< Summary of Sequences and Series >

Discussed in class:

Definition A sequence is a function of the form $f: \mathbb{N} \longrightarrow \mathbb{R}$.
(This is also denoted by $(a_n)_{n\in\mathbb{N}}$, (a_n) or $\{a_n\}$.)

Definition A sequence (a_n) converges to some real number L if, for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $n > N$ implies $|a_n - L| < \varepsilon$.
If (a_n) does not converge, then (a_n) is said to diverge.
(a_n) is said to diverge to infinity, written as $\lim_{n\to\infty} a_n = \infty$ (or $-\infty$) if, for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $n > N$ implies $a_n > \varepsilon$.

Theorem For all $x, y \in \mathbb{R}$,
1. $|xy| = |x||y|$
2. $|x \pm y| \le |x| + |y|$ (Triangular Inequality)
3. $|x| \le y$ if and only if $-y \le x \le y$.[I]

Definition A sequence (a_n) is bounded if there is some $R > 0$ such that $|a_n| < R$ for all $n \in \mathbb{N}$.

Theorem If (a_n) is a convergent sequence, then (a_n) is bounded.

Theorem If $\lim_{n\to\infty} a_n = L$ and $L \neq 0$, then $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}$.

Theorem (The Cauchy Criterion for Convergence)
A sequence (a_n) converges if and only if, for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that, for all $m, n \in \mathbb{N}$, $m, n > N$ implies $|a_m - a_n| < \varepsilon$.

Definition A series (s_n) is a sequence of the form $s_n = \sum_{i=1}^{n} a_i$, where (a_k) is some sequence.

Definition Let (a_k) be a sequence. If $\sum_{i=1}^{\infty} |a_i|$ converges, then $\sum_{i=1}^{\infty} a_i$ is said to be absolutely convergent.

Definition A geometric series is a series which takes the form of

$$s_\infty = \sum_{i=1}^{\infty} ar^i$$

where a and r are constants.

Theorem Let (s_n) be a partial geometric series. If $|r| < 1$, then

$$s_n = \sum_{i=1}^{n} ar^i = a\left(\frac{1 - r^{n+1}}{1 - r}\right)$$

Corollary If $|r| < 1$, then

$$s_\infty = \sum_{i=1}^{\infty} ar^i = a\left(\frac{1}{1 - r}\right)$$

Definition The Harmonic Series is the series

[I] A similar result holds for the strict inequality $<$.

$$\sum_{i=1}^{\infty} \frac{1}{i}$$

Theorem The Harmonic Series does not converge.

Theorem (The Comparison Test)

Let (a_n) and (b_n) be two sequences.

1. If $|a_n| \leq b_n$ for all $n \in \mathbb{N}$ and $\sum_{i=1}^{\infty} b_i$ converges, then $\sum_{i=1}^{\infty} a_i$ converges.
2. If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum_{i=1}^{\infty} a_i$ diverges, then $\sum_{i=1}^{\infty} b_i$ diverges.

Theorem (The Ratio Test)

Let (a_n) be a positive sequence.

1. If there is some θ such that $0 < \theta < 1$ and $\frac{a_{n+1}}{a_n} \leq \theta$ for all $n \in \mathbb{N}$, then $\sum_{i=1}^{\infty} a_i$ converges.
2. If there is some θ such that $\theta \geq 1$ and $\frac{a_{n+1}}{a_n} \geq \theta$ for all $n \in \mathbb{N}$, then $\sum_{i=1}^{\infty} a_i$ diverges.

Theorem (The Root Test)

Let (a_n) be a positive sequence.

1. If there is some θ such that $0 \leq \theta < 1$ and $\sqrt[n]{a_n} \leq \theta$ for all $n \in \mathbb{N}$, then $\sum_{i=1}^{\infty} a_i$ converges.
2. If there is some θ such that $\theta \geq 1$ and $\sqrt[n]{a_n} \geq \theta$ for all $n \in \mathbb{N}$, then $\sum_{i=1}^{\infty} a_i$ diverges.

Theorem (The Integral Test)

Let $f: [1, \infty) \longrightarrow [0, \infty)$. The series $\sum_{i=1}^{\infty} f(i)$ converges if and only if $\int_1^{\infty} f(x)dx$ converges.

Definition A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n$$

for some sequence (a_n) and some constant real number x.

Theorem A power series converges if

1. $|x| < \lim \left|\frac{a_n}{a_{n+1}}\right|$, or
2. $|x| < \lim \left|\frac{1}{\sqrt[n]{a_n}}\right|$

Definition The radius of convergence of a power series is

$$\lim \left|\frac{a_n}{a_{n+1}}\right| \quad \text{or} \quad \lim \left|\frac{1}{\sqrt[n]{a_n}}\right|$$

Theorem (The Cauchy Product of Series)

If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_0$ are two absolutely convergent series, then

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_0\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k}$$

Discussed in the textbook:

Definition Let $x \in \mathbb{R}$. The ceiling of x, denoted by $\lceil x \rceil$, is the smallest integer that is greater than or equal to x.

Some well-known series and power series:

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e \qquad \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2) \qquad \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)!}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n-1} = \frac{\pi}{4} \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

Important Topics:

- Zeno's Paradox, the Cantor Set and the Koch snowflake (in particular, know how the area and perimeter are derived)
- Proving convergence, divergence and divergence to infinity
- Working with inequalities and absolute values, as well as come up with functions that can be used for the purpose of bounding
- Conjecturing a limit for a sequence, series and power series
- Changing the order of addition in an infinite series can potentially result in contradictions
- Applying Cauchy's Criterion as well as the four convergence tests
- Finding the radius of convergence of a power series
- Know the derivation of some well-known constants by using power series