LINEAR MOTION
GENERAL RESULTS

- EXPLAIN HOW LINEAR MOTION IS IMPORTANT EVEN FOR MULTI-DIMENSIONAL MOTION
- SOLVE ANY ONE-D SYSTEM WHERE FORCE ONLY DEPENDS ON POSITION

LINEAR MOTION

#5

- Rearranging a bit more gives

$$dt = \frac{dx}{\sqrt{\frac{2}{M}(E-V)}}$$

so

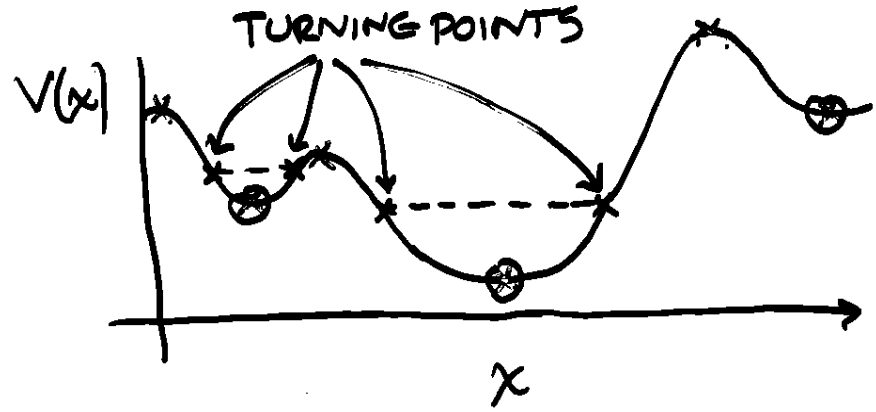
$$t_1 - t_0 = \int_{x_0}^{x_1} \frac{dx}{\sqrt{\frac{2}{M}(E-V(x))}}$$

- This integral is not always easy, so it is called a formal solution, but you could always hand it to a computer.
- More importantly, the plot of $V(x)$ can tell you the range of motion.

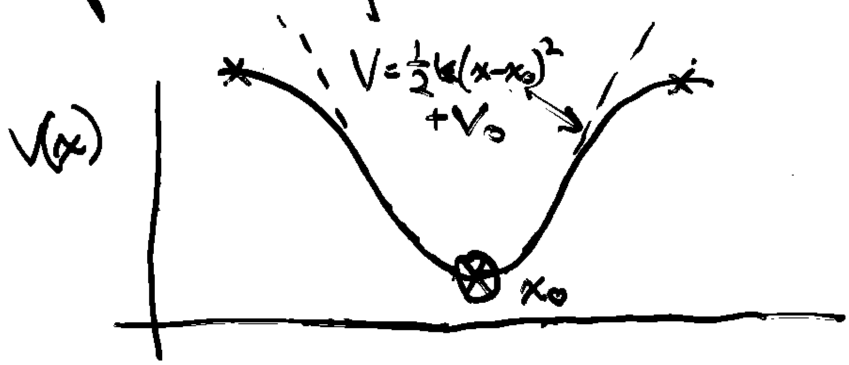
LINEAR MOTION #C

THE LAY OF THE LAND

- The potential energy curve can tell us a lot about the motion even without an exact solution.



× Equilibrium points ⊗ Stable equilibria



- Near an equilibrium the potential looks like a parabola.

LINEAR MOTION #7

HARMONIC OSCILLATORS

- EXPLAIN WHY UNDERSTANDING HARMONIC MOTION IS USEFUL EVEN THOUGH MOST SYSTEMS ARE NOT HARMONIC OSCILLATORS
- SOLVE THE HARMONIC OSCLLLATOR WITH DAMPING AND DRIVING FORCES

LINEAR MOTION #12

THE DAMPED OSCILLATOR

- If we have viscous damping the EOM is

$$m\ddot{x} + \lambda\dot{x} + kx = 0$$

where $k = V''$.

