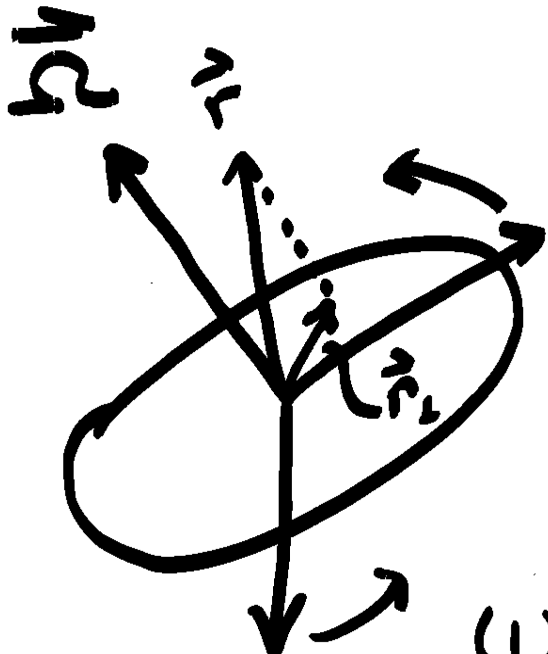
ROTATING FRAMES

WHY ARE ROTATING REFERENCE FRAMES USEFUL?

- HINT:
  THEY AREN'T JUST USEFUL FOR MERRY-GO-ROUNDS

ROTATING FRAMES



· Rotation about an axis preserves:

(1) Distance between objects

(2) Distance from the axis.

$$\vec{r}_\perp \cdot \vec{r}_\perp = \vec{r}\cdot\vec{r} - \frac{(\vec{\Omega}\cdot\vec{r})^2}{(\vec{\Omega}\cdot\vec{\Omega})}$$

$$0 = \frac{d}{dt}(\vec{r}\cdot\vec{r}) = 2\dot{\vec{r}}\cdot\vec{r} = 0$$

$$0 = \frac{d}{dt}(\vec{\Omega}\cdot\vec{r}) = \vec{\Omega}\cdot\dot{\vec{r}} = 0$$

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Therefore,

$\dot{\vec{r}}$ must be parallel to $\vec{\Omega} \times \vec{r}$

We haven't used the magnitude of $\vec{\Omega}$ so let's define:

$$\dot{\vec{r}} = \vec{\Omega} \times \vec{r}$$

where $|\Omega|$ is the rotation rate (in radians per second)

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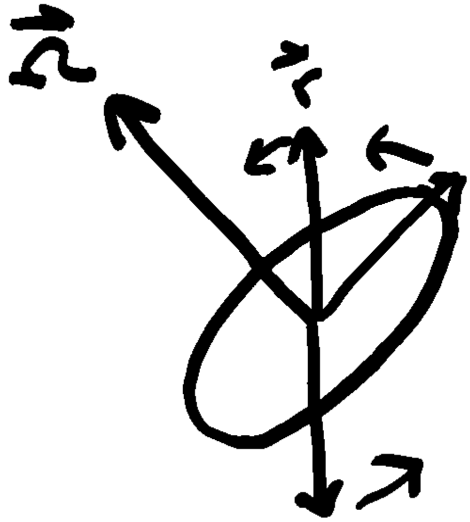
What is the velocity of an object in the rotating frame relative to an inertia frame?

Let's write $\vec{v}$ to denote the velocity measured by the rotating observer at $\vec{r}$, $\vec{u}$ for the inertial observer.



$$\vec{u} = \vec{v} + \vec{\Omega} \times \vec{r}$$

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What is the object's kinetic energy?

$$T = \frac{1}{2} m \vec{u}^2$$

$$= \frac{1}{2} m \left(\vec{v} \cdot \vec{v} + 2 \vec{v} \cdot (\vec{\Omega} \times \vec{r}) + (\vec{\Omega} \times \vec{r})^2 \right)$$

$$L = \frac{1}{2} m \left(\vec{v} \cdot \vec{v} + 2 \vec{v} \cdot (\vec{\Omega} \times \vec{r}) + (\vec{\Omega} \times \vec{r})^2 \right) - V(\vec{r})$$

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What is the object's equation of motion in the rotating coordinates?

