Zero-Age Main Sequence

The star has essentially its initial composition and the structure it will have as its completes core H burning

$M < 0.5 M_\odot$ fully convective

$0.5 < M < \cancel{1.2 M_\odot}$ $2M_\odot$ radiative core
convective envelope

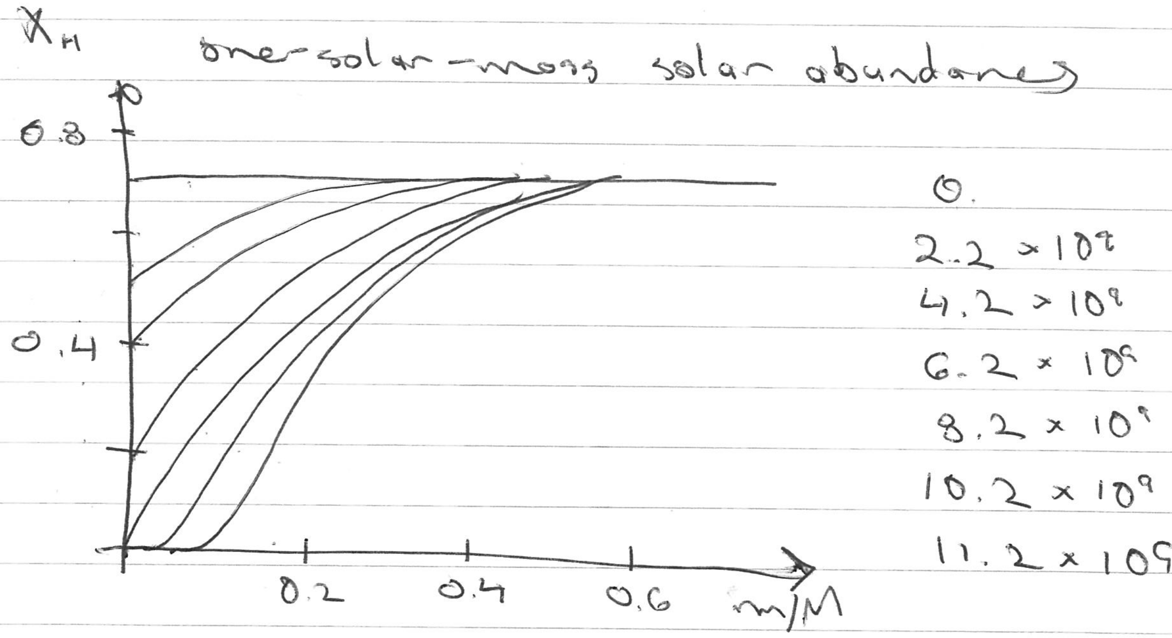
$M > \cancel{1.2 M_\odot}$ $2M_\odot$ convective core
radiative envelope

- More massive stars are much more luminous and so consume hydrogen much faster, shorter lifetimes

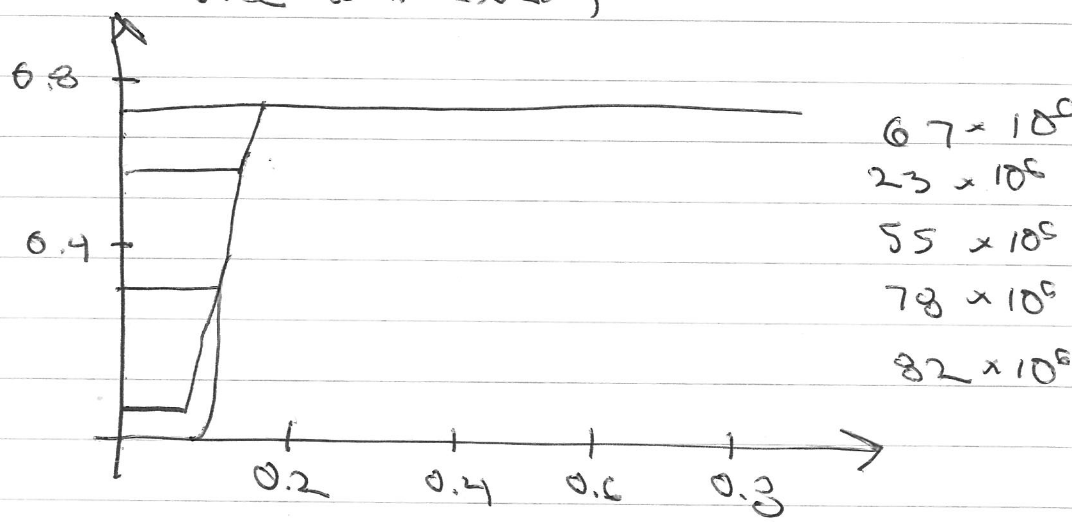
- However, stars with convective burning regions will consume all the fuel in the core (CNO is strong T dependent so ★ changes 1.14le)

- As the hydrogen is consumed the core heats up, the star brightens and gets bluer.

