Aim: To prove the following theorem:

Theorem: If R is a commutative ring, the Steinberg symbol map $R^\times \times R^\times \xrightarrow{\{\,,\,\}} K_2(R)$ satisfies the additional relations (a) $\{u,-u\}=1$ for $u \in R^\times$ (b) $\{u,1-u\}=1$ for $u \in R^\times$, $1-u \in R^\times$.

Lemma 1: Let R be a ring and $u \in R^\times$. Consider the elements

$$w_{ij}(u) = x_{ij}(u)\, x_{ji}(-u^{-1})\, x_{ij}(u)$$

$$h_{ij}(u) = w_{ij}(u)\, w_{ij}(-1).$$

Then $w_{ij}(u)^{-1} = w_{ij}(-u)$, $w_{ij}(u) = w_{ji}(-u^{-1})$, $h_{ij}(1)=1$.
In addition, if $u, v \in R^\times$ and $i \neq j$, $k \neq l$, then

$$w_{kl}(u)\, w_{ij}(v)\, w_{kl}(u)^{-1} = \begin{cases} w_{ij}(-v), & i,j,k,l \text{ all distinct} \\ w_{lj}(-u^{-1}v), & k=i;\ i,j,l \text{ distinct} \\ w_{il}(-vu), & k=j;\ i,j,l \text{ distinct} \\ w_{ji}(-u^{-1}vu^{-1}), & k=i,\ j=l. \end{cases}$$

Pf:
$$\begin{aligned} w_{ij}(u)\, w_{ij}(-u) &= x_{ij}(u)\, x_{ji}(-u^{-1})\, x_{ij}(u)\, x_{ij}(-u)\, x_{ji}(u^{-1})\, x_{ij}(-u) \\ &= x_{ij}(u)\, x_{ji}(-u^{-1}) \cdot x_{ji}(u^{-1}) \cdot x_{ij}(u) \\ &= 1. \end{aligned}$$

Here we have used $x_{ij}(u)\cdot x_{ij}(-u) = x_{ij}(0) = 1$.

Also, $h_{ij}(1) = w_{ij}(1) \cdot w_{ij}(-1) = 1$.

Recall the Steinberg relations which will be tacitly used in the sequel:

$$[x_{ij}(s), x_{kl}(t)] = \begin{cases} 1 \text{ if } i\neq l,\ j\neq k \\ x_{il}(st) \text{ if } j=k \text{ and } i\neq l \\ x_{kj}(-ts) \text{ if } j\neq k,\ i=l. \end{cases}$$

Also since $[a,b]^{-1} = [b,a]$ in St, the relation
$$[x_{ij}(s), x_{ki}(t)] = x(-ts)_{kj} \quad \text{holds.}$$

The first relation

$$[w_{kl}(u), w_{ij}(v)] = 1 \quad \text{if } i,j,k,l \text{ are distinct}$$

follows from $[x_{ij}(a), x_{kl}(b)] = 1$ if i,j,k,l distinct or $i \neq l,\ j \neq k$.

Suppose $i = k$ and i, j, l are distinct. We have

$$w_{il}(u) \cdot x_{ij}(v)\, w_{il}(u)^{-1} = x_{il}(u) \cdot x_{li}(-u^{-1})\, x_{il}(u)\, x_{ij}(v)\ x_{il}(-u) \cdot x_{li}(u^{-1}) \cdot x_{il}(-u)$$

$$= x_{il}(u)\, x_{li}(-u^{-1}) \cdot x_{ij}(v)\, x_{li}(u^{-1})\ x_{il}(u)$$
(as $[x_{il}(u), x_{ij}(v)] = 1$

$$= x_{il}(u) \cdot x_{li}(-u^{-1}) \cdot x_{ij}(v)\, x_{li}(u^{-1}) \cdot x_{il}(-u)$$

$$= x_{il}(u)\, x_{lj}(-u^{-1}v)\ x_{ij}(v) \cdot x_{il}(-u)$$ (as $[x_{li}(-u^{-1}), x_{ij}(v)] = x_{lj}(-u^{-1}v)$)

$$= x_{ij}(-v)\, x_{lj}(-u^{-1}v) \cdot x_{ij}(v)$$ (as $[x_{ij}(-v), x_{lj}(-u^{-1}v)] = 1$ and $[x_{ij}(v), x_{il}(-u)] = 1$.

$$= x_{lj}(-u^{-1}v)$$
(as $[x_{ij}(-v), x_{lj}(-u^{-1}v)] = 1$.

$$\boxed{\therefore\ w_{il}(u) \cdot x_{ij}(v) \cdot w_{il}(u)^{-1} = x_{lj}(-u^{-1}v).} \qquad (1)$$

Similarly one shows that

$$\boxed{w_{il}(u)\ x_{ji}(-v^{-1})\, w_{il}(u)^{-1} = x_{jl}(v^{-1}u)} \qquad (2).$$

We use these identities below.

