## EXTRA PROBLEMS IN GROUP THEORY

1) Show that a nonempty finite set with an associative binary operation satisfying the cancellation laws is a group.

2) For any subgroup $H$ of a group $G$, show that the intersection $\bigcap_{g \in G} gHg^{-1}$ is the largest normal subgroup contained in $H$.

Proof: Let

$$N = \bigcap_{g \in G} gHg^{-1}$$

. Then $N$ is clearly normal, and is contained in $H$. If $N'$ is another normal subgroup contained in $H$, then $gng^{-1} \in N'$ for $n' \in N'$, $g \in G$. Hence $gn'g^{-1} = h$ for $h \in H$ and hence $n' = g^{-1}Hg$ for all $g \in G$, thus $n' \in N$. Hence the theorem is proved.

3) Suppose $G$ is a group that contains a subgroup $H$ in its centre and has the property that the quotient group $G/H$ is cyclic. Then $G$ is commutative.

4) Let $p$ be the smallest prime dividing $o(G)$ for a finite group $G$. Show that any subgroup of index $p$ is normal.
Proof: Let $H \subseteq G$ be a subgroup of index $p$, so $\mid G/H \mid= p$. Consider the natural homomorphism $\phi : G \to \text{Bij}(G/H)$, then $p$ divides $\mid G \mid$ and $\mid \text{Bij}(G/H) \mid= p!$. We showed that the kernel of $\phi$ is the largest normal subgroup of $G$ contained in $H$. Now use the hypothesis that $p$ is the smallest prime dividing $\mid G \mid$ along with this fact and Lagrange's theorem to show that the kernel of $\phi$ is $H$ and hence $H$ is normal.

5) Let $G$ be a group with a subgroup of index $r$. Prove :
i) If $G$ is simple, then $o(G)$ divides $r!$.
ii) If $r = 2$, 3 or 4, then $G$ cannot be simple.

Proof: This is analogous to the applications I did in class today, using similar ideas as in 4).

6) Suppose that $N$ and $M$ are two normal subgroups of $G$ and that $N \cap M = e$. Show that for any $n \in N$, $m \in M$, we have $nm = mn$.

7) Prove, by an example, that we can find three groups $E \subset F \subset G$ where $E$ is normal in $F$, $F$ is normal in $G$ but $E$ is not normal in $G$.
Proof Consider the dihedral group $D_8$ (symmetries on a square). Take $F$ to be the subgroup

generated by the reflection and rotation by 180 degrees. This is normal in $G$, and in $F$ consider the subgroup $E$ generated by the reflection. Check that this gives an example.

8) Prove that every finite group having more than two elements has a non-trivial automorphism. You may assume that not every element has order 2 in $G$.
Proof: If $G$ is not abelian, then take any element $g$ which is not in the centre, and conjugate by $g$ i.e. consider the automorphism $c_g : G --- > G$, $c_g(x) = gxg^{-1}$. This is a non-trivial automorphism. If $G$ is abelian, then $g --- > g^-1$ is a nontrivial automorphism as there is a $g$ in $G$ with $g^2 \neq e$.

9) Let $G$ be a finite group, $T$ an automorphism of $G$ with the property that $Tx = x$ if and only if $x = e$. Suppose further that $T^2 = Id$, the identity morphism. Prove that $G$ must be abelian.

10) If $o(G) = pq$ where $p$ and $q$ are distict prime numbers and if $G$ has a normal subgroup of order $p$ and a normal subgroup of order $q$, then $G$ is cyclic.
Proof: Use the fact that the the subgroup $N$ of order $p$ and the subgroup $M$ of order $q$ are cyclic.

**Assignment solutions from Assignment 3:**

If $(ab)^3 = a^3b^3$ for any pair of elements $a, b$ in $G$, and 3 does not divide $o(G)$, show that $G$ is abelian.
Proof: Deduce first that $(ba)^2 = a^2b^2$ and $(ab)^2 = a^2b^2$. Using that $(ab)^2 = b^2a^2$, then deduce that $a^2b^3 = b^3a^2$. So every square commutes with every cube. Now consider $x \mapsto x^3$ on $G$, this is a homomorphism by the given hypothesis and is injective. Further as $G$ is finite, this homomorphism is thus an isomorphism. So any $g \in G$ is a cube. Hence squares commute with all the elements in $G$. Now $(ab)^2 = b^2a^2$, hence $(abab) = b(ba^2) = b(a^2b) = baab$; hence $ab = ba$; here we are using the fact that any element in $G$ is a cube and that squares commute with all elements of $G$.

If $aH \neq bH \iff Ha \neq Hb$, show that $gHg^1 \subset H$.
Proof: We must show that the hypothesis implies that $H$ is normal. We will use the contrapositive of this hypothesis. For $g \in G$ and every $h \in H$, we have $gH = ghH$; hence $Hg = Hgh$. This implies on multiplying both sides of the second identity by $g^{-1}$, we have $H = Hghg^{-1}$, hence $ghg^{-1} \in H$ and $H$ is normal.