1 Solutions to assignment 4, due June 9th

Problem 1.3 (a) The cardinality of $\{1, 2, 3, 4, 5\}$ is 5.

(b) The cardinality of $\{0, 2, 4, \ldots, 20\}$ is 11 (don't forget that we start counting at 0!).

(c) The cardinality of $\{25, 26, 27, \ldots, 75\}$ is 51 (Similar here. Don't forget the end points!).

(d) The cardinality of $\big\{\{1, 2\}, \{1, 2, 3, 4\}\big\}$ is 2.

(e) The cardinality of $\{\emptyset\}$ is 1.

(f) The cardinality of $\big\{2, \{2, 3, 4\}\big\}$ is also 2.

Problem 1.5 (a) If $A = \{-1, -2, -3, \ldots\}$, then we can write

$$A = \{x \in \mathbb{Z} \mid x < 0\}.$$

(b) If $B = \{-3, -2, \ldots, 3\}$ then we can also write

$$B = \{x \in \mathbb{Z} \mid |x| < 4\}.$$

(c) If $C = \{-2, -1, 1, 2\}$ then we can write this as

$$C = \{x \in \mathbb{Z} \mid |x| \leq 2 \text{ and } x \neq 0\}.$$

Note that each of these can be written in many other ways.

Problem 1.6 (a) We have $A = \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$.

(b) We have $B = \{\ldots, -8, -4, 0, 4, 8, 12, \ldots\}$.

(c) We have $C = \{\ldots, -5, -2, 1, 4, 7, 10, \ldots\}$.

Problem 1.8 (a) If we let $A = B = \emptyset$, and $C = \{0\}$, then we have $A \subseteq B \subsetneq C$.

(b) If we choose $A = \emptyset$, $B = \{\emptyset\}$, and $C = \big\{\{\emptyset\}\big\}$, then we have $A \in B$ and $B \in C$, but $A \notin C$.

(c) If we choose this time $A = \emptyset$, and $B = C = \{\emptyset\}$ then $A \in B$ and $A \subsetneq C$.

Note that examples that are simple are easy to understand!

Problem 1.10 If we work these out, we see that $A = B = D = E = \{-1, 0, 1\}$, but that $C = \{0, 1\}$.

Problem 1.13 We have that $\mathcal{P}\big(\{0, \{0\}\}\big) = \Big\{\emptyset, \{0\}, \big\{\{0\}\big\}, \big\{0, \{0\}\big\}\Big\}$. Note that as we expected, $|\mathcal{P}(A)| = 2^{|A|}$.

Problem 1.14 We find that $\mathcal{P}(\{1\}) = \big\{\emptyset, \{1\}\big\}$, and so $\mathcal{P}\big(\mathcal{P}(A)\big) = \Big\{\emptyset, \{\emptyset\}, \big\{\{1\}\big\}, \big\{\emptyset, \{1\}\big\}\Big\}$. Thus $|\mathcal{P}\big(\mathcal{P}(\{1\})\big)| = 4$.

Problem 1.15 We expect that $|\mathcal{P}(A)| = 8$, as $|A| = 3$. Writing out all possible subsets, we find

$$\mathcal{P}(A) = \Big\{\emptyset, \{0\}, \{\emptyset\}, \big\{\{\emptyset\}\big\}, \{0, \emptyset\}, \big\{0, \{\emptyset\}\big\}, \big\{\emptyset, \{\emptyset\}\big\}, \big\{0, \emptyset, \{\emptyset\}\big\}\Big\}$$

which verifies our expectation.

Problem 1.16 Recall that if $S \subset \mathcal{P}(A)$, then S is a set of subsets of A.

(a) We need any subset of $\mathcal{P}(\mathbb{N})$, so we consider $S = \emptyset$. Simple!
(b) If $S \in \mathcal{P}(\mathbb{N})$, then $S \subseteq \mathbb{N}$. So consider again $S = \emptyset$.
(c) We can't resort to such trickery this time. So we choose this time $S = \big\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\big\}$.
(d) Similarly, we choose $S = \{1, 2, 3, 4, 5\}$.

Problem 1.18 (a) Consider $A = \emptyset$, $B = \{\emptyset\}$, and $C = \{1\}$. Then all conditions are satisfied.
(b) Consider this time $A = C = \big\{\{0\}, 0\big\}$, and $B = \{0\}$. Then $B \in A$, $B \subsetneq C$, and $A \cap C \neq \emptyset$.
(c) Consider $A = \{\emptyset\}$, and $B = C = \big\{\{\emptyset\}\big\}$. Then $A \in B$, $B \subseteq C$, but $A \not\subseteq C$.

As stated before, remember, simple examples are better!

