PROBLEMS IN GROUP THEORY

1) Consider the dihedral group D_8 of order 8. Write down two nontrivial cyclic subgroups of it. Is the whole group cyclic? Justify your answer.

2) Write down a subgroup of order 4 in D_8 such that every nontrivial element in this subgroup has order 2.

3) What are all the subgroups of the alternating group A_4?

4) Prove that any normal subgroup H of a group G is a union of its conjugacy classes.

5) Prove that any subgroupp H of index 2 in a group G is normal.

6) Prove that the dihedral group D_{12} of order 12 is isomorphic to $S_3 \times \mathbb{Z}/2$.

7) Prove that if p is a prime and $\phi(n)$ denotes the Euler ϕ-function then $\phi(p^n = p^n - p^{n-1}$. Use this to count the number of elements of order 5 in $\mathbb{Z}/5$.

7) Let $G = S_3 \oplus \mathbb{Z}/5$. What are all the possible orders of elements in G. Prove that G cannot have an element of order 30 and is hence not cyclic.

And: Possible orders are 1,2, 3,5,10,15, 30 as the order of any element has to divide the order of the group which is 30. If G were cyclic, then G would be abelian, and so would all its subgroups but S_3 is not abelian.

8) For any positive integer n let $U(n)$ denote the subgroup of $\mathbb{Z}/n$ consisting of elements $a \in \mathbb{Z}/n$ such that there is an element b in $\mathbb{Z}/n$ with the *mutliplication mod n * operation. Show that $U(n)$ has $\phi(n)$ elements. For example, if $n = 5$, then $2.3 = 1$. Show that $U(p^n)$ for a prime p has $p^n - p^{n-1}$ elements.

9) Prove that if G is cyclic and H is a normal subgroup, then the quotient group G/H is cyclic.

10) Prove that if G' is the commutator subgroup of a group G, then G/G' is abelian.

11) Let $Z(G)$ denote the centre of a group G. Suppose that $G/Z(G)$ is abelian. For each $g \in G$, define a map $\psi_g : G \to G$ by $\psi_g :: x \mapsto [x, g]$ where $[x, g]$ denotes the commutator. Prove that each ψ_g is a homomorphism. Is it an automorphism? Justify your answer.

12) In the problem above, suppose that $x^m = 1$ for every $m \in Z(G)$. Prove that $y^m \in Z(G)$ for every $y \in G$.

13) Suppose that G is a group and that N is a normal subgroup of G. For each $g \in G$, let $C_G(g)$ denote the centralizer in G of g. Also let $C_N(g) = N \cap C_G(g)$. Prove that $C_N(g) \subset C_G(g)$ and that $C_G(g)/C_N(g)$ is isomorphic to a subgroup of G/N. Clearly state any isomorphism theorem to which you appeal in the course of your argument.

14) Suppose that G is a group and that its automorphism group $Aut(G)$ is finite. Prove that the index $\mid G : Z(G) \mid$ must be finite.

15) Show that every group of order 35 must be abelian.