MATH 312 – ASSIGNMENT 2 – WINTER 2012

1) As $n \in \mathbb{N}$, $Q_n = n! + 1 \geq 2$.

Thus $\exists$ a prime number p such that $p \mid Q_n$.

[This can either be proven directly or as a result of unique prime factorisation].

For $n = 1$, $Q_n = 2$ and $p = 2 > 1 = n$.

For $n \geq 2$, for integers $2 \leq j \leq n$, we see that $j \mid n! = [n(n-1)\cdots(j+1)j(j-1)\cdots 3\cdot 2\cdot 1]$, so $j \nmid (n!+1) = Q_n$. Thus, $p \neq j$ for any $2 \leq j \leq n$, which means that $p > n$.

5 marks

Suppose there are only finitely many primes and the largest prime is q. Clearly, $q \geq 2$.

Now consider the number $N = q! + 1 > 2$.

$\exists$ a prime p such that $p \mid N$. But from the first part of this question, we see that $p > q$. This contradicts the assumption that the largest prime is q. $\#$

Thus, there are infinitely many primes.

5 marks

[Total: 10 marks]

2) Suppose p is prime and $p > 3$.

Then $3 \nmid p$, so $p \equiv 1$ or $2 \pmod{3}$.

If $p \equiv 1 \pmod{3}$, then $p+2 \equiv 3 \equiv 0 \pmod{3}$.

So $3 \mid p+2$ and $p+2$ is not prime.

If $p \equiv 2 \pmod{3}$, then $p+4 \equiv 6 \equiv 0 \pmod{3}$

So $3 \mid p+4$ and $p+4$ is not prime.

Thus, p, $p+2$ and $p+4$ is not a prime triplet for $p \in \mathbb{Z}$, $p > 3$.

[Total: 10 marks]

3) Dirichlet's Theorem on Primes in Arithmetic Progressions states that for a & b relatively prime positive integers, the arithmetic progression $an+b$, $n=1,2,3,\ldots$ contains infinitely many primes.

Let $a=30$ and $b=7$.
As $\gcd(30,7)=1$, there are infinitely many primes of the form $30n+7$.

Let p be a prime of the form $p=30n+7$, $n \in \mathbb{N}$. If p was one of the primes in a pair of twin primes, then either

$$p-2 = (30n+7)-2 = 30n+5$$

or

$$p+2 = (30n+7)+2 = 30n+9$$

would be a prime. But these two numbers are obviously not prime as they are divisible by 5 and 3 respectively.
Thus, all of the (infinitely many) primes of the form $p=30n+7$ are not in a pair of twin primes.

[Total: 10 marks]

Note: Other correct choices of a & b above will also yield full credit.

4) $\cancel{50}$ $\cancel{51}$ $\cancel{52}$ 53 $\cancel{54}$ $\cancel{55}$ $\cancel{56}$ $\cancel{57}$ $\cancel{58}$ 59

$\cancel{60}$ 61 $\cancel{62}$ $\cancel{63}$ $\cancel{64}$ $\cancel{65}$ $\cancel{66}$ 67 $\cancel{68}$ $\cancel{69}$

$\cancel{70}$ 71 $\cancel{72}$ 73 $\cancel{74}$ $\cancel{75}$ $\cancel{76}$ $\cancel{77}$ $\cancel{78}$ 79

$\cancel{80}$ $\cancel{81}$ $\cancel{82}$ 83 $\cancel{84}$ $\cancel{85}$ $\cancel{86}$ $\cancel{87}$ $\cancel{88}$ 89

$\cancel{90}$ $\cancel{91}$ $\cancel{92}$ $\cancel{93}$ $\cancel{94}$ $\cancel{95}$ $\cancel{96}$ 97 $\cancel{98}$ $\cancel{99}$

$\cancel{100}$

Primes: 53, 59, 61, 67, 71, 73, 79, 83, 89, 97 [Total: 10 marks]

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5) Let $a, b \in \mathbb{Z}$ s.t. a is even & b is odd.

$b \neq 0 \Longrightarrow \gcd(a,b)$ always exists.

Suppose $2 \mid \gcd(a,b)$. Then $2 \mid b$. $\times\!\!\times$

So $2 \nmid \gcd(a,b)$. $\Longrightarrow \gcd(a,b)$ is odd. [Total: 10 marks]

6) $(a^2+b^2, a+b) = (a^2+b^2+(a+b)(a-b), a+b)$
$= (a^2+b^2+a^2-b^2, a+b)$
$= (2a^2, a+b)$. 3 marks

Note that $(a^2, a+b) = 1$.
[For if a prime p divides a^2, then $p \mid a$.
Then if $p \mid a+b$, we would have $p \mid b$ as well, which contradicts $(a,b)=1$.]

Thus, $(2a^2, a+b) = \begin{cases} 1 & \text{if } a \text{ is odd and } b \text{ even} \\ 1 & \text{if } a \text{ is even and } b \text{ odd} \\ 2 & \text{if } a \text{ and } b \text{ are both odd.} \end{cases}$

Note: a & b cannot both be even $\because (a,b)=1$. 7 marks

Note: If either a or b is 0, then the other is ± 1 since $(0,c) = |c|$ for any nonzero $c \in \mathbb{Z}$.

[Total: 10 marks]

7) The prime number theorem tells us that $\frac{x}{\log x}$ is a good approximation for $\pi(x)$ when x is large.

[Here, log refers to $\log_e$, the natural logarithm; and $\pi(x)$ refers to the number of primes not larger than x, where $x \in \mathbb{R}^+$]

We calculate that when $x = 1{,}000{,}000$, $\frac{x}{\log x} = 72{,}382.4$

and when $x = 10{,}000{,}000$, $\frac{x}{\log x} = 620{,}420.7$

This means that the $598\,709^{th}$ prime is likely to be between 1,000,000 and 10,000,000 and so is likely to have 7 digits.

[Total: 10 marks]

[Answers using other acceptable approximations for $\pi(x)$ will also be marked accordingly.]