- A long ~~time~~ during which not much happens one-solar-mass solar abundances

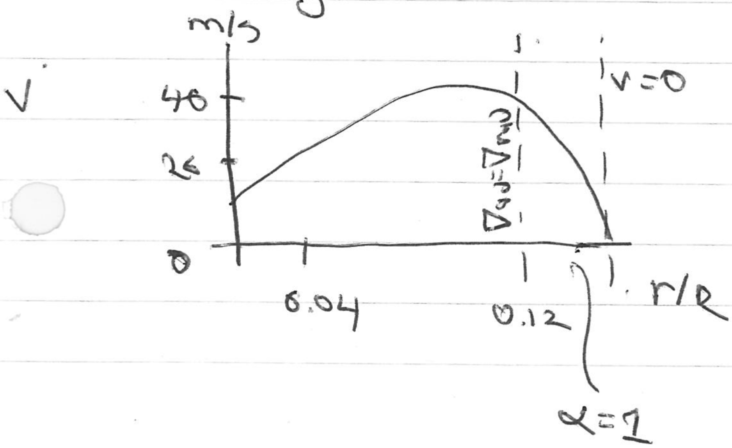


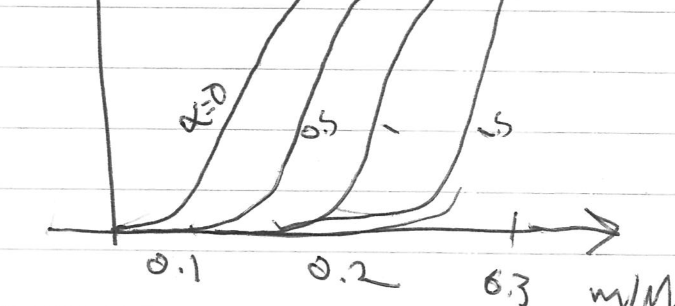
five-solar-mass



N.B. The convective core is shrinking.

Overshooting + Semiconvection





more overshoot, more fuel available

Evolution off the lower main sequence

- For stars between 0.5 and 2.0 $M_\odot$, the core of helium builds up gradually.

- No burning in core, ~~so for~~ $L_r = 0$, $\Rightarrow \frac{dT}{dr} = 0$ (isothermal core)

- The core gradually increases in mass and contracts, releasing gravitational energy → gradually heats up.

-

What happens to the star as the mass of the core increases and the radius of the core decreases? ~~Suppose it happens quickly~~

Let's look at why the core shrinks as it gets more massive?

hydrostatic eqn:

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2} \qquad \frac{dM_r}{dr} = 4\pi r^2 \rho$$

$$\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4} \qquad 4\pi r^3 \frac{dP}{dM_r} = -\frac{GM_r}{r}$$

$$\frac{d(4\pi r^3 P)}{dM_r} - \frac{d(4\pi r^3)}{dM_r} P = -\frac{GM_r}{r}$$

$$\frac{d(4\pi r^3 P)}{dM_r} - 12\pi r^2 \frac{dr}{dM_r} P = -\frac{GM_r}{r}$$

$$\frac{d(4\pi r^3 P)}{dM_r} - \frac{3P}{\rho} = -\frac{GM_r}{r}$$

Let's intergrate over the core:

$$\underbrace{\int_0^{M_{ic}} \frac{d(4\pi r^3 P)}{dM_r}\, dM_r}_{4\pi r^3 P_{ic}} - \int_0^{M_{ic}} \frac{3P}{\rho}\, dM_r = \underbrace{-\int_0^{M_{ci}} \frac{GM_r}{r}\, dM_r}_{\Omega_c}$$

