1 List of problems to consider

1. We say that two primes p, q are *twin primes* if $|p-q| = 2$. For example, $11, 13$ are twin primes, as are $29, 31$. How many primes less than 100 are twins? Less than 200? Are there infinitely many?

2. Prove or disprove: There do not exist three integers n, $n+2$, $n+4$, all of which are prime.

3. Prove or disprove: For every integer n, the integers $2n$ and $4n+3$ are relatively prime. [Hint: Remember that if $d \mid a$ and $d \mid b$, then $d \mid (sa + tb)$ for any integers s, t.]

4. Prove or disprove: For every integer n, the integers $2n+1$ and $3n+2$ are relatively prime. [Hint: Same as above.]

5. Prove or disprove: If p, q are primes with $p, q \geq 5$, then $p^2 - q^2$ is divisible by 24.

6. The greatest common divisor of a, b was defined previously as the... greatest divisor that a and b have in common. Similarly, the least common multiple is defined to be the smallest multiple of both a and b; for example, if $a = 4$ and $b = 6$, then $gcd(a, b) = 2$ and $lcm(a, b) = 12$.

   Prove or disprove: For every integers a, b, we have that

   $$lcm(a, b)gcd(a, b) = ab$$

   [Hint: Look at the fundamental theorem of arithmetic. Can you use that to figure out what $gcd(a, b)$ and $lcm(a, b)$ are?]

7. Show that there are no strictly positive integer solutions to the equation

   $$x^2 - y^2 = 1$$

   (Hint: Factor the left-hand side). Are there any solutions if we allow x, y to be negative or zero?

8. Find *all* integer solutions to the equation $x^2 - y^2 = 10$.

9. Prove or disprove that every even number greater than 2 can be written as a sum of two primes.

10. Show that there are no rational roots to the cubic equation $x^3 + x + 1 = 0$. (Hint: What would happen if you assumed there was, and cleared denominators? What can we say about the *parity* [evenness or oddness] of the result?)

11. We say that an integer a has an *inverse* mod b if there is some other integer c such that

    $$[a]_b[c]_b = [1]_b$$

    What conditions are there on a and b such that we can find such an inverse? [Hint: Look at the Euclidean algorithm]

12. We say that a number is *perfect* if the sum of its proper divisors is itself. For example, 6 and 28 are perfect, since $1+2+3 = 6$, and $1+2+4+7+14 = 28$. Prove the following: If $2^p - 1$ is prime, then $2^{p-1}(2^p - 1)$ is perfect. Is the converse true?