MATH 110/003 - Homework 7 - Solution

Problem 1

Find the points of the curve $y = x^3 + x^2$ for which the tangent line passes through the origin.

Solution

When looking at a problem like this, it is important to think of what kind of answer we're looking for. We're looking for points on a curve at which the tangent line passes through the point $(0, 0)$ (the origin). Since the point $(0, 0)$ itself is on the curve (clearly $0 = 0^3 + 0^2$), whatever its tangent line is, it already passes through the origin and is thus a solution of the problem. But could there be other points on the curve with this property?

If so, let us denote by (a, b) such a point. Since this point is on the curve, we know that actually $b = a^3 + a^2$. What is the equation of the tangent line at this point? We know we can use derivatives to find its slope. How does this work? Well, if we denote by f the function

$$f(x) = x^3 + x^2$$

Then the curve $y = x^3 + x^2$ is the graph of this function and the slope of the tangent line at a point $x = a$ is $f'(a)$. Since $f'(x) = 3x^2 + 2x$, the slope of the tangent line to this curve at the point $x = a$ is $f'(a) = 3a^2 + 2a$. This means that the equation of the tangent line to the curve at the point (a, b) looks like

$$y = (3a^2 + 2a)x + h$$

For some number h that we don't know yet. How to find it? Well, we know this line passes through the point $(a, b) = (a, a^3 + a^2)$ so we know that $x = a$, $y = a^3 + a^2$ is a solution. This means that

$$a^3 + a^2 = (3a^2 + 2a)a + h$$

And thus, we can express h in terms of a:

$$h = (a^3 + a^2) - (3a^2 + 2a)a = a^3 + a^2 - 3a^3 - 2a^2 = -2a^3 - a^2$$

This allows us finally to say that the equation of the tangent line to the curve $y = x^3 + x^2$ at a point (a, b) is

$$y = (3a^2 + 2a)x + (-2a^3 - a^2)$$

Now, we're looking for points (a, b) for which the tangent line passes through the point $(0, 0)$. In other words, $x = 0$, $y = 0$ is a solution of the equation of the line. This will tell us which values of a solve the problem. We get the equation:

$$0 = (3a^2 + 2a)0 + (-2a^3 - a^2) \qquad \iff \qquad 0 = -2a^3 - a^2 \qquad \iff \qquad a^2(2a + 1) = 0$$

Of which the solutions are $a = 0$ and $a = -1/2$, which gives the points $(0, 0)$ and $(-1/2, -1/8 + 1/4) = (-1/2, 1/8)$ on the curve $y = x^3 + x^2$. We knew since the beginning that the point $(0, 0)$ was a solution, but we now discover that there is only one other point, the point $(-1/2, 1/8)$, for which the tangent line passes through the origin.

Problem 2

The paper currency of the Kingdom of Bonoria bears the pictures of the country's monarchs. The one-bonor note carries the picture of Queen Griselda the Good. Other notes not exceeding 100 bonors bear the pictures of King Randolph the Rotten, Queen Carrie the Charming, Queen Bonita the beautiful, King Gerlad the Gross, King Waldo the Wicked, and King Hilary the Hairy. We know that:

- Together the notes on which Bonita's and Carrie's pictures appear are worth 102 bonors.
- One Gerald, Waldo, and Randolph together give 73 bonors.
- Hilary and Bonita give 22 bonors.
- Hilary and Carrie give 120 bonors.
- Hillary, Gerald, and Randolph give 43 bonors.
- Carrie, Waldo, and Randolph give 168 bonors.

On what size note does Waldo's picture appear?

Solution

Let's use some variables to denote the values of each bank note.

- $R =$ the value of one Randolph.
- $C =$ the value of one Carrie.
- $B =$ the value of one Bonita.
- $G =$ the value of one Gerald
- $W =$ the value of one Waldo.
- $H =$ the value of one Hilary.

The information that we're given yields the following system of equations:

$$\begin{cases} B + C = 102 \\ G + W + R = 73 \\ H + B = 22 \\ H + C = 120 \\ H + G + R = 43 \\ C + W + R = 168 \end{cases}$$

There are multiple ways to solve systems of equations. Here's one way to do so. Looking at the equations, we notice that the first, second and third equation, only involve B, C and H:

$$\begin{cases} B + C = 102 \\ H + B = 22 \\ H + C = 120 \end{cases}$$

If we subtract the first first equation from the third, we obtain that

$$(H + C) - (B + C) = 120 - 102 \quad \Longleftrightarrow \quad H - B = 18$$

And if we now add the second equation to this new equation, we obtain that

$$(H - B) + (H + B) = 18 + 22 \quad \Longleftrightarrow \quad 2H = 40 \quad \Longleftrightarrow \quad H = 20$$

Hence $B = 2$ and $C = 100$. Using this information, let us rewrite the three remaining equations:

$$\begin{cases} G + W + R = 73 \\ 20 + G + R = 43 \\ 100 + W + R = 168 \end{cases} \quad \Longleftrightarrow \quad \begin{cases} G + W + R = 73 \\ G + R = 23 \\ W + R = 68 \end{cases}$$

Using the third equation inside the first equation we get

$$G + 68 = 73 \quad \Longleftrightarrow \quad G = 5$$

Hence $R = 18$ and $W = 50$.

This conclude the problem and we can affirm that the bank note on which King Waldo appears is worth 50 bonors.