P_{ic} is the pressure at the surface of the core

Let's take the core to be non-degenerate

$$\frac{3P}{\rho} = \frac{3kT}{\mu_{ic} m_H}$$

$$P_{ic} = \frac{3}{4\pi R_{ic}^3}\left(\frac{M_{ic} k T_{ic}}{\mu_{ic} m_H} + \frac{1}{3}\underbrace{\Omega_c}_{-\frac{1}{5}\frac{GM_{ic}^2}{R_{ic}}}\right)$$

$$\frac{dP_{ic}}{dM_c} = \frac{3}{4\pi R_{ic}^3}\left(\frac{k_B T_{ic}}{\mu_{ic} m_H} - \frac{2}{5}\frac{GM_{ic}}{R_{ic}}\right)$$

Relationship between R_{ic}, M_{ic}, and T_{ic}

$$R_{ic} = \frac{2}{5}\frac{GM_{ic}\,\mu_{ic} m_H}{kT_{ic}}$$

$$P_{ic}^{max} = \frac{375}{64\pi}\frac{1}{G^3 M_{ic}^2}\left(\frac{kT_{ic}}{\mu_{ic} m_H}\right)^4$$

maximum pressure; notice that it decreases as $M_{ic}\Uparrow$

$$P_{env} \approx \frac{GM^2}{4\pi R^4}$$

$$T_{ic} \approx \frac{P_{env}\,\mu_{env}\, m_H}{\rho_{env}\, k} \qquad \rho_{env} \approx \frac{M}{4\pi R^3/3}$$

$$R \approx \frac{1}{3}\frac{GM}{T_{ic}}\frac{\mu_{env} m_H}{k_B}$$

$$P_{env} = \frac{81}{4\pi}\frac{1}{G^3 M^2}\left(\frac{k_B T_{ic}}{\mu_{env} m_H}\right)^4$$

$$\frac{M_{ic}}{M} < \left(0.289\right)^{1/2}\left(\frac{\mu_{en}}{\mu_{ic}}\right)^2$$

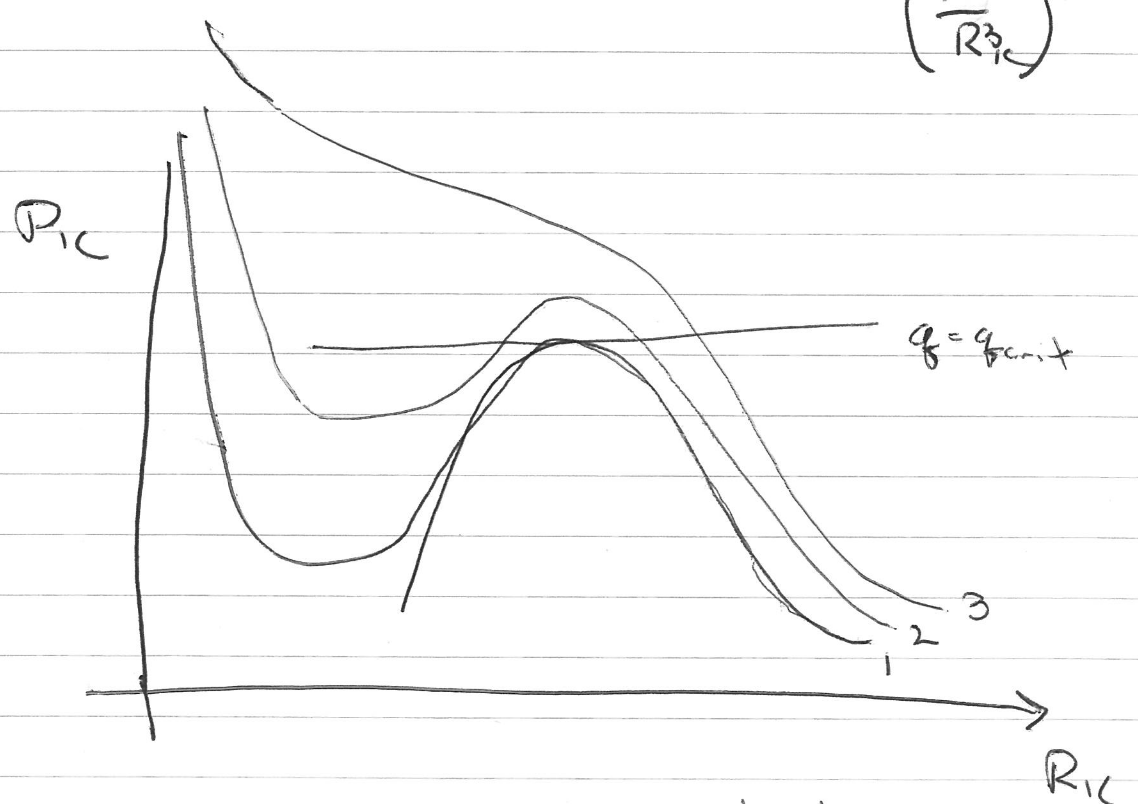
$$\frac{M_{ic}}{M} < 0.54\left(\frac{\mu_{en}}{\mu_{ic}}\right)^2$$