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$

$$\frac{d}{dt}\left(m\vec{v} + \vec{\Omega}\times\vec{r}\right) - (\vec{v}\times\vec{\Omega}) - \overbrace{\left[(\vec{\Omega}\cdot\vec{\Omega})\vec{r} - (\vec{\Omega}\cdot\vec{r})\vec{\Omega}\right]}^{\Omega\times(\vec{r}\times\vec{\Omega})} + \vec{\nabla}V = 0$$

$$m\dot{\vec{v}} = -\vec{\nabla}V - \left[2\vec{\Omega}\times\vec{v} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})\right]m$$

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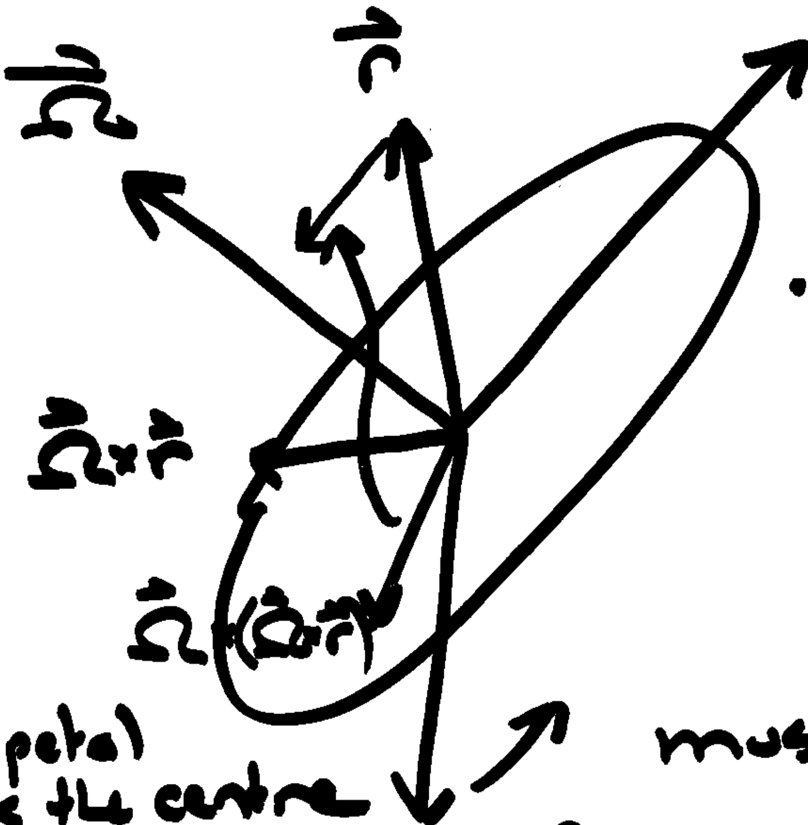
We get two additional sources of apparent acceleration in the rotating frame.



centripetal
seek the centre

• If the object is static in the rotating frame, there must be a **centripetal** force to keep it from going in a straight line.

centrifugal flee the centre.