Problem 1.19 We choose $A = \{0, 1\}$, $B = \{0, 2\}$, and $C = \{1, 2\}$. Then we have that $B - A = \{2\} = C - A$, but $B \neq C$.

Problem 1.24 We have that $A = \Big\{\emptyset, \{\emptyset\}, \big\{\{\emptyset\}\big\}\Big\}$.

(a) We have that $\emptyset \in A$, and that $\{\emptyset\} \in A$.
(b) $|A| = 3$.
(c) We have $\emptyset \subseteq A$ (since the empty set is a subset of every set), $\{\emptyset\} \subseteq A$ (Since $\emptyset \in A$), and also that $\big\{\emptyset, \{\emptyset\}\big\} \subseteq A$, since both of the elements are elements of A.
(d) $\emptyset \cap A = \emptyset$. This is true regardless of what A is.
(e) $\{\emptyset\} \cap A = \{\emptyset\}$.
(f) $\big\{\emptyset, \{\emptyset\}\big\} \cap A = \big\{\emptyset, \{\emptyset\}\big\}$, since this is a subset of A (and so its intersection with A is itself).
(g) $\emptyset \cup A = A$. Again, this is true regardless of what A is.
(h) $\{\emptyset\} \cup A = A$, since the former set is a subset of A.
(i) $\big\{\emptyset, \{\emptyset\}\big\} \cup A = A$ for the same reason.

Problem 1.27 Given $A = \{1, 2, 5\}$, $B = \{0, 2, 4\}$, and $C = \{2, 3, 4\}$, and also $S = \{A, B, C\}$ (so that S is the set whose elements are the sets A, B, C), we have that

$$\bigcup_{X \in S} X = A \cup B \cup C = \{1, 2, 3, 4, 5\}$$

and that

$$\bigcap_{X \in S} = A \cap B \cap C = \{2\}.$$

Problem 1.41 If $A = \{x, y, z\}$ and $B = \{x, y\}$, then

$$A \times B = \{(x,x),(x,y),(y,x),(y,y),(z,x),(z,y)\}.$$

This has $6 = 3 \cdot 2$ elements, as we would expect.

Problem 1.42 If $A = \Big\{1, \{1\}, \{\{1\}\}\Big\}$, then

$$\begin{aligned} A \times A = \Big\{&(1,1),(1,\{1\}),\Big(1,\{\{1\}\}\Big),\\ &(\{1\},1),(\{1\},\{1\}),\Big(\{1\},\{\{1\}\}\Big),\\ &\Big(\{\{1\}\},1\Big),\Big(\{\{1\}\},\{1\}\Big),\Big(\{\{1\}\},\{\{1\}\}\Big)\Big\} \end{aligned}$$

which has $9 = 3 \cdot 3$ elements, as we expect.

Problem 1.44 If $A = \big\{\emptyset, \{\emptyset\}\big\}$, then $\mathcal{P}(A) = \Big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\Big\}$. Thus

$$\begin{aligned} A \times \mathcal{P}(A) = \Big\{&(\emptyset,\emptyset),(\emptyset,\{\emptyset\}),\big(\emptyset,\{\{\emptyset\}\}\big),\big(\emptyset,\{\emptyset,\{\emptyset\}\}\big),\\ &(\{\emptyset\},\emptyset),(\{\emptyset\},\{\emptyset\}),\big(\{\emptyset\},\{\{\emptyset\}\}\big),\big(\{\emptyset\},\{\emptyset,\{\emptyset\}\}\big)\Big\} \end{aligned}$$

As an aside, isn't parsing brackets fun?

Problem 1.46 The graph of the circle $x^2 + y^2 = 4$ as a subset of $\mathbb{R} \times \mathbb{R}$ is

$$C = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 4\}.$$

De Morgan's Laws De Morgan's Laws are the following:

Theorem 1.1 *Let A, B be sets (contained in some larger universe U, so that we might make sense of their complements). Then*

$$(A \cup B)^c = A^c \cap B^c \qquad \text{and} \qquad (A \cap B)^c = A^c \cup B^c.$$

Proof We will first show that $(A \cup B)^c \subseteq A^c \cap B^c$. So suppose that $x \in (A \cup B)^c$. As $A \cup B$ is the collection of all elements that are either in A or in B (or possibly both), we must have that x is in neither A nor B. That is, $x \in A^c$ and $x \in B^c$. But this means that $x \in A^c \cap B^c$.

Next, we show that $A^c \cap B^c \subseteq (A \cup B)^c$. Suppose that $x \in A^c \cap B^c$. Then $x \notin A$ and $x \notin B$. But if this is true, then x cannot be in the union of A and B, and so we have that $x \in (A \cup B)^c$.

Combining these two facts, we find that $(A \cup B)^c = A^c \cap B^c$.

The proof of the other equality is similar.

Q.E.D.