ASSIGNMENT 5

DUE DATE: NOV 29, 2011

i) Let ζ be a primitive pth root of unity, where p is a prime divisor of 2^n+1 (for some $n \geq 1$). Show that $E = \mathbb{Q}(\zeta)$ is a splitting field of the quaternion algebra $\frac{(-1,-1)}{\mathbb{Q}}$. (Hint: Verify the identity $\prod(1+\zeta^{2^k}) = -\zeta^{2^n}$, where the product varies from $k=0$ to $n-1$, and note that the LHS is a sum of two squares in E.)

ii) Recall that a quaternion algebra $A = \frac{(a,b)}{F}$ over a field F of characteristic not 2 is the central simple algebra with F-basis $\{1, i, j, k\}$ such that $i^2 = a$ and $j^2 = b$, $ij = -ji$. The map $x \to \bar{x}$ which sends $(a+bi+cj+dk)$ to $(a-bi-cj-dk)$ defines a symmetric bilinear from A to F whose associated quadratic form is $x \mapsto B(x,x) = N(x) := x\bar{x}$. This is called the norm form on A. Prove that this norm form is isometric to $< 1, -a, -b, ab >$.

iii) Prove that the quaternion algebras $A = \frac{(a,b)}{F}$, $\frac{(a',b')}{F}$ are isomorphic iff their associated norm forms are isometric.

iv) Show that the quaternion alegbras $(-1,-1)$ and $(-2,-3)$ are isomorphic over $\mathbb{Q}$.. For an odd prime p show that $(-2,p)$ splits iff $p \equiv 1$ or 3 mod8.

v) Let $A = \frac{(a,b)}{F}$) and let $K = F(\sqrt{c})$. Prove that A splits over K iff the norm form of A is isometric to $< 1, -c, -d, cd >$ for some d in $F^\times$.