MATH 600D: ASSIGNMENT 2

DUE: OCT 10, 2011

(1) Let R be a commutative ring. A sequence $\sigma = (r_1, \cdots, r_n)$ of elements of R is said to be a unimodular row if there exist elements $s_1, \cdots, s_n \in R$ such that $1 = r_1 s_1 + \cdots + r_n s_n$. Prove that (a) and (b) are equivalent and verify (c) :
  (a) σ is a unimodular row.
  (b) $R^n \simeq P \oplus R$, where $P = \operatorname{Ker} \sigma$, P is projective and
$$\sigma : R^n \to R$$
$$(s_1, \cdots, s_n) \to \sum_{i=1}^{n} r_i s_i$$
  (c) There exists an $(n \times n)$ matrix over R, which is invertible and whose first row is $\sigma = (r_1, \cdots, r_n)$ iff the projective R-module P defined in (b) is free.

(2) Show that every unimodular row of length 2 may be completed to an invertible (2×2) matrix. Deduce that every stably free projective R-module of rank 1 is free (You may assume that R is a domain to define rank).

(3) Let R be a commutative ring. Show that the followings are equivalent for every R-module L:
  (a) There is an R-module M such that $L \otimes_R M \simeq R$
  (b) L is an algebraic line bundle.
  (c) L is a finitely generated R-module and $L_{\mathcal{P}} \simeq R_{\mathcal{P}}$ for every prime ideal $\mathcal{P}$ of R.

(4) Let R be a ring and I a two sided ideal of R. Define
$$D_R(I) = \{(x, y) \in R \times R \mid x - y \in I\}$$
Let π_1 be the natural first projection of $D_R(I) \to R$. Show that there is a split exact sequence
$$0 \to I \to D_R(I) \xrightarrow{\pi_1} R \to 0$$
Define
$$K_0(R, I) = \operatorname{Ker}(\pi_1)_* : \{K_0(D_R(I)) \to K_0(R)\}$$
Show that there is an exact sequence
$$0 \to K_0(R, I) \to K_0(R) \to K_0(R/I)$$
where the last map is induced by the natural surjection $R \to R/I$.