Relativity

Science is full of approximations. We treat atoms as little round ball bearings, and collections of atoms like a perfectly diffuse soup (complete with temperature, pressure and volume).
When mapping the orbits of the planets our first models treat them as single points of mass spread across the solar system and when calculating the energy of these planets, we claim that $E_k = \frac{1}{2}mv^2$.

But this is an approximation. A very very accurate approximation, but an approximation none the less.

According to our current understanding of physics, the actual equation for the total energy of a moving body is $E_{tot} = \frac{mc^2}{\sqrt{1-v^2/c^2}}$.
Most of the time relativity does not have a significant effect on our daily lives (although if you have used a GPS recently, then you will have been communicating with satellites with relativity programmed into their internal clocks), but in some cases it is important.

Question one)
For this question you are asked to derive the classical equation for kinetic energy of a moving body from the relativistic equation for kinetic energy.
We do this by assuming that light travels very very fast.

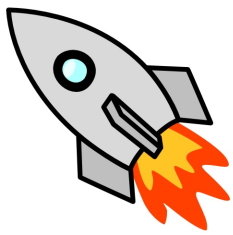
Given:

$$E_{tot} = \frac{mc^2}{\sqrt{1-v^2/c^2}} \quad , \quad E_{mass} = mc^2 \quad , \quad E_K = \frac{1}{2}mv^2$$

And assuming m and v are constant

Show that

$$\lim_{c\to\infty}\left[E_{tot} - E_{mass}\right] = E_K \ .$$



Please show ALL working.
If you decide to multiple by "one", write this step out:

$$\ldots \lim_{c\to\infty}\left[\frac{X}{Y}\right] = \lim_{c\to\infty}\left[\frac{X}{Y}\times\frac{Z}{Z}\right] = \lim_{c\to\infty}\left[\frac{XZ}{YZ}\right]\ldots$$

If you use a limit law, state which one, and comment on whether or not it is valid to use.
EG "Here I used the limit law of subtraction. This is valid because I was not attempting to subtract infinities"

If you get stuck, write out where you got to, and why you can not continue further.

HINT:
Do not be surprised to encounter the following trick somewhere in your working:

$$\frac{1-\sqrt{b}}{a}=\frac{1-\sqrt{b}}{a}\,\frac{1+\sqrt{b}}{1+\sqrt{b}}=\frac{1-b}{a\left(1+\sqrt{b}\right)}$$

Guide:

In this question we are asked "For this question you are asked to derive the classical equation for kinetic energy of a moving body from the relativistic equation for kinetic energy. We do this by assuming that light travels very very fast."

Then, the big box in the question has some equations in it.
In particular, this equation:

$$\lim_{c \to \infty} [E_{tot} - E_{mass}] = E_K$$

How do the two line up? Well, the thing on the right is the "classic equation for kinetic energy". And the thing on the left is the total relativistic energy minus the mass energy... perhaps this difference is the relativistic kinetic energy.

There's also a limit sign, with c--> inf.
This doesn't really make sense, because c is a number (not a variable) and we know what it is. BUT the question says "assume that light travels very very fast." Hmmm... I wonder why we are allowed to do this.

We are asked to SHOW that the equation works.

What does show mean?

Usually with show questions we write both sides of our equation out,and then rearrange them until we make them look the same.

$$\text{L.H.S.} = \lim_{c \to \infty} [E_{tot} - E_{mass}] \qquad \text{R.H.S.} = E_K$$

Okay, I don't think I can do anything with this without knowing more about the E values.
What do I know about the E values?
Everything! Excellent- if I know everything I can substitute in.

$$\text{L.H.S.} = \lim_{c \to \infty} [\frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2] \qquad \text{R.H.S.} = \frac{1}{2} m v^2$$

Hmmm... I want to show that these things are equal.
Now when doing a show question like this, I'll usually want to simplify both sides as much as possible.
At the moment the RHS is simple already, so I probably won't be able to change much there.
Maybe I can do something with the LHS...

Okay- so I'm trying to evaluate a limit.
The first thing I could do is try to sub in my limit value and see what happens...

$$\text{L.H.S.} = \lim_{c\to\infty}\left[\frac{mc^2}{\sqrt{1-v^2/c^2}} - m\,c^2\right]$$
$$\text{L.H.S.} = \left[\frac{m\,\infty^2}{\sqrt{1-v^2/\infty^2}} - m\,\infty^2\right]$$
$$\text{L.H.S.} = \left[\frac{m\infty^2}{\sqrt{1-0}} - m\infty^2\right]$$
$$\text{L.H.S.} = \left[\infty^2 - \infty^2\right]$$

When dealing with infinities pretty much all other numbers can be ignored. (hence why I drop “m”.)

Hmm does what I have here make sense. Is what I have a thing that's allowed to happen?

