1 Solutions to assignment 6, due June 23rd

Problem 12.1 Let $\epsilon > 0$. Let $N = \lceil \frac{1}{2\epsilon} \rceil$. Then for all $n > N$, we have that $n > \frac{1}{2\epsilon}$, and so $\epsilon > \frac{1}{2n}$. Thus:

$$\left| \frac{1}{2n} - 0 \right| = \frac{1}{2n} < \epsilon$$

for all $n > N$, and so the sequence $a_n = \frac{1}{2n}$ coverges to 0.

Problem 12.2 Let $\epsilon > 0$. We would like to define $N = \left\lceil \sqrt{\frac{1}{\epsilon} - 1} \right\rceil$, but if $\epsilon > 1$ then this would be a problem, since the term inside the square root is then negative. However, for every $n \in \mathbb{N}$, we have that $\frac{1}{n^2+1} < 1$. Thus we define instead

$$N = \begin{cases} \left\lceil \sqrt{\frac{1}{\epsilon} - 1} \right\rceil & \epsilon < 1 \\ 1 & \epsilon \geq 1 \end{cases}$$

So if $n > N$, then it follows that $\frac{1}{n^2+1} < \epsilon$. So we compute that

$$\left| \frac{1}{n^2 + 1} - 0 \right| = \frac{1}{n^2 + 1} < \epsilon$$

for every $n > N$, and so the limit is zero as claimed.

Problem 12.3 Let $\epsilon > 0$. Similar to the previous problem, we let $N = \max\left\{\lceil \log_2(\frac{1}{\epsilon}) \rceil, 1\right\}$. So for $n > N$, it follows that $n > 1$ and $n > \log_2(\frac{1}{\epsilon})$. Thus

$$\left| \left(1 + \frac{1}{2^n}\right) - 1 \right| = \frac{1}{2^n} < \epsilon$$

for all $n > N$, and so the limit is 1 as claimed.

Problem 12.4 Let $\epsilon > 0$, and let $N = \max\left\{\lceil \frac{1}{4\epsilon} - \frac{3}{2} \rceil, 1\right\}$. As before, if $n > N$ then $n > 1$ and $n > \frac{1}{4\epsilon} - \frac{3}{2}$. But we can compute easily that this is the same as $\epsilon > \frac{1}{4n+6}$. Thus

$$\left| \frac{n + 2}{2n + 3} - \frac{1}{2} \right| = \frac{1}{4n + 6} < \epsilon$$

as desired.

Problem 12.6 Let $r \in \mathbb{R}_{\geq 0}$ be arbitrary. The goal is to show that there is $N \in \mathbb{N}$ so that for all $n > N$, we have that $n^4 > r$. So choose $N = \lceil \sqrt[4]{r} \rceil$. Then it is clear that whenever $n > N$, we have that $n^4 > r$ as desired, and so n^4 diverges to ∞.

Problem 12.7 Let $r \in \mathbb{R}_{\geq 0}$ be arbitrary, and let $N = \lceil \sqrt[3]{r} \rceil$. Then for any $n > N$, it follows that $n^3 > r$. Since $\frac{n^5+2n}{n^2} = n^3 + \frac{2}{n} > n^3$, we find that for all $n > N$ that

$$\frac{n^5 + 2n}{n^2} = n^3 + \frac{2}{n} > n^3 > r$$

and so the sequence diverges to infinity.

Problem 12.8 We compute the first few partial sums to be

$$s_1 = \frac{1}{4}$$
$$s_2 = \frac{1}{4} + \frac{1}{4 \cdot 7} = \frac{2}{7}$$
$$s_3 = \frac{1}{4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} = \frac{3}{10}$$
$$s_4 = \frac{1}{4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} + \frac{1}{10 \cdot 13} = \frac{4}{13}$$

and so it appears that $s_n = \frac{n}{3n+1}$. We will prove this by induction, the base case(s) being given above.

So suppose that for some integer n that $s_n = \frac{n}{3n+1}$, and consider s_{n+1}. We have that

$$\begin{aligned} s_{n+1} &= s_n + \frac{1}{(3n+1)(3n+4)} \\ &= \frac{n}{3n+1} + \frac{1}{(3n+1)(3n+4)} \\ &= \frac{n(3n+4)+1}{(3n+1)(3n+4)} \\ &= \frac{(3n+1)(n+1)}{(3n+1)(3n+4)} = \frac{n+1}{3(n+1)+1} \end{aligned}$$

as claimed. We now claim that $\lim_{n\to\infty} s_n = \frac{1}{3}$ i.e. that $\sum_{k=1}^{\infty} \frac{1}{(3k-2)(3k+1)} = \frac{1}{3}$.

Let $\epsilon > 0$. Let $N = \lceil \frac{1}{9\epsilon} - \frac{1}{3} \rceil$. Then for all $n > N$ we have that $n > \frac{1}{9\epsilon} - \frac{1}{3}$. But this is equivalent to $\frac{1}{9n+3} < \epsilon$. So we then have that

$$\left| s_n - \frac{1}{3} \right| = \left| \frac{n}{3n+1} - \frac{1}{3} \right| = \frac{1}{9n+3} < \epsilon$$

as claimed, and so the claim follows.

Problem 12.9 As we have seen before, this is a geometric series and so we already know that the partial sums s_n are given by

$$s_n = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = 2 - \frac{1}{2^n}$$

So we will prove that the limit of this, as $n \to \infty$, is 2. This however is exactly the same proof (essentially) as problem 12.3, and so we are done.

Problem 12.10 We compute the first few terms of our sequence to be $a_1 = \frac{1}{2\cdot 3}$, $a_2 = \frac{1}{3\cdot 4}$, and $a_3 = \frac{1}{4\cdot 5}$, and so we begin by conjecturing that $a_n = \frac{1}{(n+1)(n+2)}$ for all n, which is an easy induction proof, which will not be included here.

We now compute the first few partial sums to be

$$s_1 = \frac{1}{6}$$
$$s_2 = \frac{1}{4} = \frac{2}{8}$$
$$s_3 = \frac{3}{10}$$

and so a plausible conjecture is that $s_n = \frac{n}{2n+4}$. As before, the base case is above, so we move onto the induction step.

Suppose that $s_n = \frac{n}{2n+4}$ and note that

$$s_{n+1} = s_n + \frac{1}{(n+2)(n+3)} = \frac{n}{2(n+2)} + \frac{1}{(n+2)(n+3)} = \frac{n(n+3)+2}{2(n+2)(n+3)} = \frac{n+1}{2(n+3)}$$

as claimed.

We now claim that $\lim_{n\to\infty} s_n = \frac{1}{2}$. So let $\epsilon > 0$ and let $N = \max\left\{\lceil \frac{1}{\epsilon} - 2\rceil, 1\right\}$. So it follows that for all $n > N$, we have that $n > \frac{1}{\epsilon} - 2$ or equivalently, that $\frac{1}{n+2} < \epsilon$. Thus

$$\left|\frac{n}{2n+4} - \frac{1}{2}\right| = \frac{1}{n+2} < \epsilon$$

as claimed, and so the sum $\sum_{n=1}^{\infty} a_n = \frac{1}{2}$.