Now what about degeneracy pressure:

$$\frac{P}{\rho} = \frac{k_B T_{ic}}{\mu_{ic} m_H} + \underbrace{\frac{K_{5/3}\,\rho^{2/3}}{\mu_e^{5/3}}}_{\text{extra term}}$$

$$P_{ic} = \frac{3}{4\pi R_{ic}^3}\left(\frac{M_{ic}}{\mu_{ic} m_H} - \frac{1}{5}\frac{G M_{ic}^2}{R_{ic}} + \frac{A}{3}\frac{M_{ic}^{5/3}}{R_{ic}^2}\right)$$

$$A = \frac{K_{5/3}}{\mu_e^{5/3}} \frac{\langle \rho^{2/3} \rangle}{\left(\frac{M_{ic}}{R_{ic}^3}\right)^{2/3}}$$



1, 2, 3 correspond to smaller core mass, lower T_{ic} and more degenerate

For $M \gtrsim 3M_\odot$ the curve has two turning points (rapid contraction) → heating → He burn

Look at the total energy:

$$E = \langle \Omega \rangle + \langle U \rangle + \underbrace{\int_0^t L\,dt}_{\text{luminosity}} + \underbrace{\int_0^t \epsilon\,dt}_{\text{energy generation}}$$

If $t \ll t_{KH}$, the integrals are smaller than the first terms so

$$E \approx \langle \Omega \rangle + \langle U \rangle = \text{constant}$$

Second for $t \gg t_{dyn}$ $\quad \langle \Omega \rangle + 2\langle U \rangle = 0$

Take the difference of the two equations

$$\langle U \rangle \approx -\text{constant}$$

$$\langle \Omega \rangle = 2(\text{constant}) \approx \Omega_c + \Omega_e$$

$$\Omega \approx \frac{GM_c^2}{R_c} + \frac{GM_cM_e}{R_\star}$$

$$\frac{dR_\star}{dR_c} \approx -\left(\frac{M_c}{M_e}\right)\left(\frac{R_\star}{R_c}\right)^2 \ll -1$$

if we assume M_c, M_e are approximately constant.

N.B. $\quad M_c \gg M_e$ and $R_\star \gg R_c$

so huge lever arm!!!

(58) 4/2 (04)

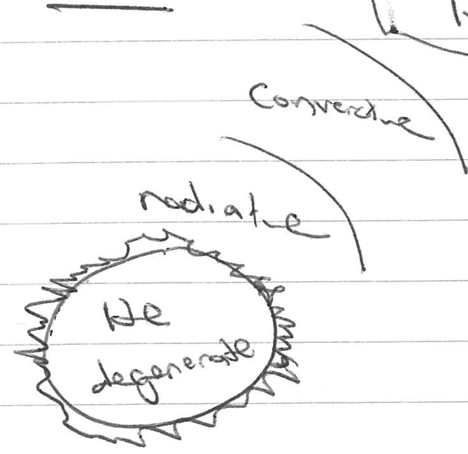
- stars above about 1.2 $M_\odot$ develop a small convective core on the MS
  ↳ H-burning shell starts suddenly

- Stars above about 3 $M_\odot$ reach the Chandrasekhar - Schonberg limit ~~and~~
  ↳ rapid evolution to RGB, ~~between~~ and up the RGB to helium burning ~~the~~ Hertzsprung Gap

