or γ) evolves in time without making the ultrarelativistic approximation.

   Answer:

   Let's start with

$$\frac{d\gamma}{dt} = -\frac{2}{3} \frac{e^4}{m_e^3 c^5} B_\perp^2 \beta^2 \gamma^2 = -A(\gamma^2 - 1)$$

   so

$$-Adt = \frac{1}{2} \left[\frac{d\gamma}{\gamma - 1} - \frac{d\gamma}{\gamma + 1} \right]$$

   and the answer upon integrating is

$$\gamma = \coth\left(\coth^{-1} \gamma_0 + At\right).$$

Chapter 8

1. **The Sunyaev-Zeldovich Effect**

(a) Let's say that you have a blackbody spectrum of temperature T of photons passing through a region of hot plasma (T_e). You can assume that $T \ll T_e \ll mc^2/k$

What is the brightness temperature of the photons in the Rayleigh-Jeans limits after passing through the plasma in terms of the Compton $y-$parameter?

Answer:

$$T_{b,\text{initial}} = \frac{c^2}{2\nu^2 k} I_\nu \tag{B.53}$$

In Compton scattering, $I = I_\nu/(h\nu)$ is constant but $\nu_f = \nu_i e^y$ so we have

$$T_{b,\text{final}} = \frac{c^2}{2\nu^2 e^{2y} k} e^y I_\nu = e^{-y} T_{b,\text{initial}} \tag{B.54}$$

(b) Let's suppose that the gas has a uniform density ρ and consists of hydrogen with mass-fraction X and helium with mass-fraction Y and other stuff Z. You can assume that $Z/A = 1/2$ is for the other stuff. What is the number density of electrons in the gas?

Answer: One gram of the gas has X grams of hydrogen which provide X/m_p electrons. It has Y grams of helium which provides $(Z/A)Y/m_p = 2/4Y/m_p$ electrons and Z grams of other stuff which provides $1/2Y/m_p$ electrons. Adding it up gives

$$n_e = \frac{\rho}{m_p}\left(X + \frac{1}{2}Y + \frac{1}{2}Z\right) = \frac{\rho}{2m_p}(2X + 1 - X) = \frac{\rho}{2m_p}(1 + X) \tag{B.55}$$

(c) If you assume that the gas is spherical with radius R, what is the value of the Compton $y-$parameter as a function of b, the distance between the line of sight and the center of the cluster? You can assume that the optical depth is much less than one.

Answer: The distance through the cluster is given by

$$l = 2\sqrt{R^2 - b^2} \tag{B.56}$$

so the optical depth is

$$\tau_{es} = l n_e \sigma_T = 2\sqrt{R^2 - b^2}\frac{\rho}{2m_p}(1 + X)\sigma_T \tag{B.57}$$

so

$$y_{NR} = \frac{4kT}{mc^2}\tau_{es} = \frac{4kT}{mc^2}\sqrt{R^2 - b^2}\frac{\rho}{m_p}(1 + X)\sigma_T \tag{B.58}$$

(d) Let's assume that the sphere contains 10^{12} $M_\odot$ of gas and that the radius of the sphere is 10 Mpc, $X = 0.7, Y = 0.27$ and $Z = 0.03$ what is the value of the $y-$parameter?

Answer: The density of the cluster gas is

$$\rho = \frac{10^{12} M_\odot (2 \times 10^{33} \text{g}/M_\odot)}{\frac{4}{3}\pi(10\text{Mpc}(3.08 \times 10^{24}\text{cm}/\text{Mpc}))^3} = 0.16 \times 10^{-31} \text{g}/\text{cm}^3 \tag{B.59}$$

This is actually really low. A realistic cluster is more massive that this. Let's plug these values in the formula for y_{NR} and pick a reasonable value for $kT = 10$ keV so we get

$$y_{NR} = 2 \times 10^{-8} M_{12} R_{10}^{-2} T_{10} \tag{B.60}$$

We can estimate the temperature of the cluster gas using the virial theorem

$$2\frac{M}{m_p} kT \approx \frac{3}{5}\frac{GM^2}{R} \tag{B.61}$$

so

$$kT \approx \frac{3}{10}\frac{GMm_p}{R} \approx 1.3\text{eV} M_{12} R_{10}^{-1} \tag{B.62}$$

(e) Let's suppose that the blackbody photons are from the cosmic microwave background. What is the difference in the brightness temperature of the photons that pass through the cluster and those that don't (including the sign)? How does this difference compare with the primordial fluctuations in the CMB? How can you tell this change in the spectrum due to the cluster from the primordial fluctuations?

Answer:

The photons that pass through the cluster have a brightness temperature that is lower by $2yT_{\text{CMB}}$. The fluctuations of the CMB are around $10^{-5} T_{\text{CMB}}$, so for such a puny cluster the S-Z would be hard to see. However, clusters are generally much more massive so the S-Z dominates over the fluctuations. Furthermore, the S-Z shifts photons to higher energies which is different than CMB fluctuations which change the temperature, so observations at energies in the Rayleigh-Jeans and Wein tail of the CMB spectrum can distinguish between the S-Z effect and primordial fluctuations.

2. **Synchrotron Self-Compton Emission Blazars**

(a) What is the synchrotron emission from a single electron passing through a magnetic field in terms of the energy density of the magnetic field and the Lorentz factor of the electron?

Answer:

$$P_B = \frac{4}{3}\gamma^2 c\beta^2 \sigma_T U_B \tag{B.63}$$

(b) The number density of the electrons is n_e and they fill a spherical region of radius R. What is the energy density of photons within the sphere, assuming that it is optically thin?

Answer: Pn_e gives the power per unit volume. To get the energy per unit volume we have to multiply by the typical time for photons to escape the spherical region typically R/c because it is optically thin so we have

$$U_{\text{photon}} = \frac{4}{3}\gamma^2 \sigma_T c \beta^2 U_B n_e \frac{R}{c} \tag{B.64}$$

(c) What is the inverse Compton emission from a single electron passing through a gas of photons field in terms of the energy density of the photons and the Lorentz factor of the electron?

Answer:

$$P_{\text{IC}} = \frac{4}{3}\gamma^2 c \beta^2 \sigma_T U_{\text{photon}} \tag{B.65}$$

(d) What is the total inverse Compton emission from the region if you assume that the synchrotron emission provides the seed photons for the inverse Compton emission?

Answer:

$$P_{\text{IC}} = \frac{4}{3}\gamma^2 c \beta^2 \sigma_T \left(\frac{4}{3}\gamma^2 c \beta^2 \sigma_T U_B n_e \frac{R}{c} \right) n_e V \tag{B.66}$$

so

$$P_{\text{IC}} = \frac{64}{27}\gamma^4 \beta^4 c \sigma_T^2 U_B n_e^2 R^4 \tag{B.67}$$

Chapter 9

1. **Particles in a Box**

A reasonable model for the neutrons and protons in a nucleus is that they are confined to a small region. Let's take a one-dimensional model of this. The potential is $V(x)$ is zero everywhere for $0 < x < l$ and infinite otherwise. This means that

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E_n\psi \text{ if } 0 < x < l \tag{B.68}$$

and $\psi = 0$ if $x < 0$ or $x > l$. What are the energy levels of this system?

