UBC MATH 312 ASSIGNMENT 4 22 OCT 2012

MARKING SCHEME

① There are no integral solutions. Writing $6(10x+3y) = 97$, we see that for any $x, y \in \mathbb{Z}$, $6 \mid \text{LHS}$ but $6 \nmid \text{RHS}$ of the equation.	5 marks
② As in Prof. Sujatha's Midterm Review problems (see Course Website), this question reduces to finding the highest power of 5 that divides 1000! That is: $\left[\frac{1000}{5}\right] + \left[\frac{1000}{5^2}\right] + \left[\frac{1000}{5^3}\right] + \left[\frac{1000}{5^4}\right] + \cdots$ $= 200 + 40 + 8 + 1 + 0 + 0 + \cdots$ $= 249$	5 marks

③

$\cdot$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{4}$	$\bar{0}$	$\bar{2}$	$\bar{4}$
$\bar{3}$	$\bar{0}$	$\bar{3}$	$\bar{0}$	$\bar{3}$	$\bar{0}$	$\bar{3}$
$\bar{4}$	$\bar{0}$	$\bar{4}$	$\bar{2}$	$\bar{0}$	$\bar{4}$	$\bar{2}$
$\bar{5}$	$\bar{0}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$

{Multiplication modulo 6}

The set of elements with multiplicative inverses (modulo 6) is $\{\bar{1}, \bar{5}\}$.

5 marks

④ $x^2 \equiv x \pmod{p}$

$\Rightarrow x^2 - x \equiv 0 \pmod{p}$

$\Rightarrow x(x-1) \equiv 0 \pmod{p}$

$\Rightarrow p \mid x(x-1)$

As p is prime,

Either $p \mid x$ Or $p \mid (x-1)$

$\Downarrow$ $\Downarrow$

$x \equiv 0 \pmod{p}$ $x-1 \equiv 0 \pmod{p}$

$\Downarrow$

$x \equiv 1 \pmod{p}$

5 marks

(5) First, we note that $5(3) = 15 \equiv 1 \pmod 7$.

We're searching for solutions

$$x, y \in \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\}$$

but in our working, for convenience, we'll drop the residue class bars above the numbers.

If $x \equiv 0$, then $2(0) + 3y \equiv 1 \pmod 7$

$$3y \equiv 1 \pmod 7$$

Multiplying by 5, $5(3)y \equiv 5 \pmod 7$

$$y \equiv (1)y \equiv 15y \equiv 5 \pmod 7$$

So one solution is $\boxed{x = \bar{0},\ y = \bar{5}} \pmod 7$.

By applying exactly the same method for the other possible values of x, you can obtain the other solutions:

$$\boxed{\begin{array}{l} x = \bar{1}, \quad y = \bar{2} \\ x = \bar{2}, \quad y = \bar{6} \\ x = \bar{3}, \quad y = \bar{3} \\ x = \bar{4}, \quad y = \bar{0} \\ x = \bar{5}, \quad y = \bar{4} \\ x = \bar{6}, \quad y = \bar{1} \end{array}}$$

(all modulo 7)

5 marks.

(6) This question is very similar to one of the Midterm Review Problems (see course website). We notice that

$$2^{12} = 4096 \equiv 7 \pmod{47}$$

and $$7^2 = 49 \equiv 2 \pmod{47}.$$

Thus, $$2^{300} = (2^{12})^{25} \equiv 7^{25} \pmod{47}$$
$$\equiv 7^{24} \cdot 7^1 \pmod{47}$$
$$\equiv (7^2)^{12} \cdot 7 \pmod{47}$$
$$\equiv 2^{12} \cdot 7 \pmod{47}$$
$$\equiv 7 \cdot 7 \pmod{47}$$
$$\equiv 2 \pmod{47}.$$

So the least positive residue of $2^{300} \pmod{47}$ is 2. 5 marks

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Remark: For a <u>much</u> simpler and more straightforward way to solve similar congruences (modulo a prime), read ahead in the textbook $\longrightarrow$ "Fermat's Little Theorem."