1 Solutions to assignment 6, due June 23rd

Problem 10.2 We have that A and C are disjoint, since $A \subset \mathbb{R}_{>0}$ and $C \subset \mathbb{R}_{<0}$. Moreover, as each of these sets are denumerable[1], we have bijective functions $\mu_A : \mathbb{N} \to A$ and $\mu_C : \mathbb{N} \to C$.

Define then the function $\mu : \mathbb{N} \to A \cup C$ via

$$n \mapsto \begin{cases} \mu_A\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd} \\ \mu_C\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \end{cases}.$$

I claim that this function is bijective.

First, suppose that $\mu(n) = \mu(m)$. Then n and m are either both even or odd (since A, C are disjoint). So suppose that $m = 2k$, $n = 2\ell$. Then we have

$$\mu_C(k) = \mu(2k) = \mu(2\ell) = \mu_C(\ell).$$

But since μ_C is injective, it follows that $k = \ell$, and thus that $n = m$. The case that n, m are both odd is dealt with similarly.

I now claim that μ is surjective. Let $x \in A$. Then there is some $n \in \mathbb{N}$ with $\mu_A(n) = x$, since μ_A is surjective. However, this implies that $\mu(2n+1) = \mu_A(n) = x$. Similarly, let $y \in C$. Then there is some $m \in \mathbb{N}$ such that $\mu_C(m) = y$ (due again to surjectivity). Thus $\mu(2m) = \mu_C(m) = y$. Together these imply that μ is surjective as claimed, and thus that it is bijective.

Problem 10.3 Given

$$S = \left\{ x \in \mathbb{R} \,\middle|\, x = \frac{n^2 + \sqrt{2}}{n} \text{ for some } n \in \mathbb{N} \right\}$$

and $f : \mathbb{N} \to S$ given by $n \mapsto \frac{n^2+\sqrt{2}}{n}$.

(a) The elements $1 + \sqrt{2}$, $\frac{4+\sqrt{2}}{2}$, and $\frac{9+\sqrt{2}}{3}$ are all in S.

(b) The function f is injective, since if $f(n) = f(m)$ we would have

$$n + \frac{\sqrt{2}}{n} = m + \frac{\sqrt{2}}{m}$$

or, rearranging,

$$n - m = \sqrt{2}\left(\frac{n-m}{nm}\right).$$

If we had that $n \neq m$, then we could rewrite this last eqution as $\sqrt{2} = nm$ for these two integers n, m, which is clearly absurd.

(c) This is onto; let $x \in S$. Since $x = \frac{n^2+\sqrt{2}}{n}$ for some integer n, it follows that $f(n) = x$.

(d) Since the function f is bijective, it follows that S is denumberable.

[1] Why is C denumerable?

Problem 10.4 The function f is defined by

$$f(n) = \frac{1 + (-1)^n(2n-1)}{4}$$

We first show that it is injective.

Suppose first that $f(n) = f(m)$. Then this is equivalent (after some simplification) $(-1)^n(2n-1) = (-1)^m(2m-1)$. This implies that n, m are both of the same parity (i.e. both even or both odd), and so we can cancel the $(-1)^n$ and $(-1)^m$. Thus we find the $f(n) = f(m)$ if and only if $2n - 1 = 2m - 1$ which is if and only if $n = m$, and so f is injective.

So to show surjectivity, note that for $n = 2k$, we find that $f(n) = \frac{1+(4k-1)}{4} = k$, and for $n = 2k+1$ that $f(n) = \frac{1-(4k+1)}{4} = -k$. Thus it is clear that for any positive integer k, that $f(2k) = k$, and for any non-negative integer ℓ (i.e. $\ell \leq 0$), that $f(-2\ell + 1) = \ell$, and so the function is surjective.

Problem 10.6 We can say that either A is finite, or A is denumerable itself. This is since $A = \bigcup_{b \in B} f^{-1}(b)$. Since each $f^{-1}(b)$ is either empty or a one element set, and since B is denumerable, it follows that A is the (at most countable) union of single points. Thus A must be denumerable.

Problem 10.8 We will first prove the following lemma.

Lemma 1.1 *Let $A = \bigcup_{i \in \mathbb{N}} A_i$ with $A_i \cap A_j = \emptyset$ and each A_i finite. Then A is countable.*

Proof Write each set A_i as $A_i = \{a_{i,1}, a_{i,2}, \ldots a_{i,c_i}\}$ with $|A_i| = c_i$. Then we can write A as the list

$$A = \{a_{1,1}, a_{1,2}, \ldots, a_{1,c_1}, a_{2,1}, a_{2,2}, \ldots a_{2,c_2}, a_{3,1}, \ldots\}$$

It is clear that this list is equal to all of A, and hence by the lemma from class, that A is countable.

Q.E.D.