Answer: The harmonic functions the sine and cosine have the property that the second derivative is proportional to the function itself. We have $\psi = 0$ at $x = 0$ and at $x = l$ so

$$\psi_n = N \sin\left(\frac{\pi n x}{l}\right) \tag{B.69}$$

where $n = 1, 2, 3, \ldots$. Let's calculate,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = \frac{\hbar^2}{2m}\frac{\pi^2 n^2}{l^2} N \sin\left(\frac{\pi n x}{l}\right) = \frac{\hbar^2}{2m}\frac{\pi^2}{l^2} n^2 \psi \tag{B.70}$$

so

$$E_n = \frac{\hbar^2}{2m}\frac{\pi^2}{l^2} n^2 \tag{B.71}$$

2. **Hyperfine Transition**

Calculate the energy and wavelength of the hyperfine transition of the hyodrgen atom. You may use the following formula for the energy of two magnets near to each other

$$E = -\frac{\mu_1 \cdot \mu_2}{r^3} \tag{B.72}$$

We are looking for an order of magnitude estimate of the wavelength. I got 151 cm which is in the ballpark.

Answer: First let's write the values of the magnetic moments,

$$\mu_1 = \mu_p = g_p \frac{e}{2Mc}\frac{\hbar}{2} \tag{B.73}$$

and

$$\mu_2 = \mu_e = g_e \frac{e}{2mc}\frac{\hbar}{2} \tag{B.74}$$

The spins can be aligned or antialigned so the energy difference is $2\mu_1\mu_2/r^3$ so we get

$$\Delta E \sim \frac{g_p g_e}{8}\frac{e^2}{mc^2}\frac{\hbar^2}{Mr^3} \tag{B.75}$$

Let's take $r = a_0 = \hbar^2/(me^2)$ to get

$$\Delta E \sim \frac{g_p g_e}{8}\frac{e^2}{mc^2}\frac{m^3 e^6}{\hbar^4 M} = \frac{g_p g_e}{8}\frac{\alpha\hbar c}{mc^2}\frac{m^3\alpha^3\hbar^3 c^3}{\hbar^4 M} = \frac{g_p g_e}{8}\alpha^4\frac{m}{M}mc^2 = 10^{-6}\text{ eV} \tag{B.76}$$

so $\lambda = 123$ cm.

A Better Answer:

First let's write the values of the magnetic moments,

$$\mu_1 = \mu_p = g_p\frac{e}{2Mc}\hbar\mathbf{s}_1 \tag{B.77}$$

and

$$\mu_2 = \mu_e = g_e\frac{e}{2mc}\hbar\mathbf{s}_2 \tag{B.78}$$

so we get

$$E = \frac{g_p g_e}{4}\frac{e^2}{mc^2}\frac{\hbar^2}{Mr^3}\mathbf{s}_1\cdot\mathbf{s}_2 \tag{B.79}$$

Let's take $r = a_0 = \hbar^2/(me^2)$ to get

$$E = \frac{g_p g_e}{4}\frac{e^2}{mc^2}\frac{m^3 e^6}{\hbar^4 M} = \frac{g_p g_e}{4}\frac{\alpha\hbar c}{mc^2}\frac{m^3\alpha^3\hbar^3 c^3}{\hbar^4 M} = \frac{g_p g_e}{4}\alpha^4\frac{m}{M}mc^2\left(\mathbf{s}_1\cdot\mathbf{s}_2\right). \tag{B.80}$$

Let's calculate $\mathbf{F} = \mathbf{s}_1 + \mathbf{s}_2$ and square it

$$\mathbf{F}\cdot\mathbf{F} = (\mathbf{s}_1+\mathbf{s}_2)^2 = \mathbf{s}_1^2 + \mathbf{s}_2^2 + 2\mathbf{s}_1\cdot\mathbf{s}_2 \tag{B.81}$$

$$F(F+1) = S_1(S_1+1) + S_2(S_2+1) + 2\mathbf{s}_1\cdot\mathbf{s}_2 \tag{B.82}$$

$$F(F+1) = \frac{3}{4} + \frac{3}{4} + 2\mathbf{s}_1\cdot\mathbf{s}_2 \tag{B.83}$$

so

$$\mathbf{s}_1\cdot\mathbf{s}_2 = \frac{1}{2}F(F+1) - \frac{3}{4} = -\frac{3}{4}, \frac{1}{4} \tag{B.84}$$

so

$$\Delta E_{F=0,F=1} = \frac{g_p g_e}{4}\alpha^4\frac{m}{M}mc^2 = 2\times 10^{-6}\text{eV} \tag{B.85}$$

and $\lambda = 60$ cm.

3. **Density and Ionization**

Calculate the ionized fraction of pure hydrogen as a function of the density for a fixed temperature. You may take $U(T) = g_0 = 2$ and $U^+(T) = g_0^+ = 2$.

Answer:

Let's take the Saha equation,

$$\frac{N^+ N_e}{N} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}\frac{2U^+(T)}{U(T)}e^{-E_I/kT}. \tag{B.86}$$

Let ξ be the ionized fraction,

$$\xi = \frac{N^+}{N+N^+} = \frac{N^+}{N_{\rm tot}} \tag{B.87}$$

so using the values of $U(T)$ and $U^+(T)$ given in the problem

$$\frac{\xi^2 N_{\rm tot}^2}{(1-\xi)N_{\rm tot}} = 2\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-E_I/kT}. \tag{B.88}$$

Rearranging

$$\frac{\xi^2}{1-\xi} = 2\frac{2}{N_{\rm tot}}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-E_I/kT} = 2y \tag{B.89}$$

so

$$\xi = \sqrt{y^2+2y} - y \approx \sqrt{2y} \propto N_{\rm tot}^{-1/2} \tag{B.90}$$

Chapter 10

1. **Lifetime**

   Derive the lifetime of the $n = 2, l = 1, m = 0$ state of hydrogen to emit a photon and end up in the $n = 1, l = 0, m = 0$ state.

   Answer:

   The Einstein A-coefficient gives the rate of spontaneous emission for a state

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21} = \frac{32\pi^3\nu^3}{3\hbar c^3} |\mathbf{d}_{if}|^2 = \frac{32\pi^3\nu^3}{3\hbar c^3} e^2 a_0^2 \left| \frac{\mathbf{r}_{if}}{a_0} \right|^2 \qquad \text{(B.91)}$$

   Let's calculate everything expect the matrix element to be sure of the units. We know that

$$h\nu = 2\pi\hbar\nu = \frac{e^2}{2a_0}\left(1 - \frac{1}{2^2}\right) = \frac{3e^2}{8a_0} \qquad \text{(B.92)}$$

   so we get

$$A_{21} = \frac{9e^8}{128\hbar^4 c^3 a_0} \left| \frac{\mathbf{r}_{if}}{a_0} \right|^2 = \frac{9\alpha^4}{128} \frac{c}{a_0} \left| \frac{\mathbf{r}_{if}}{a_0} \right|^2 = 1.13 \times 10^9 \text{s}^{-1} \left| \frac{\mathbf{r}_{if}}{a_0} \right|^2 \qquad \text{(B.93)}$$

   where we used $\alpha = e^2/(\hbar c)$, so the units are clearly right!

