MATH 110/003 - Homework 8 - Solution

Problem 1

Write a proof for the fact I claimed while proving the Power Rule which states that the derivative of $h(x) = xf(x)$ is $h'(x) = f(x) + xf'(x)$.

Solution

Using the power rule isn't the point here (we don't have a proof for it), we can do things simply by hand using the limit definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Since the function that we're looking at is also named h, let's avoid the confusion and give it a new name, say g, so we'll write that $g(x) = xf(x)$.[1]

Let's compute the derivative of g using the limit definition:

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

But since $g(x) = xf(x)$ we have that:

$$= \lim_{h \to 0} \frac{(x+h)f(x+h) - xf(x)}{h}$$

Multiplying out, we get:

$$= \lim_{h \to 0} \frac{xf(x+h) + hf(x+h) - xf(x)}{h}$$

We know we're aiming to find $f(x) + xf'(x)$, so let's break the limit in two:

$$= \lim_{h \to 0} \frac{hf(x+h)}{h} + \lim_{h \to 0} \frac{xf(x+h) - xf(x)}{h}$$

Now we can simplify the limit on the left and factor an x in the limit on the right:

$$= \lim_{h \to 0} f(x+h) + x \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The limit on the left clearly is simply $f(x)$ and the one on the right yields $f'(x)$ hence:

$$= f(x) + xf'(x)$$

Which terminates our proof.

[1] Collision of notations happens all the time, it is your job to avoid it, just relabel things if needed and stay out of trouble.

Problem 2

Salted water containing 5 grams of salt per litre is being poured at a rate of 10 litres per hour into a tank that initially contains 10 litres of pure unsalted water. Give an expression for the function that gives the concentration of salt in the tank after t hours and then explain what will the concentration of salt in the tank be after a very long period of time.

Solution

To compute the concentration of salt in the tank, we simply compute how much salt and how much water there is in the tank after t hours.

How much salt is there in the tank? The tank is filled with water that has 5 grams of salt per litre and 10 litres are poured in per hour, hence there are 50 grams of salt poured in the tank every hour. If we denote by $s(t)$ the amount of salt in grams in the tank after t hours, we can clearly claim that $s(t) = 50t$ since we start with no salt in the water and pour 50 grams every hour.

Now, how much water is there in the tank? Let us denote by $w(t)$ the amount of water in litres in the tank after t hours. Since we started with 10 litres and are adding an extra 10 litres per hour, we can claim that $w(t) = 10t+10$.

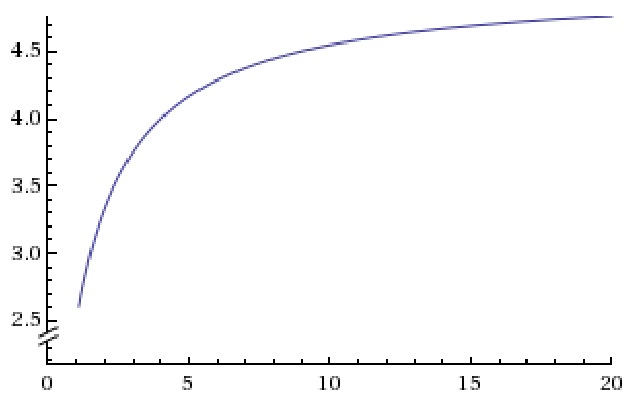
The concentration of salt in the tank being simply the ration of the amount of salt to the amount of water; if we denote by $c(t)$ the concentration of salt in the tank in grams per litres, we have that

$$c(t) = \frac{s(t)}{w(t)} = \frac{50t}{10t+10}$$

Now, what will happen after a very long time? We can rephrase this question by asking what will the limit of the function $c(t)$ be as t tends to infinity. Let's compute this limt:

$$\lim_{t\to+\infty} c(t) = \lim_{t\to+\infty} \frac{50t}{10t+10} = \lim_{t\to+\infty} \frac{t(50)}{t(10+\frac{10}{t})} = \lim_{t\to+\infty} \frac{50}{10+0} = 5$$

This means that the more time we wait, the closer the concentration of salt in the tank will get to 5 grams per litre. This makes a lot of sense, since we've started with some finite amount of water which gets dwarfed by the salted water that we're pouring in (and which has a concentration of 5 grams of salt per litre). Another way to see this is to say that if graphed the function $c(t)$ we would see a horizontal asymptote to the right at $y = 5$.



Problem 3

The silver currency of the Kingdom of Bonoria consists of glomeks, nindars and morms. Four golmeks are equal in value to seven nindars; and one glomek and one nindar together are worth thirty-three morms.

On my last visit to Bonoria, I entered a bank, handed the teller some glomeks and nindars and asked him to change them into morms. He told me that if I had twice as many glomeks, he could give me 120 morms; and if I had twice as many nindars he could give me 114 morms. How many morms did I get in the end?

Solution

What can be tricky here in setting your variables is making the difference between the value of say a glomek and then number of glomeks we have to trade at the bank. If you write something like *"denote by g the glomeks"* it is unclear who is who and this confusion can lead into being unable to solve the problem. So, let's use the following variables:

- $G =$ the value of a glomek.
- $N =$ the value of a nindar.
- $M =$ the value of a morm.
- $a =$ the number of glomeks to exchange.
- $b =$ the number of nindars to exchange.

Now what information do we have. The first part of the question tells us that:

$$\begin{cases} 4G = 7N \\ G + N = 33M \end{cases}$$

The second part of the problem is slightly more complicated to write. If we have a glomeks, then their value is aG (the number of glomeks multiplied by the value of a glomek). Also, the teller tells us about what if we exchanged different amount of glomeks and nindars than what we have. Using our variables, the teller's information can be written as:

$$\begin{cases} 2aG + bN = 120M \\ aG + 2bN = 114M \end{cases}$$

Now is a good time to stop for a while and ask ourselves: *"What are we trying to find?"*. We're supposed to find the number of morms that the teller gave back in exchange for the glomeks and nindars. If we denote this number by c, it means this number satisfies the equation:

$$aG + bN = cM$$

But looking at the equations that we got out of the teller's information shows that we almost have this already. Let's just add these two equations together, we obtain:

$$(2aG + bN) + (aG + 2bN) = 120M + 114M \quad \Longleftrightarrow \quad 3aG + 3bN = 234M$$

We can now simply divide by 3 on both sides and we have that:

$$aG + bN = 78M$$

And hence we received 78 morms in exchange of our a glomeks and b nindars. It is interesting to notice that in this problem, we haven't found the value of any of the variables a, b, G, N or M and we haven't used all the information that we had either.