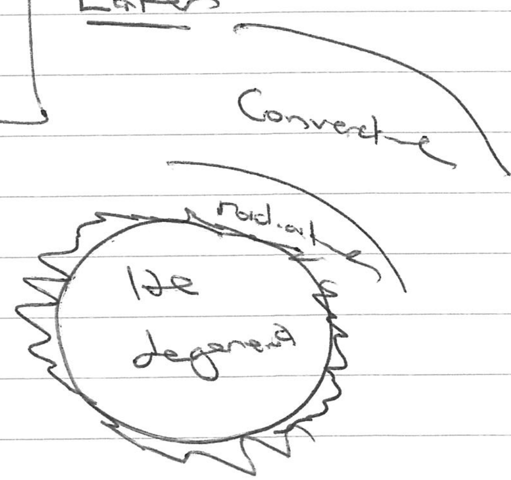
- What happens below 3 $M_\odot$, especially about 1 $M_\odot$? The turn-off of old open clusters and globular clusters

Structure ($\boxed{\text{going backwards along Henyey}}$

Initial

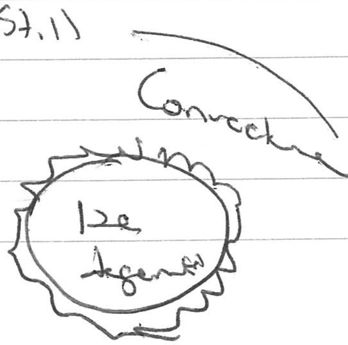


Later



Later Still

$\boxed{\text{Going up Hayashi}}$



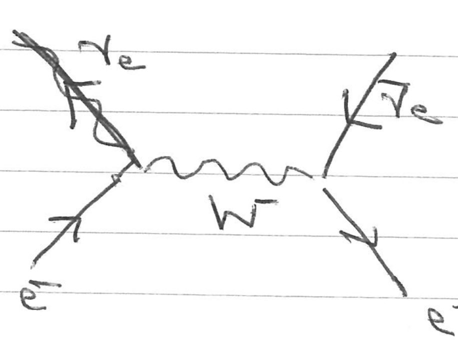
When radiative layer disappears we get a loop!

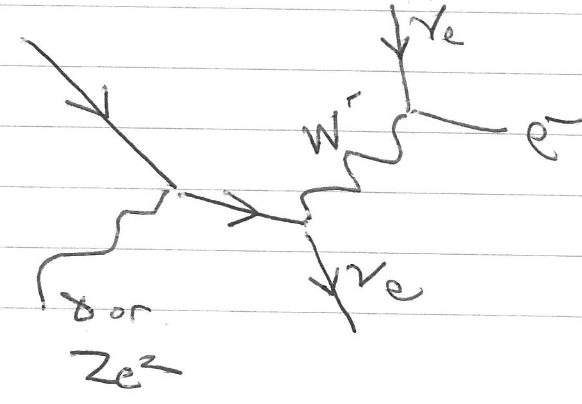
Approaching helium burning in low-mass stars