   The last step is to calculate the matrix element. We will choose the electron to initially be in the $m = 0$ state so the x and y components of the dipole matrix element will be zero, so we are left with

$$\frac{\mathbf{r}_{if}}{a_0} = \frac{2\pi}{a_0^5} \int_0^\infty r^2 dr \int_{-1}^1 d\mu \frac{1}{\sqrt{\pi}} e^{-r} (r\mu) \frac{1}{4\sqrt{2\pi}} r e^{-r/2} \mu \qquad \text{(B.94)}$$

$$= \frac{1}{2\sqrt{2}a_0^5} \int_0^\infty dr r^4 e^{-3r/2} \int_{-1}^1 d\mu \mu^2 = \frac{2^7\sqrt{2}}{3^5} = 0.745 \qquad \text{(B.95)}$$

$$\text{(B.96)}$$

   The lifetime is

$$\frac{1}{A_{21}} = \left(\frac{3}{2}\right)^8 \frac{a_0}{c} \frac{1}{\alpha^4} = \left(\frac{3}{2}\right)^8 \frac{\hbar}{m_e c^2} \frac{1}{\alpha^5} = 1.58 \text{ ns} \qquad \text{(B.97)}$$

2. **Hydrogen-Like Absorption**

   How much energy does a photon need to ionize the following atoms by removing a K-shell electron?

   Hydrogen, Helium, Carbon, Oxygen, Iron

   Using the formula that I derived in class, draw an energy diagram that shows the total cross section for one gram of gas as a function of energy between 10eV and 10keV. It would be great if you used the initial expression in Eq. (72) for the dipole matrix element rather than the final answer given by Eq. (73).

Consider that the mass fraction of the different atoms are hydrogen (0.7), helium (0.27), carbon (0.008), oxygen (0.016) and iron (0.004).

Answer:

Let's first get the units right like in the previous quesiton. Using equations (72) and (61) we get

$$\sigma_{bf} = \frac{2pVm\omega}{3c\hbar^3}\frac{256\pi}{V}\left(\frac{Z}{a_0^2}\right)^5\left(\frac{Z}{a_0^2}+q^2\right)^{-6}e^2q^2 \tag{B.98}$$

Let's relate $p = \hbar q$ to the energy of the photon, we have

$$E = \hbar\omega = \frac{p^2}{2m} + E_I = \frac{p^2}{2m} + \frac{Z^2\alpha^2}{2}mc^2 \tag{B.99}$$

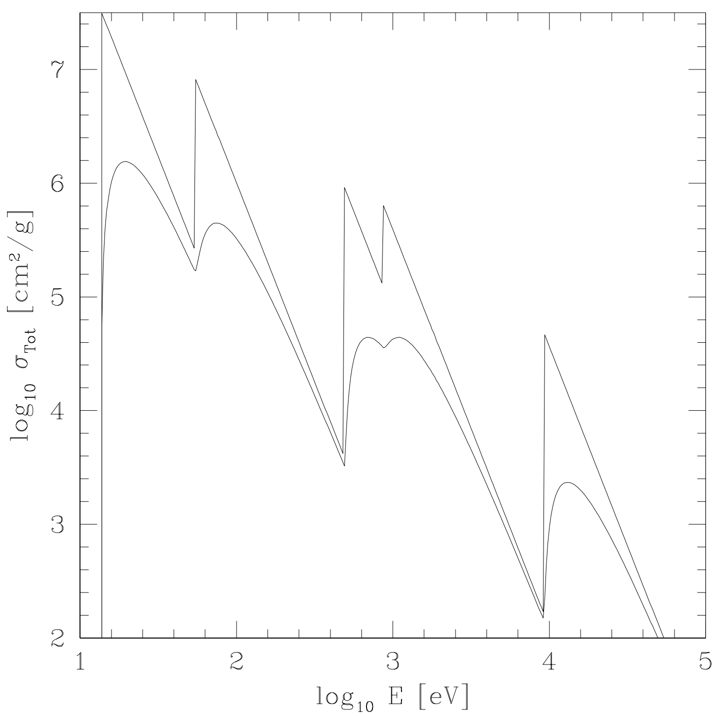
Let's define $x = E/E_I$ to get

$$\sigma_{bf} = \frac{256\pi}{3Z^2}\frac{1}{\alpha}\left(\frac{\hbar}{mc}\right)^2(x-1)^{3/2}x^{-5} = \frac{32}{Z^2\alpha^3}\sigma_T(x-1)^{3/2}x^{-5} \tag{B.100}$$

I know that $\sigma_T/m_p = 0.4\ \text{cm}^2\text{g}^{-1}$ so the total cross section per gram of matertial is

$$\sigma_{bf,\text{Total}} = \sum_i X_i \frac{\sigma_T}{m_p}\frac{32}{A_i Z_i^2\alpha^3}(x_i-1)^{3/2}x_i^{-5} \tag{B.101}$$

where X_i is the mass fraction of the species, A_i is its atomic weight, Z_i is its atomic number and $x_i = E/(Z^2 13.6\ \text{eV})$.



log10 σTot [cm²/g]
7
6
5
4
3
2
1
2
3
4
5
log10 E [eV]

Chapter 11

1. **The Number of Levels**

I fit a Morse function to the potential of H_2^+. The parameters were

$$E_{n,0} = -0.065\frac{e^2}{a_0}, B_n = 0.07\frac{e^2}{a_0}, \beta_n = 0.7a_0^{-1}, R_0 = 2.5a_0 \quad \text{(B.102)}$$

How many vibrational levels does H_2^+ have? How many rotational levels does each vibrational level typically have?

Answer:

Let's first to the rotational levels. When we increase the value of the angular momentum from L to $L+1$ and the energy of the molecule decreases, we have reached the maximum value of L. From Eq. (14) we have

$$4\hbar^2 \frac{(L+1)^2}{k\mu_{AB}R_0^4} = 1 \quad \text{(B.103)}$$

so

$$L_{\max} = \left(\frac{k\mu_{AB}R_0^4}{4\hbar^2}\right)^{1/2} - 1 \quad \text{(B.104)}$$

We need to determine k. This is related to the parameters that I gave in the question, we know that $\omega^2 = k/\mu_{AB}$ and in the Morse potential $\omega^2 = 2\beta_n^2 B_n$ so

$$k = 2\beta_n^2 B_n \quad \text{(B.105)}$$

and

$$L_{\max} = \left(\frac{2\beta_n^2 B_n \mu_{AB} R_0^4}{4\hbar^2}\right)^{1/2} - 1 \quad \text{(B.106)}$$

I'm going to substitute the units for the various quantities into the expression above

$$L_{\max} = \left(\frac{\beta_n B_n e^2 m_p a_0 R_0^4}{4\hbar^2}\right)^{1/2} - 1 = \left(\frac{\beta_n B_n m_p R_0^4}{4m_e}\right)^{1/2} - 1 \quad \text{(B.107)}$$

where I used $\mu_{AB} = m_p/2$ and $e^2 a_0 = \hbar^2/m_e$. The expression in the parenthesis is dimensionless! I get

$$L_{\max} = 23.79. \quad \text{(B.108)}$$

Because L ranges from zero to $L_{\max}$, I have 24 or 25 levels.