We will show that $S = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid i \leq j\}$ is a countable union of finite sets.

For each $j \in \mathbb{N}$, let $S_j = \{(i,j) \mid i \leq j\}$. Then clearly it is the case that $S = \bigcup_{j \in \mathbb{N}} S_j$. Moreover, we have that $|S_j| = j$ for each j (since they consist of the elements $\{(1,j), (2,j), \ldots, (j-1,j), (j,j)\}$). Thus S is the countable union of finite sets as in the lemma, and so it is countable.

Problem 10.10 We consider the chain of maps

$$\mathbb{N} \times \mathbb{N} \xrightarrow{g} \mathbb{Z} \times \mathbb{Z} \xrightarrow{h} \mathcal{G}$$

given by $g(n,m) = \big(f(n), f(m)\big)$ (where f is given in problem 10.4), and $h(a,b) = a+bi$. These are both bijections, so it remains to show that $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$.

Consider $N_k = \{(i,j) \in \mathbb{N} \times \mathbb{N} | i + j = k\}$. For example, $N_4 = \{(3,1), (2,2), (1,3)\}$. It is clear that $\bigcup_{k \in \mathbb{N}} N_k = \mathbb{N}$, and moreover that $|N_k| = k - 1$ and so is finite. Thus $\mathbb{N} \times \mathbb{N}$ is countable as desired, and so we have that $\mathcal{G}$ is as well.

Problem 10.14 (a) We show that $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{2\}$ is bijective by showing that it has an inverse. Consider the function $g : \mathbb{R} - \{2\} \to \mathbb{R} - \{1\}$ given by $g(x) = \frac{x}{x-2}$. Then

$$f\big(g(x)\big) = \frac{2\frac{x}{x-2}}{\frac{x}{x-2} - 1} = \frac{2x}{x - (x-2)} = \frac{2x}{2} = x$$

and

$$g\big(f(x)\big) = \frac{\frac{2x}{x-1}}{\frac{2x}{x-1} - 2} = \frac{2x}{2x - 2(x-1)} = \frac{2x}{2} = x.$$

Since these functions are inverses of each other, it follows that f is bijective.

(b) As there is a bijective function from one to the other, it follows by definition that $|\mathbb{R} - \{1\}| = |\mathbb{R} - \{2\}|$.

Problem 10.15 Suppose to the contrary that the set $\mathfrak{I}$ of all irrational numbers is countable. Then it would follow that $\mathbb{R} = \mathbb{Q} \cup \mathfrak{I}$ is the union of two countable sets, and so it would also be countable. As $\mathbb{R}$ is uncountable, this is a contradiction.

Problem 10.18 (a) The function $f : (0,1) \to (0,2)$ given by $x \mapsto 2x$ is injective, since if $f(x) = f(y)$ we would have $2x = 2y \iff x = y$. It is surjective, since for any $r \in (0,2)$ we have that $\frac{r}{2} \in (0,1)$. Since $f(\frac{r}{2}) = r$, it follows that f is also surjective.

(b) These sets have the same cardinality since there is a bijective function between the two of them.

(c) We will exhibit a bijective function $g : (0,1) \to (a,b)$.
Define g by

$$x \mapsto (b-a)x + a$$

It is easy to see (See part (a)) that this is injective, and similarly that it is surjective. So $|(0,1)| = |(a,b)|$.

Problem 10.19 (a) This is false. $\mathcal{P}(\mathbb{R})$ is uncountable, but $|\mathcal{P}(\mathbb{R})| > |\mathbb{R}|$.

(b) This is false, since $\mathbb{Q}$ is countable, as it would imply that $\mathbb{R}$ is also countable.

(c) This is true. Subsets of denumberable sets are either finite or denumberable themselves. Since B has a denumerable subset, it cannot be finite, and so it is denumerable.

(d) This is true. We can define $f : \mathbb{N} \to S$ by $n \mapsto \frac{\sqrt{2}}{n}$. This is a bijection.

(e) This is true. Consider the set S from (d); as $\sqrt{2}$ is irrational, so is $\frac{\sqrt{2}}{n}$. Thus S is a set of irrational numbers, and as seen before, it is denumerable.

(f) This is false. $\mathbb{R}$ is uncountable, and so cannot be a subset of a denumerable set.

(g) This is false. Consider the inclusion of the set $A = \{1\}$ into the set $B = \{1,2\}$ (i.e the function which maps $1 \mapsto 1$). This function is injective, but $|A| = 1$ and $|B| = 2$.

Problem 10.22 (a) The set B in this instance is $B = \{a,c\} = A_d$ (since $A_a = \emptyset$, and $a \notin \emptyset$, etc.)

(b) This set illustrates that none of a, b, or c map under the function g to B i.e. that B is *not* in the image of g, and so $|\mathcal{P}(A)| > |A|$.