Again this is a LDEWCC, so we can try $x(t) = Ae^{pt}$ to get

$$mAp^2e^{pt} + \lambda pAe^{pt} + kAe^{pt} = 0$$

$$mp^2 + \lambda p + k = 0$$

$$\Downarrow$$

$$p = \frac{-\lambda \pm \sqrt{\lambda^2 - 4km}}{2m}$$

$$= -\underbrace{\frac{\lambda}{2m}}_{\substack{\gamma \\ \text{damping} \\ \text{constant}}} \pm \sqrt{\left(\frac{\lambda}{2m}\right)^2 - \underbrace{\frac{k}{m}}_{\substack{\omega_0^2 \\ \text{undamped} \\ \text{frequency}}}}$$

STRONG DAMPING

- If $\gamma > \omega_0$ then both solutions are real

$$p = -\gamma_\pm \quad \text{with} \quad \gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

$$x = \underbrace{\tfrac{1}{2} A e^{-\gamma_+ t}}_{\text{Dies faster}} + \underbrace{\tfrac{1}{2} B e^{-\gamma_- t}}_{\text{Dominates at late times}}$$

WEAK DAMPING

- If $\gamma < \omega_0$ then the solutions are complex

$$p = -\gamma \pm i\omega \quad \text{with} \quad \omega = \sqrt{\omega_0^2 - \gamma^2}$$

$$x = \tfrac{1}{2} A e^{i\omega t - \gamma t} + \tfrac{1}{2} B e^{-i\omega t - \gamma t}$$

$$= a e^{-\gamma t} \cos(\omega t - \Theta)$$

$$= A e^{-\gamma t} \cos(\omega t) + B e^{-\gamma t} \sin(\omega t)$$

- If $\gamma = \omega_0$ then $\omega \to 0$ **CRITICAL**

$$x = A e^{-\gamma t} + B\omega t\, e^{-\gamma t} \rightarrow x = (a + bt) e^{-\gamma t}$$

DRIVEN OSCILLATORS

- What if we apply a force to the oscillator?

$$m\ddot{x} + \lambda\dot{x} + kx = F(t)$$

Because the equation is linear we can add the solution to the unforced oscillation to the particular solution with the forcing.

Let's try $F(t) = F_1 e^{i\omega_1 t}$ because any $F(t)$ can be written as a sum of harmonic oscillations.

$$m\ddot{x} + \lambda\dot{x} + kx = F_1 e^{i\omega_1 t}$$

and a solution $x_p = A_1 e^{i\omega_1 t}$ to get

$$-m\omega_1^2 A_1 + i\lambda A_1\omega_1 + kA_1 = F_1$$

$$A_1 = \frac{F_1}{m}\frac{1}{\omega_0^2 - \omega_1^2 + i(2\gamma\omega_1)}$$

$$A_1 = a_1 e^{-i\Theta_1} \quad a_1 = \frac{F_1}{m}\frac{1}{\sqrt{(\omega_0^2-\omega_1^2)^2 + 4\gamma^2\omega_1^2}}$$

INELASTIC COLLISION

- The collision could reduce the relative velocity of the objects.

e.g. They could stick.

$$v_2 - v_1 = 0(u_1 - u_2)$$

Coefficient of restitution

In general we have $v_2 - v_1 = e(u_1 - u_2)$

$\Downarrow$

$$v_1 = \frac{m_1 - e m_2}{m_1 + m_2} u_1 \qquad v_2 = \frac{(1+e) m_1}{m_1 + m_2} u_1$$

The energy decreases:

$$\boxed{\frac{T - T'}{T} = (1 - e^2) \frac{m_2}{m_1 + m_2}}$$

E&J #1

ORGANIZING CONCEPTS

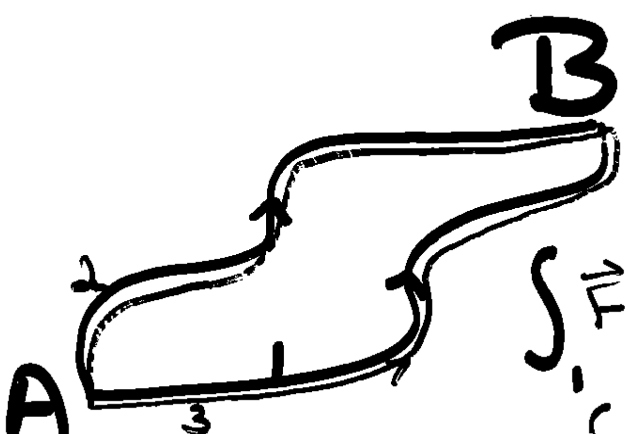
$\boxed{\text{ENERGY AND ANGULAR MOMENTUM}}$

- EXPLAIN UNDER WHAT CIRCUMSTANCES ARE ENERGY, MOMENTUM AND ANGULAR MOMENTUM CONSERVED.
- DETERMINE THE EQUATIONS OF MOTION WITH LAGRANGE'S EQUATIONS

