PERMUTATION GROUPS

Let $G = S_n$, the symmetric group consisting of one to one bijective maps on the set $\{1, 2, \cdots, n\}$. Recall that any element σ in S_n can be expressed as a product of disjoint cycles, and the element has type $(a_1, a_2, \cdots, a_k)$ if the integers a_i are the lengths of the cycles τ_i where

$$\sigma = \tau_1.\tau_2.\cdots\tau_k$$

is the product expression of σ as disjoint cycles. We show that this product expression is uniques. Note that the factors commute i.e. $\tau_i\tau_j = \tau_j\tau_i$ as the cycles are disjoint. Suppose there are two expressions for σ,

$$\sigma = \tau_1.\tau_2.\cdots\tau_k \text{ and } \pi_1.\pi_2...\pi_j.$$

Write $\tau_1 = (i_1, \cdots, i_k)$ where $k = a_i$ and $\pi_1 = (i'_1, \cdots, ..i'_k)$. (We can assume that an element τ_1 and π_1 have the same length by moving the elements π_i and also assume that $i_1 = i'_1$. Then $i_2 = pi(i_1) = i'_2 = \pi_1(i'_1)$, similarly $i_3 = i'_3$, etc. We can deal with the other cycles in a similar manner and hence the two expressions for σ are identical.

A two cycle i.e. a cycle of the form (a, b) is called a *transposition.* Every cycle in S_n can be written as a product of transpositions. This is because any cycle

$$(i_1 i_2 \cdots i_{r-1} i_r) = (i_1 i_r)(i_1 i_{r-1} \cdots (i_1 i_3)(i_1 i_2).$$

This expression of any cycle as a product of transpositions is not unique. However the parity of the number of transpositions that occur (i.e. whether even number or odd number) is always well-defined and this is called the *sign of a permutation.* A permutation is *even* if it is a product of an even number of transpositions and *odd* if it is a product of an odd number of transpositions. The sign can be determined by the number of intersections in the crossover diagram.

Important points to remember about S_n:
i) $\mid S_n \mid = n!$.
ii) Two cycles in S_n are conjugate if and only if they have the same type.
iii) Every element in S_n can be written uniquely as a product of disjoint cycles.
iv) Every element can be written as a product of transpositions.
v) The sign of a permutation σ is the parity of the number of transpositions that occurs in an expression of σ as a product of transpositions.
vi) Two elements of S_n are conjugate if and only if they have the same cycle type.
vii) The number of conjugacy classes of S_n coincides with the number of partitions of n.

Examples:
(1653) is odd in S_6 as (1653) = (16)(15)(!3). The cycle (13567) = (13)(15)(16)(17) is even.

Let n be a positive integer. If n is odd, is an n-cycle an odd or even permutation? Same question for n even.
Writing the n-cycle as a product of two cycles, it can be expressed as product of $(n-1)$ two cycles; if n is odd, then $n-1$ is even, so odd cycles have even length.
In S_n, let α be an r-cycle, β be an s-cycle and γ be a t-cycle. Then check that $\alpha\beta$ is even if and only if $(r+s)$ is even, $\alpha\beta\gamma$ is even if and only if $r+s+t$ is even.
Show that D_{24} and S_4 are not isomorphic. What are the cardinalities of the two groups?
Ans: The two groups D_{24} and S_4 both have 24 elements. The group D_{24} has an element t of order 12, namely the element corresponding to the rotation r_n. If there is an isomorphism, then this should give us an element of order 12 in S_4. However, S_4 has no such elements as the order of any element in S_n is the least common multiple of the integers occurring in its type. However the elements of largest order in S_4 are the 4-cycles, which have order 4.