A formula for the number of vibrational levels is given explicitly in Eq. (19). The number of levels is

$$\frac{(2\mu_{AB}B_n)^{1/2}}{\beta_n \hbar} + \frac{1}{2} = \frac{B_n^{1/2}}{\beta_n}\left(m_p \frac{e^2}{a_0} a_0^2 \hbar^{-2}\right)^{1/2} + \frac{1}{2} = \frac{B_n^{1/2}}{\beta_n}\left(\frac{m_p}{m_e}\right)^{1/2} + \frac{1}{2} = 16.7 \quad \text{(B.109)}$$

2. **Nuclear Overlap**

Consider two deuterons bound by a single electron as in question (1). What is the probability that the two deuterons lie on top of each other, *i.e.* that $R < 4$ fermi, the diameter of the deuteron? What is the probability if the two deuterons are bound by a single muon, $m_\mu \approx 207 m_e$? You can find the eigenfunctions of the Morse potential on Wikipedia.

If you assume that whenever the deuterons overlap they fuse and that you get to "roll the dice" once each oscillation period, calculate the fusion rate in both cases.

Answer:

The probability of overlap is simply the squared modulus of the nuclear wavefunction evaluated at $r = 0$ integrated over the volume $4/3\pi(4 \text{ fermi})^3$. The nuclear wavefunction is given by

$$\Psi_n(z) = N_n z^{\lambda-n-\frac{1}{2}} e^{-z/2} L_n^{2\lambda-2n-1}(z) \tag{B.110}$$

where $\lambda = \sqrt{2MB_n}/(\beta_n \hbar)$ and the normalization

$$N_n = n! \left[\frac{\beta_n(2\lambda - 2n - 1)}{\Gamma(n+1)\Gamma(2\lambda - n)} \right]^{1/2} \tag{B.111}$$

and L_n^α is a Laguerre polynomial and $z = 2\lambda e^{-(x-x_e)}$ and $x = \beta_n r$. This wavefunction is in terms of r as a one-dimensional coordinate; it is analogous to the function $R(r)$ in the expansion of the atomic wavefunction in spherical symmetry. The complete wavefunction is

$$\psi(r, \theta, \phi) = \frac{1}{\sqrt{4\pi}} r^{-1} \Psi_0(z), \tag{B.112}$$

so the probability of the two nuclei being within 4 fermi of each other is given by

$$P = \int_0^{4 \text{ fermi}} dr \left| \Psi\left(2\lambda \exp\left[\beta_n R_0\right]\right) \right|^2 = \frac{4 \text{ fermi}}{a_0} \left| \Psi\left(2\lambda \exp\left[\beta_n R_0\right]\right) \right|^2 \tag{B.113}$$

Since we are interested in the ground state, $n = 0$ so

$$L_n^{2\lambda-2n-1} = 1 \text{ and } N_n = \left[\frac{\beta_n(2\lambda - 1)}{\Gamma(2\lambda)} \right]^{1/2} \tag{B.114}$$

which simplifies matters. What remains is to calculate determine how the value of λ depends on the mass of the binding particle muon or electron. We have

$$\lambda = \frac{\sqrt{2MAe^2/a_0}}{Ba_0^{-1}\hbar} = \frac{\sqrt{2A}}{B} \sqrt{\frac{M}{m}} \tag{B.115}$$

where M is the reduced mass of the pair of deuterons and m is the mass of the muon or electron. The constants A and B are simply the numerical constants 0.07 and 0.7 that define the parameters of the Morse potential in dimensionless units. For the electronically bound system $\lambda = 22.9$ and for the muonically bound system $\lambda = 1.59$.

What remains is to evaluate the wavefunctions in both cases, for the electron we have

$$\Psi(R = 0) = 7.5 \times 10^{-30} \tag{B.116}$$

and for the muon we have

$$\Psi(R = 0) = 2.0 \times 10^{-3}. \tag{B.117}$$

Converting these to probabilities yields

$$P_{\text{electron}} = 5 \times 10^{-63}, P_{\text{muon}} = 6 \times 10^{-8}. \tag{B.118}$$

To get a fusion rate we should multiply these by the typical frequency of the systems say $\omega = 2\beta_n^2 B_n / M = 2AB^2e^2/(a_0^3 m_D/2)$ or 1.2×10^{16} Hz for the electron and 3.6×10^{19} Hz for the muon. Therefore, we get a rate of three deuterium fusions over the age of the universe in one ton of deuterium for electronically bound molecules or 2×10^{12} Hz for the muonically bound molecule or about 4 million times over the 2.2μs lifetime of the muon.

It turns out that the rate-limiting step in muonic fusion is the formation of muonic molecules which takes about one thousand times longer than the fusion, but even this is not the killer. It is the fact that about one percent of the time the muon stays stuck to the fusion product so cannot catalyse another reaction. The first person to consider muon-catalysed fusion was John David Jackson, and Eugene Wigner suggested that "alpha sticking" could be a problem. This process was the original "cold fusion," and it almost breaks even (within a factor of a few).

Chapter 12

1. **Maximum Flux**

Calculate from the Euler equation and the continuity equation, at what velocity does the flux (ρV) reach its maximum for fluid flowing through a tube of variable cross-sectional area? At which velocities does the flux vanish? You can consider the flow to be adiabatic.

Answer:

From Euler's equation we have

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla P}{\rho} \tag{B.119}$$

so

$$v\frac{dv}{dx} = -\frac{1}{\rho}\frac{dP}{dx} \tag{B.120}$$

and we have

$$\frac{dP}{d\rho} = c_s^2 \tag{B.121}$$

Combining these two gives

$$\frac{d\rho}{dv} = -\rho\frac{v}{c_s^2} \tag{B.122}$$

We have

$$\frac{d(\rho v)}{dv} = \rho + v\frac{d\rho}{dv} = \rho\left(1 - \frac{v^2}{c_s^2}\right) \tag{B.123}$$

This function reaches an extremum at $v = c_s$. Because the flux is zero for $v = 0$ and increases with v for $v \ll c_s$, this must be a maximum for jv.

