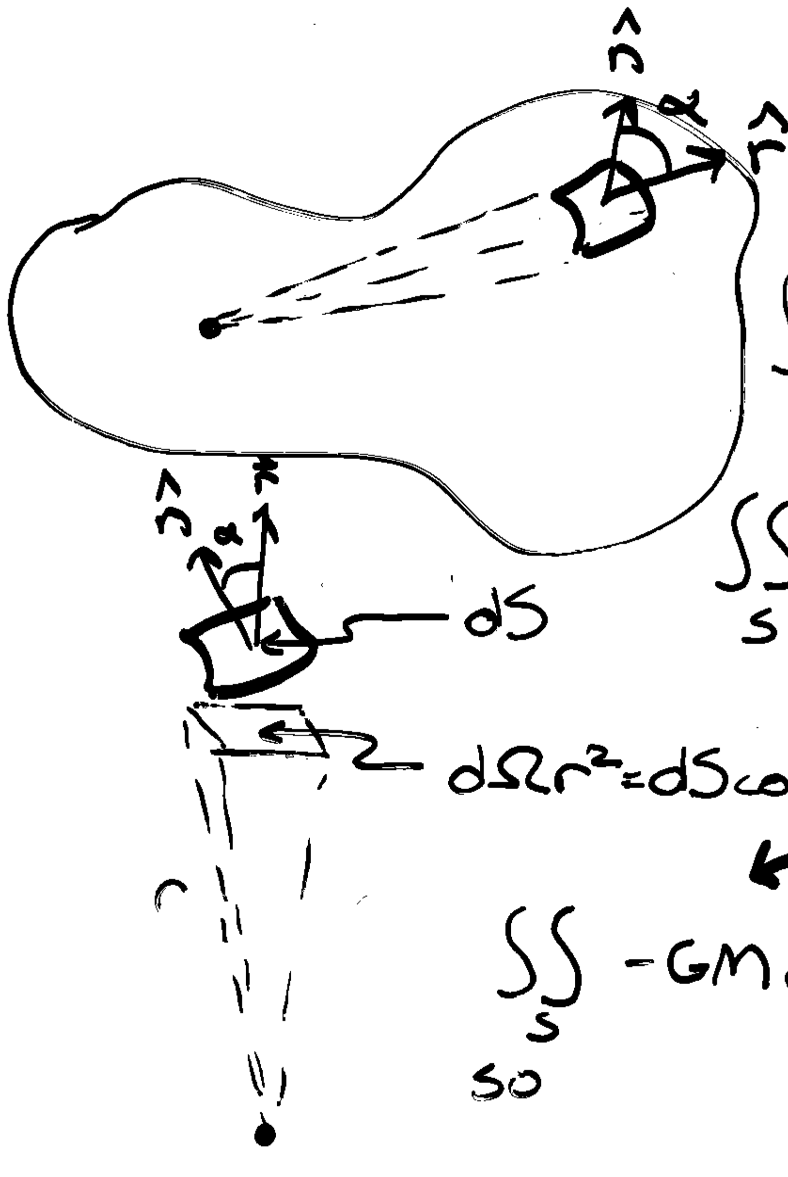
POTENTIAL THEORY

FIELD EQUATIONS 1



$$\vec{g} = -\frac{GM}{r^2}\hat{r}$$

$$\iint_S \vec{g}\cdot\hat{n}\,dS$$

$$\Downarrow$$

$$\iint_S -\frac{GM}{r^2}\cos\alpha\,dS$$

$$\iint_S -GM\,d\Omega = -4\pi GM$$

SO

$$\iint_S \vec{g}\cdot\hat{n}\,dS = -4\pi G \iiint_V \rho(\vec{r})\,d^3\vec{r}$$

POTENTIAL THEORY

FIELD EQUATIONS 2

$$\iint_S \vec{g}\cdot\hat{n}\, dS = \iiint_V \vec{\nabla}\cdot\vec{g}\, dV$$

$$\iiint_V \vec{\nabla}\cdot\vec{g}\, dV = \left[\iiint_V \rho(\vec{r})\, d^3\vec{r}\right](-4\pi G)$$

$$\iiint_V \left(\vec{\nabla}\cdot\vec{g} + 4\pi G\rho\right) d^3\vec{r} = 0$$

but V is arbitrary so the integrand must vanish

$$\vec{\nabla}\cdot\vec{g} + 4\pi G\rho = 0$$

$$\vec{\nabla}\cdot\left(-\vec{\nabla}\phi\right) = -4\pi G\rho$$

$$\boxed{\nabla^2\phi = 4\pi G\rho}$$

POTENTIAL THEORY

SPHERICAL CHARGE DISTRIBUTIONS

Let $\vec{r} \parallel \hat{z}$

$$\phi(\vec{r}) = G \iiint \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d^3\vec{r}\,'$$

$$d^3\vec{r}\,' = r'^2 dr' d\varphi' \, d(\cos\theta')$$

$\mu = \cos\theta$

$$= G \int dr' \int_0^{2\pi}\int_{-1}^{1} \frac{d\varphi' d\mu' \rho(\vec{r}\,')}{(r^2 - 2r'r\mu' + r'^2)^{1/2}}$$

$$= 2\pi G \int \rho(r') dr' \left[- \frac{(r^2 - 2r'r\mu' + r'^2)}{rr'} \right]_{\mu=-1}^{\mu=1}$$

$$= 2\pi G \int \rho(r') dr' \frac{|r+r'| - |r-r'|}{rr'}$$

$\frac{1}{r}$ for $r > r'$; $\frac{1}{r'}$ for $r < r'$

POTENTIAL THEORY
(THE STUDY OF HARMONIC FUNCTIONS)