Ed. J #2

CONSERVATIVE FORCES

- In one dimension, all forces that depend only on position are conservative.
- In one dimension, there is only one way from A → B
- For a force to be conservative, the work must not depend on path.



$$\int_1 \vec{F}\cdot d\vec{s} - \int_2 \vec{F}\cdot d\vec{s} = 0$$

$$\oint_3 \vec{F}\cdot d\vec{s} = 0$$

Stokes' theorem

$$\int (\vec{\nabla}\times\vec{F})\cdot d\vec{A} = 0$$

- For this to be true for all paths $\boxed{\vec{\nabla}\times\vec{F} = 0}$

E6 $\vec{J}$ #3

TORQUES & ANGULAR MOMENTUM

- Torque is defined as

$$\vec{\tau} = \vec{r} \times \vec{F}$$

- Angular momentum is

$$\vec{J} = \vec{r} \times \vec{p}$$

$$\frac{d\vec{J}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

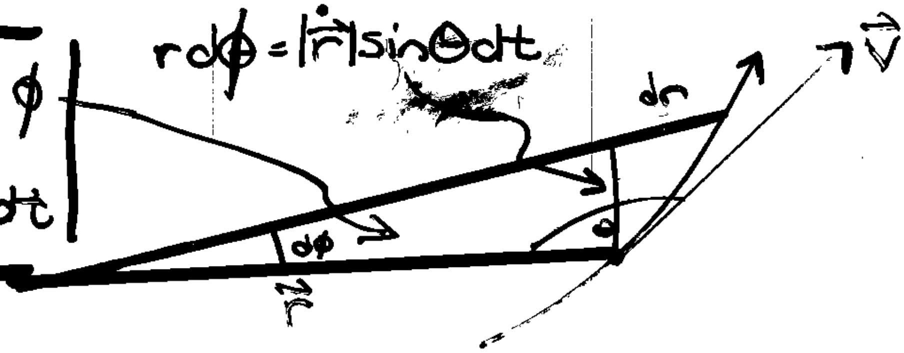
- Both the direction and magnitude are constant.

$\vec{r}, \vec{p}$ define a plane. This plane is constant.

$$|\vec{J}| = m r |\dot{\vec{r}}| \sin\Theta = m r v_\phi = m r^2 \dot{\phi}$$

$$\boxed{\begin{aligned} dA &= \frac{1}{2} r^2 d\phi \\ dA &= \frac{1}{2} \frac{|\vec{L}|}{m} dt \end{aligned}}$$

$$r\,d\phi = |\dot{\vec{r}}| \sin\Theta\, dt$$



EG3 #6

THE MAIN RESULT

The function $y(x)$ that minimizes

$$I = \int_{x_0}^{x_1} f(y, y')\,dx$$

and satisfies $y(x_0) = y_0$, $y(x_1) = y_1$ also satisfies the differential equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

EULER-LAGRANGE EQUATION

and vice versa.

Integral Statement $\Leftrightarrow$ Differential Equation

EL5 #7

THE AMAZING THINGS

- Particles and fields evolve in time so as to minimize (extremize)

$$S = \int (T-V)\,dt = \int L\,dt$$

just by following $\ddot{x} = -\frac{V'}{m}$

- An analogy to Fermat's principle tells you that particles accumulate a <u>phase</u>

$$\boxed{\begin{aligned}\Delta\phi &\propto \int_{t_1}^{t_2} (T-V)\,dt \\ \Delta\phi &= \int dt \times \omega\end{aligned}}$$

$$\Delta\phi = \frac{S}{\hbar}$$

PLAN OF ATTACK.

What do we already know about the solution?

(1) The motion is restricted to a plane. $3 \to 2$

(2) The orbit sweeps out equal area in equal time. $2 \to 1$

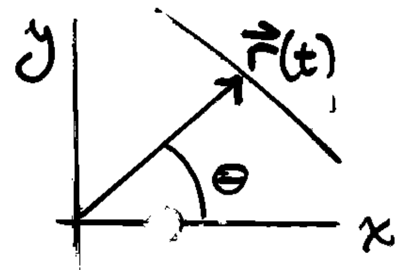
Although the motion is 3-D, we only have to worry about the radial motion!

CENTRAL FORCES 3

ANGULAR MOTION

- Because the force is central, we have conservation of angular momentum

$$J = mr^2\dot{\Theta}$$



so once we know $r(t)$ we can solve for $\Theta(t)$ by integration.