If we assume that the sound speed is constant (isothermal gas), this integrates to give

$$\rho v = \rho_0 v e^{-v^2/(2c_s^2)} \tag{B.124}$$

where ρ_0 is the density at zero velocity. This has a maximum of $\rho_0 c e^{-1/2}$ at $v = c_s$. We can let the gas expand and accelerate to arbitrarily high velocities. In a more realistic situation, the sound speed is a function of density

$$c_s^2 = c_{s,0}^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}, \tag{B.125}$$

and we have

$$\frac{dj}{dv} = \rho\left[1 - \frac{v^2}{c_{s,0}^2}\left(\frac{\rho}{\rho_0}\right)^{1-\gamma}\right] = \frac{j}{v}\left[1 - \frac{v^2}{c_{s,0}^2}(\rho_0 v)^{\gamma-1} j^{1-\gamma}\right] \tag{B.126}$$

with $j = \rho v$. This differential equation has the following solution

$$j(v) = \rho_0 v \left[1 + (1-\gamma)\frac{v^2}{2c_{s,0}^2}\right]^{1/(\gamma-1)} \quad \text{(B.127)}$$

If we take $\gamma \to 1$ we get the solution above for the isothermal case. The flux reaches a maximum of

$$j_{\max} = \rho_0 c_{s,0} \left(\frac{2\gamma}{\gamma+1}\right)^{1/(\gamma-1)} \frac{1}{\sqrt{2\gamma+2}} \quad \text{(B.128)}$$

at a velocity of

$$v = \sqrt{\frac{2}{\gamma+1}} c_{s,0}. \quad \text{(B.129)}$$

Unlike the isothermal case, the flux vanishes at $v = 0$ and $v = c_{s,0}\sqrt{2/(\gamma-1)}$. How can we understand this second velocity when the flux vanishes? Along a streamline of the gas we have

$$\frac{v^2}{2} + w = w_0 = \frac{P+\epsilon}{\rho} = \frac{P + (\gamma-1)^{-1}P}{\rho} = \frac{\gamma}{\gamma-1}\gamma^{-1}c_{s,0}^2 \quad \text{(B.130)}$$

The maximum velocity that the gas can attain is

$$v = \sqrt{2w_0} = \sqrt{2c_{s,0}^2/(\gamma-1)} = c_{s,0}\sqrt{2/(\gamma-1)} \quad \text{(B.131)}$$

2. **Stream Bed**

Fig. 12.2 shows how the level of the surface changes for a flow passing over an obstacle. For an initial depth of $z_0 = 1$ and $g = 10$ and a bump height of $y(x) = 0.1e^{-x^2}$, find the solutions to Bernoulli's equation (Eq. 12.80) for z as a function of x and the initial velocity v_0. You may find several solutions for a given x. Also you should only worry about the positive real solutions for z. What are the values of the critical velocities v_0?

Answer:

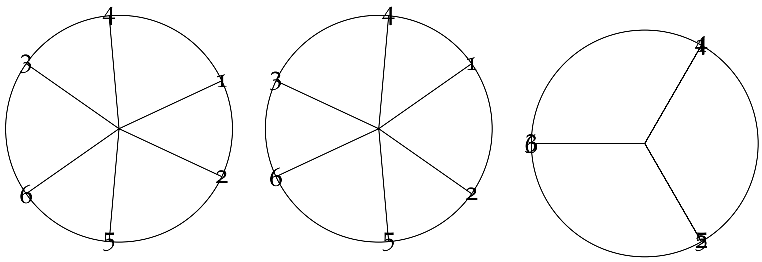
The solution follows from Eq. 12.84 by plugging in the values of $y(x)$, $z_0 = 1$, $g = 10$ and v_0 which you are going to vary to look at the different solutions. This yields

$$A = \frac{v_0^2}{2}, B = \frac{v_0^2}{2} + 10\left[1 - 0.1e^{-x^2}\right], C = 10. \quad \text{(B.132)}$$

Next we use Eq. 12.86 to find the value of $\cos 3t$. This equation will yield several values of $3t$ because the cosine function is symmetric and periodic. They are

$$3t = 3t_1, 3(-t_1), 3\left(t_1 + \frac{2}{3}\pi\right), 3\left(\frac{2}{3}\pi - t_1\right), 3\left(t_1 + \frac{4}{3}\pi\right), 3\left(\frac{4}{3}\pi - t_1\right). \quad \text{(B.133)}$$

Because we are interested in the value of $\cos t$, the first two results yield the same value. Let's draw a picture with the various possibilities numbered:



Because we are only interested in the $x-$coordinate (the cosine of the angle), we see that solutions 1 and 2, 3 and 6 and 4 and 5 are equivalent, so we only need to keep the solutions with t between zero and π. In the left diagram we used $t = 25°$ so we only have a single value with $\cos t$ greater than zero. We can discard negative values of $\cos t$ because that would yield that the surface of the water lies underneath the surface of the bottom.

For $t_1 > 30°$ there are two positive solutions. The centre diagram has $t_1 = 35°$. These solutions coincide for $t_1 = 60°$ (right diagram). Where this condition holds the flow is travelling at the critical velocity. The value of v_0 that causes the flow to travel at the critical velocity over the peak of the bump is the critical value of v_0. In general, we only have to be concerned with solutions (1) and (4), the rest are repeats or negative.

3. **Sound Velocity**

Show that for a linear sound wave *i.e.* one in which $\delta\rho \ll \rho$ that the velocity v of fluid motion is much less than c_s. Estimate the maximum longitudinal fluid velocity in the case of a sound wave in air at STP in the case of a disturbance which sets up pressure fluctuations of order 0.1%.

Answer:

Starting with Eq. 12.71 we can relate the velocity of the fluid in the wave to the pressure disturbance,

$$\mathbf{v}' = \frac{p'}{\rho_0}\frac{\mathbf{k}}{\omega}, v' = \frac{p'}{\rho_0}\frac{1}{c_s} = c_s\frac{\rho'}{\rho_0} = \frac{p'}{p_0}\frac{c_s}{\gamma} \tag{B.134}$$

where $p' = c_s^2\rho'$ because $c_s^2 = \partial p/\partial\rho$. Furthermore, the adiabatic exponent is given by $\gamma = \partial \ln p/\partial \ln \rho = (\rho/p)c_s^2$.

Chapter 13

1. **Shock Entropy** Show that the entropy of the fluid increases as it passes through a shock. Hint: the equation of state of an isentropic fluid is $P = K\rho^\gamma$ where the value of K increases with increasing entropy.

   Answer:

   The simplest way to solve this is to look at the shock adiabat and see that the entropy increases along it, but let's be a bit more rigorous. The value of K is a function of entropy alone, so let's look at how K changes across the shock. Specifically what is P/ρ^γ on each side of the shock? We have

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{2+M_1^2(\gamma-1)}, \frac{P_2}{P_1} = \frac{1-\gamma+2M_1^2\gamma}{(\gamma+1)} \tag{B.135}$$

   so

$$\frac{K_2}{K_1} = \frac{1-\gamma+2M_1^2\gamma}{(\gamma+1)}\left[\frac{(\gamma+1)M_1^2}{2+M_1^2(\gamma-1)}\right]^{-\gamma} \tag{B.136}$$

   Let's expand this ratio for $M_1^2 \approx 1$ to understand the change in entropy for a weak shock,

$$\frac{K_2}{K_1} = 1 + \frac{2\gamma(\gamma-1)}{3(\gamma+1)^2}\left(M_1^2-1\right)^3 + \mathcal{O}\left(M_1^2-1\right)^4. \tag{B.137}$$

   The value of K increases across the shock for $\gamma > 1$, therefore the entropy increases. To make this precise we know that for an ideal gas, $s = c_V \ln K + s_0$, so

$$\Delta s = c_v\frac{\Delta K}{K} = \frac{2\gamma(\gamma-1)}{3(\gamma+1)^2}\left(M_1^2-1\right)^3 c_V + +\mathcal{O}\left(M_1^2-1\right)^4. \tag{B.138}$$

2. **Bomb Yield**

   Fig. 13.5 shows shocked air heated to incandescence about two milliseconds after the detonation of a nuclear bomb. The height of the device was 90 meters. What was the approximate yield of the device?

