LINEAR MOTION Q1

FREE FALL

HOW LONG TO FALL FROM THE MOON TO THE EARTH?

A ONE HOUR

B ONE DAY

C ONE WEEK

D ONE MONTH

E ONE YEAR

FREE FALL

- How long to fall from the Moon to the Earth's surface?

$$m\ddot{x} = -\frac{GMm}{x^2} \Rightarrow V = -\frac{GMm}{x}$$

- Initially, $x = d_{moon}$, $E = -\frac{GMm}{d_{moon}}$
- Finally, $x = r_\oplus$
- Putting it together,

$$t_1 - t_0 = -\int_{d_{moon}}^{r_\oplus} \frac{dx}{\sqrt{2\left(-\frac{GM}{d_{moon}} + \frac{GM}{x}\right)}}$$

$$= -\frac{1}{\sqrt{GM}} \int_{d_{moon}}^{r_\oplus} \sqrt{\frac{x}{1 - \frac{x}{d_{moon}}}}\, dx$$

$$= -\sqrt{\frac{d_{moon}^3}{GM}} \int_1^{r_\oplus/d_{moon}} \sqrt{\frac{u}{1-u}}\, du$$

LINEAR MOTION

Q3

FREE FALL

- We consult the integral tables:

$$\int \sqrt{\frac{x}{a-x}}\,dx = -\sqrt{x}\sqrt{a-x} - a\tan^{-1}\left(\frac{\sqrt{x}\sqrt{a-x}}{x-a}\right)$$

This gives

$$t_1 - t_0 = \sqrt{\frac{d_{moon}^3}{GM_\oplus}}\left[\frac{\pi}{2} + \sqrt{\frac{r_\oplus}{d_{moon}}}\sqrt{1-\frac{r_\oplus}{d_{moon}}} - \tan^{-1}\left(\frac{\sqrt{\frac{r_\oplus}{d_{moon}}}}{\sqrt{1-\frac{r_\oplus}{d_{moon}}}}\right)\right]$$

$$\frac{r_\oplus}{d_{moon}} = 1.6\%$$

$$2\pi\sqrt{\frac{d_{moon}^3}{GM_\oplus}} = 27\,\text{days} \rightarrow \frac{\pi}{2}\sqrt{\frac{d_{moon}^3}{GM_\oplus}} = 6\tfrac{3}{4}\,\text{days}$$

The last two terms sum to $0.00136 \ll \frac{\pi}{2}$

LINEAR MOTION
GENERAL RESULTS

- EXPLAIN HOW LINEAR MOTION IS IMPORTANT EVEN FOR MULTI-DIMENSIONAL MOTION
- SOLVE ANY ONE-D SYSTEM WHERE FORCE ONLY DEPENDS ON POSITION

LINEAR MOTION

Let's start with Newton's 2nd Law:

$$m\ddot{x} = F(x)$$

If the force only depends on position we can find a constant of the motion.

Let's multiply both sides by $\dot{x}$ to yield

$$m\ddot{x}\dot{x} = F(x)\dot{x}$$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2\right) = \frac{d}{dt}\left(\int_{x_0}^{x} F(x')\,dx'\right)$$

These two quantities are so important, we make the definitions:

$$\boxed{T = \frac{1}{2}m\dot{x}^2, \quad V = -\int_{x_0}^{x} F(x')\,dx'}$$

ENERGY

Taking these definitions in $\vec{F} = m\vec{a}$ gives

$$\frac{d}{dt}T = -\frac{d}{dt}V \Rightarrow \frac{d}{dt}(T+V) = 0$$

where the sum of kinetic and potential energy is conserved:

$$E = T + V \text{ is constant for linear motion if } F(x).$$

Although this result seems to be of restricted use, many multidimensional systems can be split into several one-dimensional systems where these results apply!

SOLUTION BY QUADRATURE

- The conservation of energy allows us to find a <u>formal</u> solution to all one-D systems with $F = F(x)$.
- A formal solution means that we can replace the differential equation with an integral.
- We know that E is constant so

$$E = T + V = \frac{1}{2} m \dot{x}^2 + V(x)$$

$$\dot{x}^2 = \frac{2}{M}(E - V)$$

Because $\dot{x}^2 \geq 0$ we know that the motion must be bounded in a region where $E \geq V(x)$

LINEAR MOTION

#5

- Rearranging a bit more gives

$$dt = \frac{dx}{\sqrt{\frac{2}{M}(E-V)}}$$

so

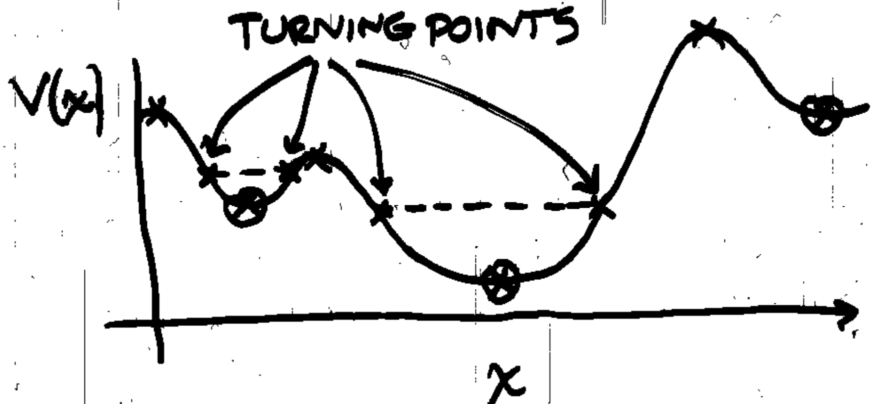
$$t_1 - t_0 = \int_{x_0}^{x_1} \frac{dx}{\sqrt{\frac{2}{M}(E-V(x))}}$$

- This integral is not always easy, so it is called a formal solution, but you could always hand it to a computer.
- More importantly, the plot of $V(x)$ can tell you the range of motion.

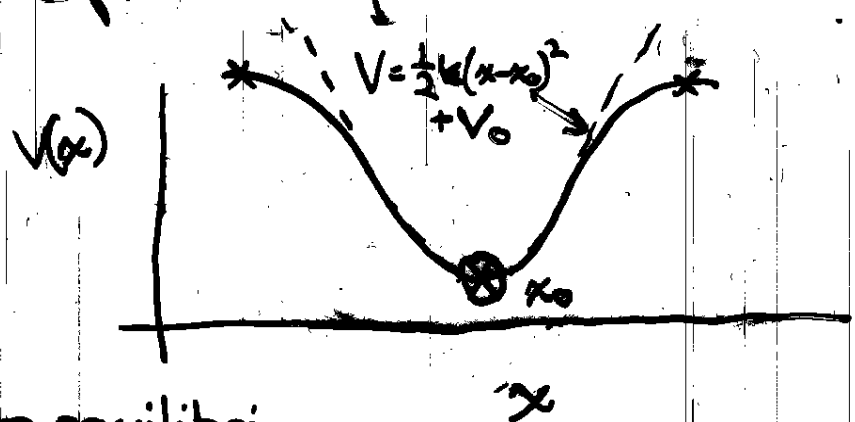
LINEAR MOTION #C

THE LAY OF THE LAND

- The potential energy curve can tell us a lot about the motion even without an exact solution.



× Equilibrium points ⊗ Stable equilibria



- Near an equilibrium the potential looks like a parabola.