Let's look at our list of “Things that can happen with infinity”

$\infty + \infty = \infty$ ... Good, no problem
$n/\infty = 0$ ... Good, no problem
$\infty^0 = ???$ ... Bad, indeterminant form
$\infty^2 = \infty$ ... Good, no problem
$\frac{\infty}{\infty} = ???$ ... Bad, indeterminant form
$\infty - \infty = ???$ ... Bad, indeterminant form
$\infty \times 0 = ???$ ... Bad, indeterminant form

The thing we have is infinity minus infinity- this is a bad thing, because its impossible to tell what the result is.

So that means that we ARE NOT allowed to evaluate our limit straight away!

… drat.
Okay then,
what can we do?
What techniques do we have so that our equation is no longer like A-B?
…
Well, we can cancel things. Or regroup things... or take common factors, or combine fractions or expand brackets or... what else? Can you think of any other things we could do?

Let's go back to the last time our equation made sense, and apply one of those techniques.

Lets try taking a common factor out, and see if that helps.

$$\text{L.H.S.} = \lim_{c\to\infty}\left[\frac{mc^2}{\sqrt{1-v^2/c^2}} - m\,c^2\right]$$
$$\text{L.H.S.} = \lim_{c\to\infty} mc^2\left[\frac{1}{\sqrt{1-v^2/c^2}} - 1\right] = \infty \times \left(\frac{1}{\sqrt{1-0}} - 1\right) = \infty \times 0$$

Now we have a different sort of result... but it is still on our list of “bad sorts of infinities” so we still can't evaluate our limits just yet.

Does that mean our step was in the wrong direction?
Hmmm... well, one thing I know is that by the time all this is over, I need to have a v^2 somehow. I also know that I DON'T want a c^2. The only place I'm going to find a v^2 is inside that square root... which probably means I'll have to open it up somehow...
Probably I need to do something to the square root, so probably getting it by itself in the brackets is a good thing.

Let's trying playing with the bracket a little to see if we can get anything.

What kinds of tools can we apply here..
... cancel things. regroup things... expand brackets...common factors...combine fractions …

Hmmm.. well, the 2nd thing in the bracket isn't a fraction yet, but maybe it should be.

$$\text{L.H.S.} = \lim_{c\to\infty} mc^2\left[\frac{1}{\sqrt{1-v^2/c^2}} - \frac{\sqrt{1-v^2/c^2}}{\sqrt{1-v^2/c^2}}\right]$$
$$\text{L.H.S.} = \lim_{c\to\infty} mc^2\left[\frac{1-\sqrt{1-v^2/c^2}}{\sqrt{1-v^2/c^2}}\right]$$

Hmmm... did that help? I don't know. I still can't stick $c=\infty$ into my equation

I know the thing I'm meant to finish with needs to have a v in it... but at the moment the only v around is inside that square root.
I'd really like to get rid of the square root.
Have I seen any techniques for cancelling out nasty square roots from the tops of fractions?

Yes!
$$\frac{1-\sqrt{b}}{a} = \frac{1-\sqrt{b}}{a}\frac{1+\sqrt{b}}{1+\sqrt{b}} = \frac{1-b}{a(1+\sqrt{b})}$$ (hopefully you have seen something like this in class...)

Can we use that here?
Let's try it.
$$\text{L.H.S.} = \lim_{c\to\infty} mc^2\left[\frac{1-\sqrt{1-v^2/c^2}}{\sqrt{1-v^2/c^2}}\right] = \lim_{c\to\infty} mc^2\left[\frac{1-\sqrt{1-v^2/c^2}}{\sqrt{1-v^2/c^2}}\frac{1+\sqrt{1-v^2/c^2}}{1+\sqrt{1-v^2/c^2}}\right]$$
$$\text{L.H.S.} = \lim_{c\to\infty} mc^2\left[\frac{1^2-(1-v^2/c^2)}{\sqrt{1-v^2/c^2}(1+\sqrt{1-v^2/c^2})}\right] = \lim_{c\to\infty} mc^2\left[\frac{v^2/c^2}{\sqrt{1-v^2/c^2}(1+\sqrt{1-v^2/c^2})}\right]$$
The bottom of my equation is now a mess, but at least the tops okay. As an added bonus, I've now got a v^2 floating around.
We can cancel out c^2 here, and take lots of stuff outside our limit signs..
$$\text{L.H.S.} = mv^2\lim_{c\to\infty}\left[\frac{1}{\sqrt{1-v^2/c^2}(1+\sqrt{1-v^2/c^2})}\right]$$ … this is looking promising...

We look like we are very close to where we need to be.
Perhaps NOW we can evaluate our limit signs...

$$\text{L.H.S.} = mv^2 \lim_{c\to\infty}\left[\frac{1}{\sqrt{1-v^2/c^2}\left(1+\sqrt{1-v^2/c^2}\right)}\right] = mv^2\left[\frac{1}{\sqrt{1-v^2/\infty^2}\left(1+\sqrt{1-v^2/\infty^2}\right)}\right]$$

$$\text{L.H.S.} = mv^2\left[\frac{1}{\sqrt{1-0}\left(1+\sqrt{1-0}\right)}\right] = \frac{1}{2}mv^2 = \text{R.H.S.}$$

And now we realise that we have reached the RHS, thus showing that the two are equal.