Consider

$$w_{il}(u)\, w_{ij}(v)\, w_{il}(u)^{-1}$$

$$= w_{il}(u)\, x_{ij}(v)\, x_{ji}(-v^{-1}) \cdot x_{ij}(v) \cdot w_{il}(u)^{-1} \quad \text{(by defn)}$$

$$\text{(by (1))} = x_{lj}(-u^{-1}v) \cdot w_{il}(u) \cdot x_{ji}(-v^{-1}) \cdot x_{ij}(v) \cdot w_{il}(u)^{-1}$$

$$\text{(by (2))} = x_{lj}(-u^{-1}v)\, x_{jl}(v^{-1}u) \cdot w_{il}(u) \cdot x_{ij}(v) \cdot w_{il}(u)^{-1}$$

$$\text{(by (1))} = x_{lj}(-u^{-1}v) \cdot x_{jl}(v^{-1}u) \cdot x_{lj}(-u^{-1}v)$$

$$= w_{lj}(-u^{-1}v). \quad \text{(2}^{\text{nd}}\text{ relation)}$$

Similarly, we have

$$w_{il}(u)\, w_{ji}(v)\, w_{il}(u)^{-1} = w_{jl}(-vu) \quad \text{(3}^{\text{rd}}\text{ relation)}.$$

Let l now be distinct from i and j. By what we have shown,

$$w_{ij}(v) = w_{il}(1)\, w_{lj}(v)\, w_{il}(-1).$$

Hence,

$$w_{ij}(u)\, w_{ij}(v)\, w_{ij}(u)^{-1} = w_{ij}(u)\left(w_{il}(1)\, w_{lj}(v)\, w_{il}(-1)\right)\left(w_{ij}(u)\right)^{-1}$$

Changing l to j in (1) and (2) gives:

$$w_{ij}(u) \cdot w_{il}(v) \cdot w_{ij}(u)^{-1} = w_{jl}(-u^{-1}v) \quad (1)'$$

$$w_{ij}(u)\, w_{ji}(v)\, w_{ij}(u)^{-1} = w_{lj}(-vu) \quad (2)'$$

To prove $w_{ij}(u) = w_{ji}(-u^{-1})$, we first prove

Consider

$$w_{il}(u)\, w_{ij}(v)\, w_{il}(u)^{-1}$$

$$= w_{il}(u)\, x_{ij}(v)\, x_{ji}(-v^{-1}) . x_{ij}(v) . w_{il}(u)^{-1} \quad \text{(by defn)}$$

$$\text{(by (1))} = x_{lj}(-u^{-1}v) . w_{il}(u) . x_{ji}(-v^{-1}) . x_{ij}(v) . w_{il}(u)^{-1}$$

$$\text{(by (2))} = x_{lj}(-u^{-1}v)\, x_{jl}(v^{-1}u) . w_{il}(u) . x_{ij}(v) . w_{il}(u)^{-1}$$

$$\text{(by (1))} = x_{lj}(-u^{-1}v) . x_{jl}(v^{-1}u) . x_{lj}(-u^{-1}v)$$

$$= w_{lj}(-u^{-1}v) . \quad (2^{nd} \text{ relation})$$

Similarly, we have

$$w_{il}(u)\, w_{ji}(v)\, w_{il}(u)^{-1} = w_{jl}(-vu) \quad (3^{rd} \text{ relation}).$$

Let l now be distinct from i and j. By what we have shown,

$$w_{ij}(v) = w_{il}(1)\, w_{lj}(v)\, w_{il}(-1).$$

Hence,

$$w_{ij}(u)\, w_{ij}(v)\, w_{ij}(u)^{-1} = w_{ij}(u)\left(w_{il}(1) w_{lj}(v) w_{il}(-1)\right)\left(w_{ij}(u)\right)^{-1}$$