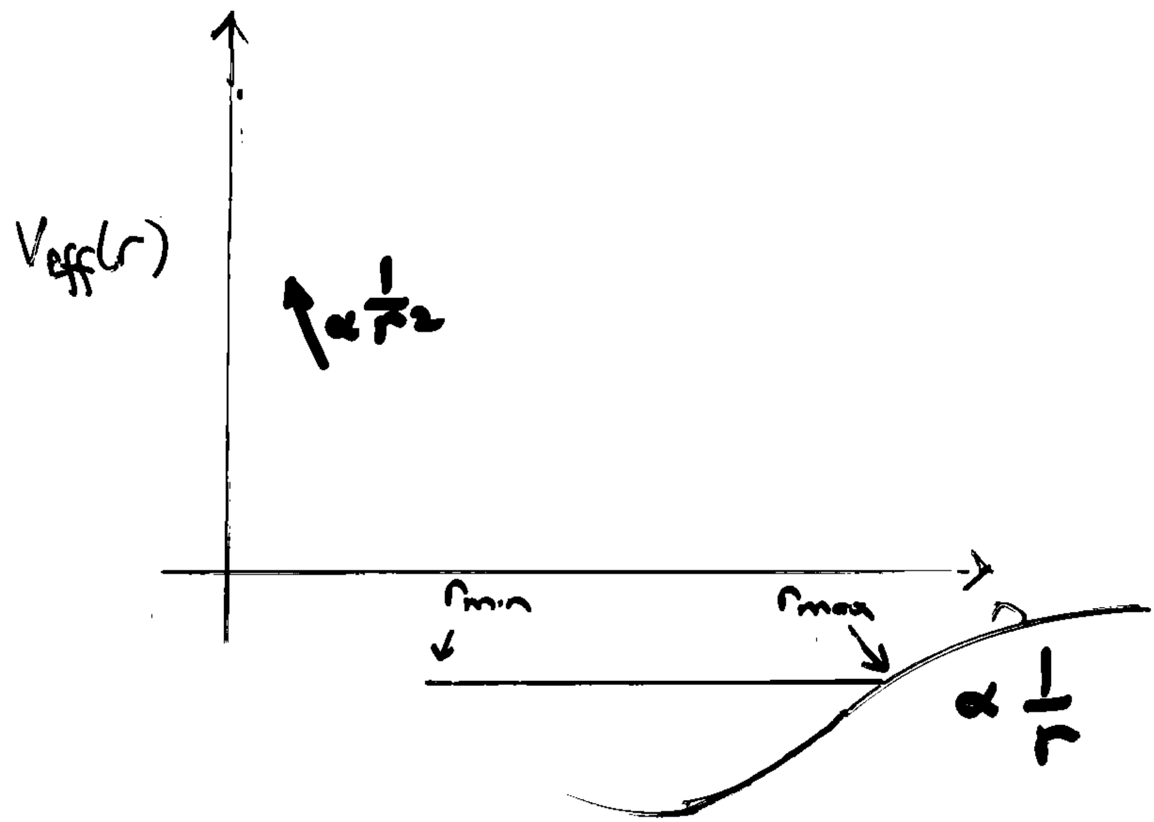
$$\frac{d\Theta}{dt} = \frac{J}{mr^2}$$

$$\Theta - \Theta_0 = \int_{t_0}^{t} \frac{J}{m} \frac{dt}{r^2(t)}$$

THE EFFECTIVE POTENTIAL

- If we look at the radial motion there is an effective potential.

$$V_{eff}(r) = \frac{\vec{J}^2}{2mr^2} + V(r)$$



TOWARD A GENERAL SOLUTION

- Using V_{eff} we can write:

$$\dot{r}^2 = \frac{2}{m}\left(E - V_{eff}(r)\right)$$

so
$$\frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E - V_{eff}\right)}$$

and the radial period is

$$P_r = 2\int_{r_{min}}^{r_{max}} \frac{dr}{\sqrt{\frac{2}{m}\left(E - V_{eff}\right)}}$$

THE SHAPE OF THE ORBIT

Through what angle has the object travelled during a radial period.

$$\frac{d\theta}{dt} = \frac{J}{mr^2} \quad \text{so} \quad \frac{dr}{d\theta} = \frac{mr^2}{J}\sqrt{\frac{2}{m}\left(E - V_{eff}\right)}$$

$$\Delta\theta = 2\int_{r_{min}}^{r_{max}} \frac{dr}{\frac{mr^2}{J}\sqrt{\frac{2}{m}\left(E - V_{eff}\right)}}$$