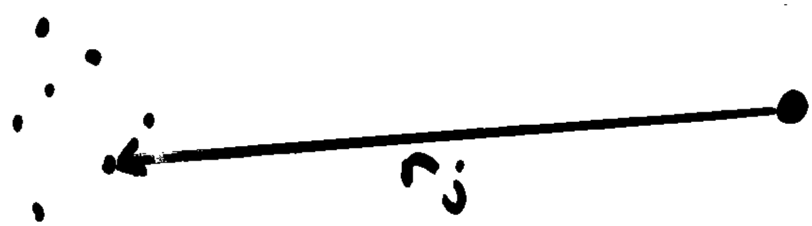
OR

HOW TO MAKE REALLY GOOD APPROXIMATIONS?

APPLICATIONS TO.
- SOLAR SYSTEM
- ELECTROMAGNETISM

POTENTIAL THEORY

WHAT IS THE FORCE ON A PARTICULAR CHARGE FROM A BUNCH OF CHARGES?

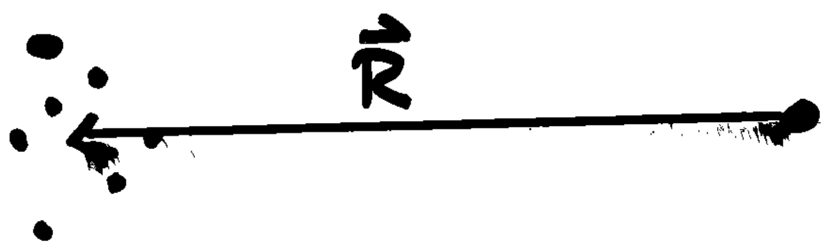


$$\vec{F}_j = \frac{q q_j}{r_j^2}\hat{r}_j = \frac{q q_j}{r_j^3}\vec{r}_j$$

$$\vec{F} = \sum_j \vec{F}_j$$

POTENTIAL THEORY

WHAT IF YOU NEEDED ONLY AN APPROXIMATE ANSWER?



$$\vec{F} = \frac{Q}{R^3}\vec{R} \quad \text{where}$$

$$Q = \sum q_j$$

$$\vec{R} = \frac{1}{Q}\sum q_j \vec{r}_j$$

POTENTIAL THEORY

LET'S LOOK AT THE POTENTIALS

$$V_j = \frac{q q_j}{|\vec{r} - \vec{r}_j|}$$

$$V = \sum_j \frac{q q_j}{|\vec{r} - \vec{r}_j|} \approx \frac{q Q}{|\vec{r} - \vec{R}|}$$

where $\vec{r}$ is the location of the black charge and $\vec{r}_j$ are the locations of the other charges.

POTENTIAL THEORY

HOW FAR OFF ARE WE?

Let's take $\vec{R}=0$ to make life easier

$$\Delta V = \sum \frac{q q_j}{\sqrt{(\vec{r}-\vec{r}_j)^2}} - \frac{qQ}{r}$$

$$= \sum \frac{q q_j}{\sqrt{r^2 - 2\vec{r}\cdot\vec{r}_j + r_j^2}} - \frac{qQ}{r}$$

$$= \sum \frac{q q_j}{r}\left(1 - 2\frac{\vec{r}\cdot\vec{r}_j}{r^2} + \left(\frac{r_j}{r}\right)^2\right)^{-1/2} - \frac{qQ}{r}$$

THE TAYLOR SERIES

$$V = \sum \frac{q q_j}{r}\left(1 - \frac{2\vec{r}\cdot\vec{r}_j}{r^2} + \left(\frac{r_j}{r}\right)^2\right)^{-1/2}$$

$$V = \frac{q}{r}Q + \frac{q}{r^3}\vec{r}\cdot\overbrace{\sum q_j\vec{r}_j}^{\vec{d}}$$

$$+ \frac{q}{r^5}\left(\underbrace{\sum\left[\frac{3}{2}(r\cdot r_j)^2 - \frac{1}{2}r_j^2 r^2\right]q_j}_{\frac{q}{r^5}\vec{r}\cdot\overleftrightarrow{Q}\cdot\vec{r}}\right) + \cdots$$

$$\frac{1}{\sqrt{1-x}} \approx 1 + \frac{x}{2} + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots$$

$$\overleftrightarrow{Q} = \sum \frac{q_j}{2}\begin{bmatrix} 3x_j^2 - r_j^2 & 3x_jy_j & 3x_jz_j \\ \cdots & 3y_j^2 - r_j^2 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

MONOPOLE, DIPOLE, QUADRUPOLE..

If the total charge does not vanish, can take origin to lie at the centre of charge, so $\vec{d} = 0$.

For gravity there is no negative mass, so take origin to be centre of mass, leaving

$$V = \frac{qQ}{r} + \frac{q}{r^5} \vec{r} \cdot \overleftrightarrow{Q} \cdot \vec{r} + \cdots$$

POTENTIAL THEORY

FORCE FIELDS