Changing l to j in (1) and (2) gives:

$$w_{ij}(u) . w_{il}(v) . w_{ij}(u)^{-1} = w_{jl}(-u^{-1}v) \qquad (1)'$$

$$w_{ij}(u)\, w_{ji}(v)\, w_{ij}(u)^{-1} = w_{lj}(-vu) \qquad (2)'$$

To prove $w_{ij}(u) = w_{ji}(-u^{-1})$, we first prove

(*) $w_{ij}(u)\, w_{lj}(v)\, w_{ij}(u)^{-1}\, w_{ij}(u)\, w_{il}(-1)\, w_{ij}(u)^{-1} = w_{li}(vu^{-1})\, w_{jl}(u^{-1})$

Now $(2)' \Rightarrow w_{ij}(-u) \cdot w_{lj}(-vu) \cdot w_{ij}(u) = w_{li}(vu^{-1})$,

hence we get

$$w_{ij}(u)^{-1}\, w_{lj}(-v'u) \cdot w_{ij}(u) = w_{li}(-v') \quad (v' = -vu^{-1})$$
$$w_{ij}(-u)\, w_{lj}(vu^{-1}u) \cdot w_{ij}(u) = w_{ij}(-u) \cdot w_{lj}(v) \cdot w_{ij}(u) = w_{li}(vu^{-1})$$

Hence LHS of (*) $= w_{li}(vu^{-1}) \cdot w_{ij}(u)\, w_{il}(-1)\, w_{ij}(u)^{-1}$.

To establish (*), it therefore suffices to show

(**) $w_{ij}(u)\, w_{il}(-1)\, w_{ij}(u)^{-1} = w_{jl}(u^{-1})$

LHS of (**) $= w_{jl}(u^{-1})$; this follows on putting $u = u$ and $v = -1$ in ($1'$), viz. in

$w_{ij}(u)\, w_{il}(-1)\, w_{ij}(u)^{-1} = w_{jl}(u^{-1})$. Substituting in (*), we get LHS = RHS of (*).

Finally, taking $u = v$ in $w_{ij}(u)\, w_{ij}(v)\, w_{ij}(u)^{-1} = w_{ji}(-u^{-1}vu^{-1})$, we get

$$\boxed{w_{ij}(u) = w_{ji}(-u^{-1}).} \quad \text{——— } (3)$$

__Theorem__: For any commutative ring R, we have (a) $\{u, -u\} = 1$ for $u \in R^\times$ (b) $\{u, 1-u\} = 1$ for $u \in R^\times$, $(1-u) \in R^\times$.

pf. Recall that if $u, v \in R^\times$, then

$$h_{12}(uv) = h_{12}(u)\, h_{12}(v)\, \{u,v\}^{-1}.$$

Hence to show (a), it suffices to show that

$$h_{12}(u)\, h_{12}(-u) = h_{12}(-u^2).$$

$$\begin{aligned} \text{LHS} &= \omega_{12}(u)\,\omega_{12}(-1) \cdot \omega_{12}(-u) \cdot \omega_{12}(-1) \\ &= \omega_{12}(u^{-2})\,\omega_{12}(-1) \\ &= \omega_{12}(-u^2)\,\omega_{12}(-1) = h_{12}(-u^2). \end{aligned}$$

To show (b) we need to prove that

$$h_{12}(u)\, h_{12}(1-u) = h_{12}(u-u^2).$$

LHS

$= \omega_{12}(u)\,\omega_{12}(-1)\,\omega_{12}(1-u)\,\omega_{12}(-1)$

$= \omega_{12}(u)\,\omega_{21}(1)\,\omega_{12}(1-u)\,\omega_{12}(-1)$ (by ③)

$= \omega_{12}(u)\, x_{21}(1)\, x_{12}(-1)\, x_{21}(1)\, \omega_{12}(1-u)\, \omega_{12}(-1)$ (by defn. of w_{ij})

$= (\omega_{12}(u)\, x_{21}(1)\, \omega_{12}(-u)) \cdot \omega_{12}(u)\, x_{12}(-1) \cdot \omega_{12}(1-u) \cdot \omega_{12}(u-1) \cdot$
$x_{21}(1) \cdot \omega_{12}(1-u) \cdot \omega_{12}(-1)$

$= x_{12}(-u^2) \cdot \omega_{12}(u) \cdot x_{12}(-1)\, \omega_{12}(1-u)\, x_{12}(-(1-u)^2) \cdot \omega_{12}(-1)$

(by defn. of $\omega_{12}(u)$)

$= x_{12}(-u^2) \cdot x_{12}(u) \cdot x_{21}(-u^{-1})\, x_{12}(u)\, x_{12}(-1)\, \omega_{12}(1-u) \cdot x_{12}(-(1-u)^2) \cdot \omega_{12}(-1)$