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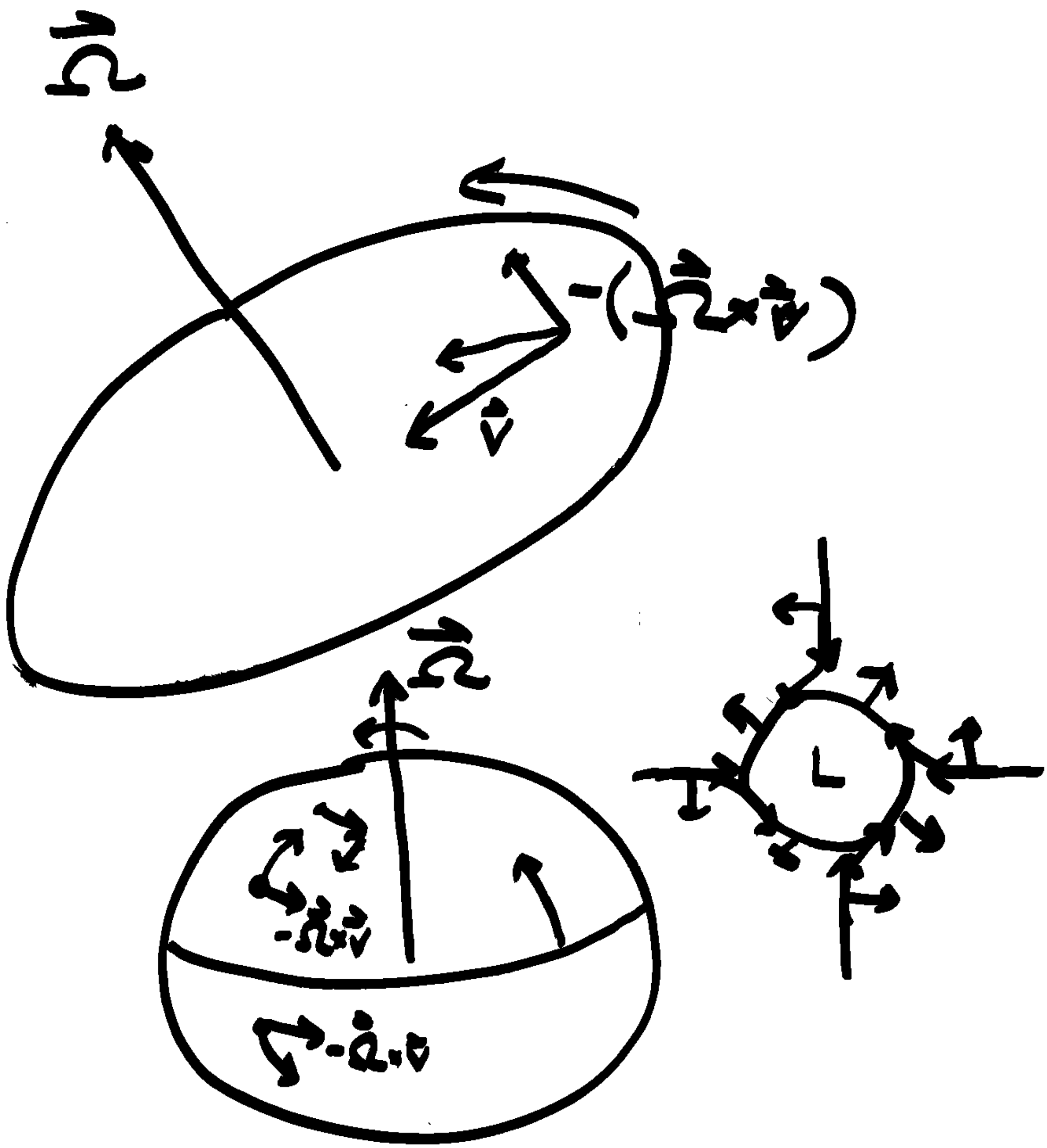
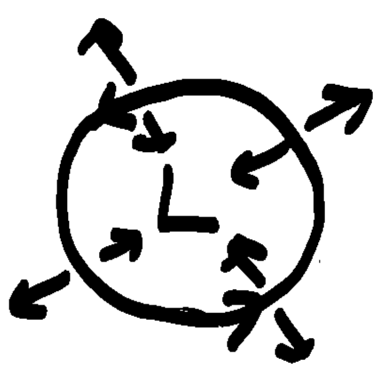


Why are there small-scale cyclones (e.g. tornados) but not small-scale anticyclones?



MOTION IN A ROTATING FRAME

$$\dot{\vec{v}} = 2\vec{v} \times \vec{\Omega}$$

The change in the velocity is perpendicular to $\vec{\Omega}$

$$\Downarrow$$

$v_{\parallel}$ is constant.

The change is perpendicular to $\vec{v}$, so $|\vec{v}|$ is constant.

$$\vec{v}_\perp = |\vec{v}_\perp| \left(\hat{i} \sin f(t) + \hat{j} \cos f(t) \right)$$

if we take $\vec{\Omega} \parallel \hat{k}$

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$$\dot{\vec{v}}_\perp = |\vec{v}_\perp| \left(+\hat{\imath} \cos f(t) - \hat{\jmath} \sin f(t) \right) \dot{f}$$

$$\dot{\vec{v}}_\perp = 2 \vec{v}_\perp \times \vec{\Omega}$$
$$= 2\left(\hat{\imath} \cos f(t) - \hat{\jmath} \sin f(t) \right) \Omega |\vec{v}_\perp|$$

so $\dot{f} = +2\Omega$

$$\vec{v}_\perp = |\vec{v}_\perp| \left(\hat{\imath} \sin 2\Omega t + \hat{\jmath} \cos 2\Omega t \right)$$

$$\vec{r} = |v_\parallel| t \hat{k} + \frac{|\vec{v}_\perp|}{2\Omega} \times \left(-\hat{\imath} \cos 2\Omega t + \hat{\jmath} \sin 2\Omega t \right)$$

IT GOES IN A CIRCLE!

ON THE EARTH:

In the Earth's atmosphere and oceans, the vertical component of the Coriolis force is balanced by bouyancy!

Therefoe, let's focus on vertical componebt of $\vec{\Omega}$

$$\Omega_{\text{vertical}} = \Omega \sin(\text{latitude})$$

For a given $|\vec{v}_\perp|$ radius↑ latitude↓

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ON THE EARTH

$$Ro = \frac{V}{2\Omega_z L}$$

Air and water will travel in inertial circles in the absense of other horizontal forces.

ON A FLY

Flies vibrate their antennae to sense their angular rotation.