   Answer:

   Eq. 13.26.

3. **Relativistic Shock**

   Find the incoming and outgoing velocity of a relativistic shock in terms of the energy density and pressure on either side of the shock.

   Answer:

Start with Eq. 13.50 and 13.51,

$$w_1 U_1 \gamma_1 = w_2 U_2 \gamma_2, w_1 U_1^2 + p_1 = w_2 U_2^2 + p_2. \qquad (B.139)$$

Let's use the second equation to solve for U_2^2

$$U_2^2 = \frac{w_1 U_1^2 + p_1 - p_2}{w_2} \qquad (B.140)$$

and rewrite γ_2^2 in terms of U_2

$$\gamma_2^2 = \frac{1}{1-\beta^2} = \frac{\beta^2 + 1 - \beta^2}{1-\beta^2} = U_2^2 + 1. \qquad (B.141)$$

The square of the first equation yields

$$w_1^2 U_1^2 \left(U_1^2 + 1\right) = w_2^2 U_2^2 \left(U_2^2 + 1\right) \qquad (B.142)$$

$$w_1^2 U_1^2 \left(U_1^2 + 1\right) = w_2^2 \frac{w_1 U_1^2 + p_1 - p_2}{w_2} \left(\frac{w_1 U_1^2 + p_1 - p_2}{w_2} + 1\right) \qquad (B.143)$$

$$U_1^2 = \frac{(p_1 - p_2)^2 - p_1 w_2 - p_2 w_2}{w_1 \left[w_2 - w_1 + 2(p_1 - p_2)\right]} \qquad (B.144)$$

$$\left(\frac{v_1}{c}\right)^2 = \frac{(p_1 - p_2)(p_1 - p_2 + w_2)}{(p_1 - p_2 - w_1)(p_1 - p_2 + w_2 - w_1)} \qquad (B.145)$$

$$\left(\frac{v_1}{c}\right)^2 = \frac{(p_2 - p_1)(\epsilon_2 + p_1)}{(\epsilon_2 - \epsilon_1)(\epsilon_1 + p_2)} \qquad (B.146)$$

and we obtain v_2 by swapping the one and two indicies in the previous equaiton, yielding

$$\frac{v_1}{c} = \sqrt{\frac{(p_2 - p_1)(\epsilon_2 + p_1)}{(\epsilon_2 - \epsilon_1)(\epsilon_1 + p_2)}} \qquad (B.147)$$

$$\frac{v_2}{c} = \sqrt{\frac{(p_2 - p_1)(\epsilon_1 + p_2)}{(\epsilon_2 - \epsilon_1)(\epsilon_2 + p_1)}} \qquad (B.148)$$

4. **Relativistic Bernoulli**

Find the relativistic generalisation of Bernoulli's equation for a streamline (you can neglect gravitiy).

Answer:

For the Bernoulli equaion we must assume that all time derivatives vanish and look at the properties of the fluid along a flow line. We can use the shock jump conditions as a starting point, (e.g. Eq. 13.49 and 13.50), because they must hold along a streamline as well as across a discontinuity. We have

$$Un = \text{constant}, wU\gamma = \text{constant}. \qquad (B.149)$$

Using the first equation to eliminate U from the second yields

$$\frac{\gamma w}{n} = \text{constant.} \tag{B.150}$$

This doesn't look much like the non-relativistic Bernoulli equation. Let's make some substitutions. We have

$$\frac{1}{n}\left(1+\frac{v^2}{2c^2}\right)\left(\rho c^2 + w_{NR,V}\right) + \text{Higher order in velocity} = \text{constant.} \tag{B.151}$$

Now let's divide both sides by the rest mass of the particles

$$\frac{1}{\rho}\left(1+\frac{v^2}{2c^2}\right)\left(\rho c^2 + w_{NR,V}\right) + \text{Higher order in velocity} = \text{constant} \tag{B.152}$$

and expand, dropping higher-order terms

$$c^2 + \frac{v^2}{2} + \frac{w_{NR,V}}{\rho} = \text{constant} \tag{B.153}$$

and

$$\frac{v^2}{2} + w = \text{constant} \tag{B.154}$$

where w is the enthalpy per unit mass. This is the non-relativistic Bernoulli equation. For the classical result with an incompressible fluid we have $w = P/\rho$.

5. **Bathtub Physics**

When water flows into a bathtub, a circular hydraulic jump forms around the incoming stream of water. If you assume that the flow rate is constant and the flow is initially vertical, calculate the height of the water downstream of the jump as a function of the radius of the jump and the flow rate. You may neglect friction. If the bathtub is large compared to the radius of the jump and the walls are vertical, how does the radius of the jump change with time?

Answer:

Here the flow rate and downstream height are given. We have

$$j = \frac{Q}{2\pi r} \tag{B.155}$$

where Q is the volumetric flow rate and r is the radius of the jump. What is the height of the upsteam flow h_1 in terms of the downstream height h_2? We have

$$v_1^2 h_1 + \frac{1}{2} g h_1^2 = v_2^2 h_2 + \frac{1}{2} g h_2^2 \tag{B.156}$$

so

$$\frac{j^2}{h_1} + \frac{1}{2} g h_1^2 = \frac{j^2}{h_2} + \frac{1}{2} g h_2^2 \tag{B.157}$$

and with rearranging

$$j^2 (h_1 - h_2) + \frac{1}{2} g \left(h_2^3 h_1 - h_1^3 h_2 \right) = 0. \tag{B.158}$$

We can factor this to give

$$\frac{1}{2} (h_1 - h_2) \left(g h_2 h_1^2 + g h_2^2 h_1 - 2 j_2 \right) = 0 \tag{B.159}$$

so we have the positive solutions

$$h_1 = h_2, h_1 = \frac{h_2}{2} \left(\sqrt{1 + \frac{8 j^2}{g h_2^3}} - 1 \right). \tag{B.160}$$

Now we use that v_1 is constant to eliminate $h_1 = j/v_1 = Q/(2\pi r v_1)$. Furthermore, $h_2 = Qt/A$ where A is the cross-sectional area of the jump. Putting all of these into the equation above yields

$$\frac{Q}{2\pi r v_1} = \frac{Qt}{2A} \left(\sqrt{1 + \frac{8}{g} \left(\frac{Q}{2\pi r} \right)^2 \left(\frac{A}{Qt} \right)^3} - 1 \right). \tag{B.161}$$

and solving for the radius r yields

$$r = \frac{A \left(2 A v_1^2 - t Q g \right)}{2\pi t^2 Q g v_1} = \frac{A^2 v_1}{Q g \pi} \frac{1}{t^2} - \frac{A}{2\pi v_1} \frac{1}{t}. \tag{B.162}$$

Chapter 14

1. **Exact Solutions**

For which values of γ can the Bernoulli equation (Eq. 14.9) be solved using elementary methods (linear, quadratic and cubic equations of the form in Eq. 12.85). There are many, however only a few have $1 < \gamma < 5/3$.