$= x_{12}(u-u^2) \cdot x_{21}(-u^{-1}) \cdot x_{12}(u-1) \cdot \omega_{12}(1-u) \cdot x_{12}(-(1-u)^2) \cdot \omega_{12}(-1)$

(by defn. of $\omega_{12}(1-u)$)

$= x_{12}(u-u^2)\, x_{21}(-u^{-1}) \cdot x_{12}(u-1) \cdot x_{12}(1-u)\, x_{21}(-(1-u)^{-1}) \cdot x_{12}(1-u)$
$\cdot\, x_{12}(-(1-u)^2) \cdot \omega_{12}(-1)$

$= x_{12}(u-u^2)\, x_{21}(-u^{-1}) \cdot x_{21}(-(1-u)^{-1}) \cdot x_{12}(u-u^2) \cdot \omega_{12}(-1)$

$= \omega_{12}(u(1-u)) \cdot \omega_{12}(-1) = h_{12}(u-u^2) = \text{RHS}.$

$\underline{\text{pf}}$. Recall that if $u, v \in R^{\times}$, then

$$h_{12}(uv) = h_{12}(u)\, h_{12}(v)\, \{u,v\}^{-1}.$$

Hence to show (a), it suffices to show that

$$h_{12}(u)\, h_{12}(-u) = h_{12}(-u^2).$$

$$\begin{aligned} \text{LHS} &= \omega_{12}(u)\,\omega_{12}(-1).\,\omega_{12}(-u).\,\omega_{12}(-1) \\ &= \omega_{12}(u^{-2})\,\omega_{12}(-1) \\ &= \omega_{12}(-u^2)\,\omega_{12}(-1) = h_{12}(-u^2). \end{aligned}$$

To show (b) we need to prove that

$$h_{12}(u)\, h_{12}(1-u) = h_{12}(u-u^2).$$

LHS

$$\begin{aligned} &= \omega_{12}(u)\,\omega_{12}(-1)\,\omega_{12}(1-u)\,\omega_{12}(-1) \\ &= \omega_{12}(u)\,\omega_{21}(1)\,\omega_{12}(1-u)\,\omega_{12}(-1) \quad (\text{by } 3) \\ &= \omega_{12}(u)\,x_{21}(1)\,x_{12}(-1)\,x_{21}(1)\,\omega_{12}(1-u)\,\omega_{12}(-1) \quad (\text{by defn. of } \omega_{ij}) \\ &= (\omega_{12}(u)\,x_{21}(1)\,\omega_{12}(-u)).\,\omega_{12}(u)\,x_{12}(-1).\,\omega_{12}(1-u).\,\omega_{12}(u-1). \\ &\qquad x_{21}(1).\,\omega_{12}(1-u).\,\omega_{12}(-1) \\ &= x_{12}(-u^2).\,\omega_{12}(u).\,x_{12}(-1)\,\omega_{12}(1-u)\,x_{12}(-(1-u)^2).\,\omega_{12}(-1) \end{aligned}$$

(by defn. of $\omega_{12}(u)$)

$$\begin{aligned} &= x_{12}(-u^2).\,x_{12}(u).\,x_{21}(-u^{-1})\,x_{12}(u)\,x_{12}(-1)\,\omega_{12}(1-u).\,x_{12}(-(1-u)^2).\,\omega_{12}(-1) \\ &= x_{12}(u-u^2).\,x_{21}(-u^{-1}).\,x_{12}(u-1).\,\omega_{12}(1-u).\,x_{12}(-(1-u)^2).\,\omega_{12}(-1) \end{aligned}$$

(by defn. of $\omega_{12}(1-u)$)

$$\begin{aligned} &= x_{12}(u-u^2)\,x_{21}(-u^{-1}).\,x_{12}(u-1).\,x_{12}(1-u)\,x_{21}(-(1-u)^{-1}).\,x_{12}(1-u) \\ &\qquad .\,x_{12}(-(1-u)^2).\,\omega_{12}(-1) \\ &= x_{12}(u-u^2)\,x_{21}(-u^{-1}).\,x_{21}(-(1-u)^{-1}).\,x_{12}(u-u^2).\,\omega_{12}(-1) \\ &= \omega_{12}(u(1-u)).\,\omega_{12}(-1) = h_{12}(u-u^2) = \text{RHS}. \end{aligned}$$