Answer:

Let's start with the Bernoulli equation,

$$\frac{v^2}{2} + \frac{c_s^2 - c_s^2(\infty)}{\gamma - 1} - \frac{GM}{r} = 0. \tag{B.163}$$

First let's divide both sides by c_s^2 to get

$$\frac{1}{2}\frac{v^2}{c_s^2} + \frac{1 - c_s^2(\infty)/c_s^2}{\gamma - 1} - \frac{GM}{rc_s^2} = \frac{1}{2}\frac{v^2}{c_s^2} + \frac{1 - c_s^2(\infty)/c_s^2}{\gamma - 1} - \frac{r_c}{r}\frac{4}{5-3\gamma}\frac{c_s^2(\infty)}{c_s^2} = 0. \tag{B.164}$$

We need to express the sound speed in terms of r and v. We have

$$P = K\rho^\gamma \tag{B.165}$$

so

$$c_s^2 = \gamma K \rho^{\gamma-1} = c_s^2(\infty)\left(\frac{\rho}{\rho(\infty)}\right)^{\gamma-1} \tag{B.166}$$

We can relate v, r and ρ through $\dot{M} = 4\pi r^2 v\rho$ We are interested in a plot of $y = v/c_s$ versus $x = r/r_c$, so let's substitute for x and y to get

$$\frac{y^2}{2} + \frac{1}{\gamma - 1} - \left(\frac{1}{\gamma - 1} - \frac{1}{x}\frac{4}{5-3\gamma}\right)\frac{c_s^2(\infty)}{c_s^2} = 0. \tag{B.167}$$

Now we need to find $c_s^2(\infty)/c_s^2$ in terms of x and y. We can determine ρ at any point through $\dot{M} = 4\pi\rho v$ and the formula above. The key is to write $\dot{M} = \alpha\dot{M}_{\rm crit}$. First, we have

$$\dot{M} = 4\pi r^2 v\rho = 4\pi\alpha r_c^2 c_s(r_c)\rho(r_c) \tag{B.168}$$

so

$$\frac{\rho}{\rho(r_c)} = \frac{\alpha}{x^2 y}\frac{c_s(r_c)}{c_s} = \frac{\alpha}{x^2 y}\left(\frac{\rho(r_c)}{\rho}\right)^{(\gamma-1)/2} = \left(\frac{\alpha}{x^2 y}\right)^{2/(\gamma+1)}. \tag{B.169}$$

and

$$\frac{\rho}{\rho(\infty)} = \frac{\rho}{\rho(r_c)}\frac{\rho(r_c)}{\rho(\infty)} = \left(\frac{\alpha}{x^2 y}\right)^{2/(\gamma+1)}\left(\frac{2}{5-3\gamma}\right)^{1/(\gamma-1)} \tag{B.170}$$

using Eq. 14.13 and giving an expression for

$$\frac{c_s^2}{c_s^2(\infty)} = \left(\frac{\alpha}{x^2 y}\right)^{2(\gamma-1)/(\gamma+1)}\frac{2}{5-3\gamma} \tag{B.171}$$

Let's substitute this into the Bernoulli equation to yield

$$y^{2(1-\gamma)/(\gamma+1)}\left(\frac{y^2}{2}+\frac{1}{\gamma-1}\right)\frac{2\alpha^{2(\gamma-1)/(\gamma+1)}}{5-3\gamma}-\\ x^{4(\gamma-1)/(\gamma+1)}\left(\frac{1}{\gamma-1}+\frac{4}{x(5-3\gamma)}\right)=0 \qquad \text{(B.172)}$$

There could be many possibilities: we could solve for x or y in terms of the other, and the resulting equation could be a cubic, quadratic or linear in the three terms. Let's begin with solving for x.

Solution for x:

On the second line there are two terms in x that differ by a single power of x. We can make substitutions of the form $x = u^{\pm 2}$, $x = w^{\pm 3}$ that will transform the equation into a quadratic or cubic with the correct choice of γ, or we could solve for x directly which would require that $4(\gamma-1)/(\gamma+1) = 2, 1, 0, -1, -2$, yielding quadratic, linear, linear, quadratic and cubic equations, respectives. A value of $4(\gamma-1)/(\gamma-1) = 3$ would also yield a cubic equation but not of the simply solvable form (Eq. 12.85). There are also simply solvable quartics, but this is beyond the scope of the question.

$\frac{4(\gamma-1)}{\gamma-1}$	γ	Type	Substitution	Comment
2	3	quadratic	x	Non-Ideal
1	5/3	linear	x	Divergent
0	1	linear	x	Divergent
-1	3/5	quadratic	x	Unphysical
-2	1/3	cubic	x	Unphysical
1/2	9/7	quadratic	$x = u^2$	Good
$-1/2$	7/9	cubic	$x = u^{-2}$	Unphysical
2/3	7/5	cubic	$x = w^3$	Good
1/3	13/11	cubic	$x = w^{-3}$	Good

Solution for y:

We can also solve for y in terms of x. Here the two terms in y differ by two powers of y; this naturally leads to a quadratic without substitution. Some of the various possibilities are listed in the following table.

4. **Bondi Solution**

Generate a picture like the figure in the lecture notes for the Bondi solution to spherical accretion. Use $\gamma = 9/7$.

Answer:

$\frac{2(1-\gamma)}{\gamma+1}$	γ	Type	Substitution	Comment
2	0	quadratic	$w = y^2$	Unphysical
1	$1/3$	cubic	y	Unphysical
0	1	linear	$w = y^2$	Divergent
$-2/3$	2	cubic	$w^3 = y^2$	Non-Ideal
-1	3	quadratic	y	Non-Ideal
$-4/3$	5	cubic	$w^{-3} = y^2$	Non-Ideal

The answer starts as the first question up to the Bernoulli equation (Eq. B.172) where we substitute $\gamma = 9/7$ to yield

$$\frac{7}{2}x^{1/2} - \frac{7}{2}x^{-1/2} - \frac{7}{8}\alpha^{1/4}y^{7/4} - \frac{49}{8}\frac{\alpha^{1/4}}{y^{7/4}} = 0 \qquad \text{(B.173)}$$

Let $u = \sqrt{x}$ and multiply everything by u to give a quadratic equation for u

$$\frac{7}{2}u^2 - \left[\frac{7}{8}\frac{\alpha^{1/4}}{y^{1/4}}(y^2+7)\right]u - \frac{7}{2} = 0 \qquad \text{(B.174)}$$

5. **Bondi Solution — Harder**

   Generate a picture like the figure in the lecture notes for the Bondi solution to spherical accretion. Use $\gamma = 7/5$.

   Answer:

   The answer starts as the first question up to the Bernoulli equation (Eq. B.172) where we substitute $\gamma = 7/5$ to yield

$$\frac{5}{2}x^{2/3} - 5x^{-1/3} - \frac{5}{4}\alpha^{1/3}y^{5/3} - \frac{25}{4}\frac{\alpha^{1/3}}{y^{1/3}} = 0. \qquad \text{(B.175)}$$

   Let $w^3 = x$ and multiply everything by u to give a cubic equation for w

$$\frac{5}{2}w^3 - \frac{5}{4}\frac{\alpha^{1/3}}{y^{1/3}}\left(y^2 - 5\right)w - 5 = 0. \qquad \text{(B.176)}$$

   of the form Eq. 12.85. This equation can be solved analytically

6. **Accretion Energetics**

   (a) $T = \frac{GMm}{R}$

   (b) $T = \frac{GMm}{2R}$

   (c) $T_{\rm NS}/m = 2 \times 10^{20}$ erg/g, $T_{\rm WD}/m = 8 \times 10^{16}$ erg/g. The accretion energy for a neutron star greatly exceeds the nuclear energy. The opposite holds for a white dwarf.

   (d) The total energy per gram is essentially the value given in part c). The Eddington luminosity is 1.8×10^{38} erg/s for a neutron star (see problem 3.6). This yields an Eddington accretion rate of approximately 10^{18} g/s.

7. **A simplified accretion disk**

(a) $dE = -\frac{GMdm}{2r}$

(b) $\frac{d}{dr}dE = \frac{GMdm}{2r^2}$, $\frac{dE}{drdt} = \frac{GM}{2r^2}\frac{dm}{dt}$

(c) $\frac{dE}{dAdt} = \frac{GM}{4\pi r^3}\frac{dm}{dt}$

(d) $\sigma T^4 = \frac{GM}{4\pi r^3}\frac{dm}{dt}$

(e) $\frac{dE}{dt} = \int_{r_0}^{r_A} \frac{GM}{2r^2}\frac{dm}{dt}dr = \frac{GM}{2}\frac{dm}{dt}\left(r_0^{-1} - r_A^{-1}\right)$, the peak temperature is at r_0 and the minimum temperature is at r_A.

(f) To sketch the spectrum we will assume that the BB emission emerges exclusively at the peak of the BB, so we need to translate $dE/(drdt)$ to $dE/(dTdt)$.

$$\frac{dP}{dT} = \frac{dP}{dr}\frac{dr}{dT} = \frac{GM}{2r^2}\frac{dm}{dt}\left(\frac{\sigma T^3 4\pi r^4}{3GM\dot{m}}\right) = \frac{4\pi r^2 \sigma T^3}{3}$$

and substituting yields

$$\frac{dP}{dT} = \frac{4\pi\sigma}{3}\left(\frac{GM}{4\pi\sigma}\dot{m}\right)^{2/3} T^{1/3}$$

or $F_\nu \propto \nu^{1/3}$.

(g) If the accretion rate exceeds the Eddington rate, some matter must be expelled.

(h) Viscosity

Chapter 15

1. **X-ray Bursts:**

We will try to model Type-I X-ray bursts using a simple model for the instability. We will calculate how much material will accumulate on a neutron star before it bursts.

(a) Let us assume that the star accretes pure helium, that the temperature of the degenerate layer is constant down to the core (T_c), how much luminosity emerges from the surface of the star?

(b) Let us assume that the helium layer has a mass, dM, and that the enregy generation rate for helium burning is given by

$$\epsilon_{3\alpha} = 3.5 \times 10^{20} T_9^{-3} \exp(-4.32/T_9) \mathrm{erg s^{-1} g^{-1}}$$

where $T_9 = T/10^9$K. The energy generation rate is a function of density too, but let's forget about that to keep things simple. How much power does the helium layer generate as a function of dM?

(c) Equate your answer to (a) to the answer to (b) and solve for dM. This is the thickness of a layer in thermal equilibrium.

(d) Let's assume that the potential burst starts by the temperature in the accreted layer jiggling up by a wee bit. If the surface luminosity increases faster with temperature than the helium burning rate, then the layer is stable. Calculate dL_{surface}/dT and dP_{helium}/dT.

(e) Calculate the value of dM for which dP_{helium}/dT exceeds dL_{surface}/dT and the layer bursts.

(f) Equate your value of dM in (c) and (e) and solve for T. What is dM? How long will it take for such a layer to accumulate if the star is accreting at one-tenth of the Eddington accretion rate?

Answer:

(a) If you assume free-free opacity you get using results from Chapter 1

$$L_{\gamma,ff} = 2.35 \times 10^8 \mathrm{erg/s} \left(\frac{T}{1\mathrm{K}}\right)^{7/2}$$

or if you used the black-body formula you get

$$L_{\gamma,BB} = 7 \times 10^8 \mathrm{erg/s} \left(\frac{T}{1\mathrm{K}}\right)^{4}$$

(b)

$$P_{\mathrm{He}} = \epsilon_{3\alpha} dM = 3.5 \times 10^{20} T_9^{-3} \exp(-4.32/T_9) \mathrm{erg s^{-1} g^{-1}} dM$$

(c)
$$dM_{ff} = 2.12 \times 10^{19} T_9^{13/2} \exp(4.32/T_9) \text{g}$$
and
$$dM_{BB} = 2 \times 10^{24} T_9^7 \exp(4.32/T_9) \text{g}$$

(d)
$$\frac{dL_{\gamma,ff}}{dT} = 8.2 \times 10^8 \text{erg/s/K} \left(\frac{T}{1\text{K}}\right)^{5/2}$$
and
$$\frac{dL_{\gamma,BB}}{dT} = 2.8 \times 10^9 \text{erg/s/K} \left(\frac{T}{1\text{K}}\right)^3 .$$
For the helium burning we get
$$\frac{dP_{\text{He}}}{dT} = 4.2 \times 10^{10} \text{erg/s/g/K} T_9^{-5} \exp(4.32/T_9)(36 - 25T_9) dM.$$

(e) Let's solve for dM again where the various derivatives are equal
$$dM_{ff} = 6.19 \times 10^{20} T_9^{15/2} \exp(4.32/T_9)(36 - 25T_9)^{-1} \text{g}$$
and
$$dM_{ff} = 6.67 \times 10^{25} T_9^8 \exp(4.32/T_9)(36 - 25T_9)^{-1} \text{g}.$$

(f) We find that $T_9 = 0.664$ for the free-free opacity and $T_9 = 0.617$ for the BB-case (no insulation). The layer thicknesses are
$$dM_{ff} = 10^{21} \text{ g}$$
and
$$dM_{BB} = 7 \times 10^{25} \text{ g},$$
yielding accretion times of 2.8 hours and 24 years, respectively. The insulation of the envelope makes a big difference. Type-I bursts typically recur on a timescale of hours at one-tenth of the Eddington accretion rate.