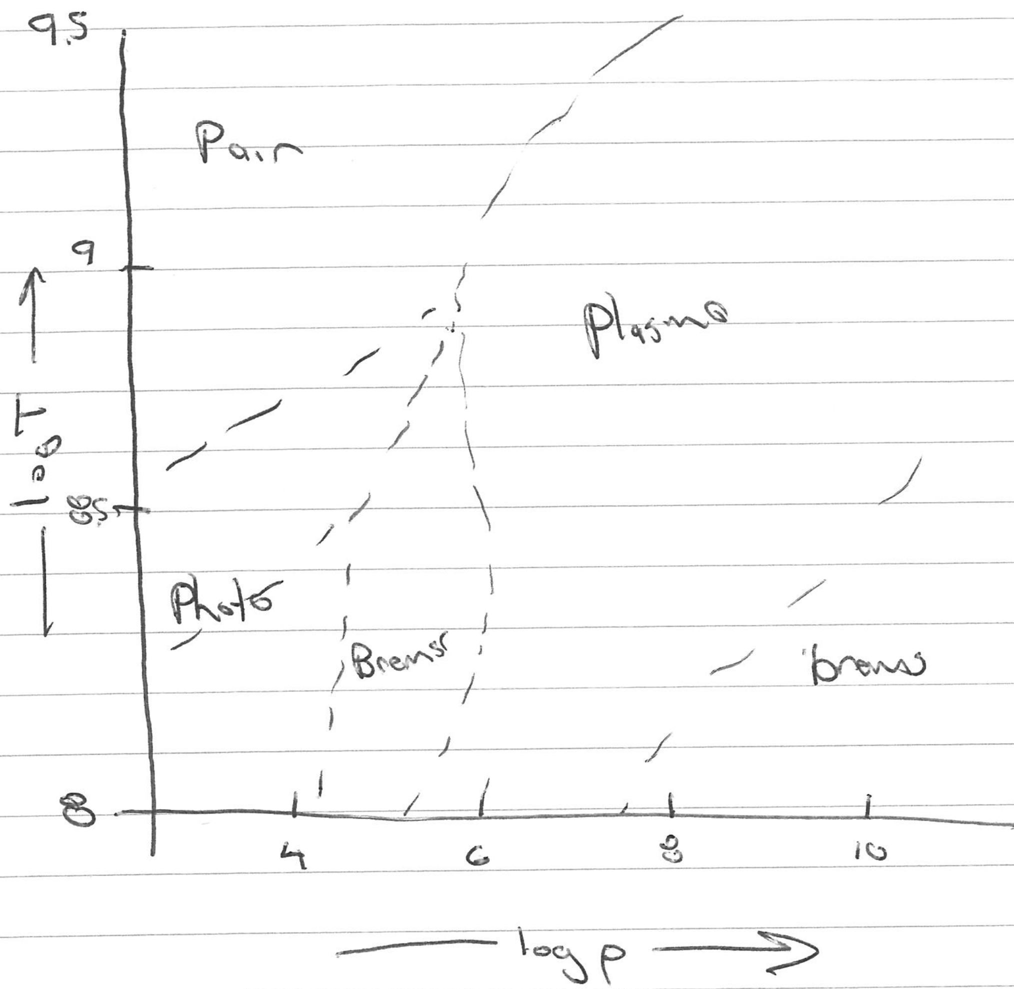
- the ~~te~~ pressure at the surface of the core increases as the mass grows
- temperature increases, star gets more luminous, neutrino emission ~~&~~ becomes important to energy balance

- Let's talk about neutrinos.

Pair annihilation:	$e^- + e^+ \rightarrow \nu + \bar{\nu}$
Photo neutrino	$\gamma + e^- \rightarrow e^- + \nu + \bar{\nu}$
plasmon	$\gamma \rightarrow \nu + \bar{\nu}$
bremss	$e^- + Z \rightarrow e^- + Z + \nu + \bar{\nu}$







- In the core of the red giant the density is high and the temperature is low → brems + plasmon neutrinos

- Once helium burning starts, the core expands so photo neutrinos

How does helium burning start?

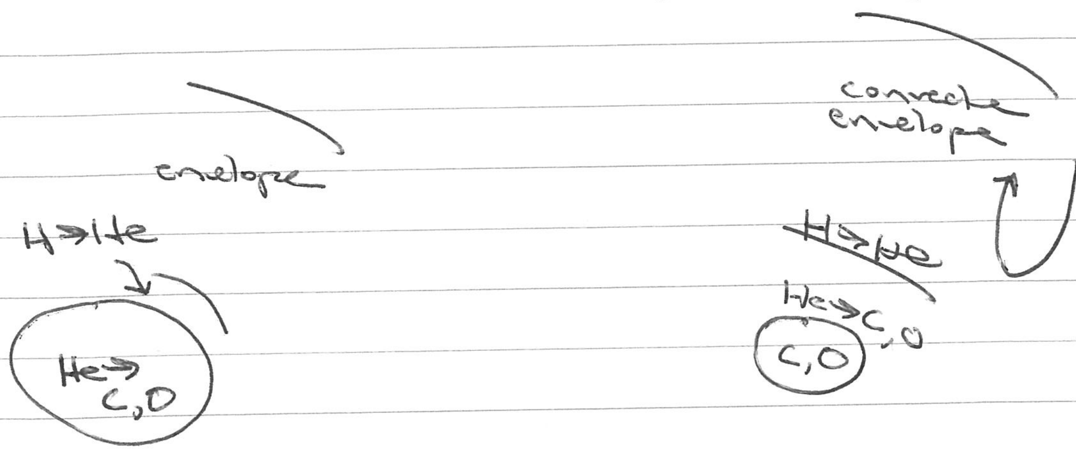
NB (1) neutrino rates increases with density so centre of core is colder than on the outer region

(2) Helium burning is strongly sensitive to the temperature so helium burning is strongest away from the core.

- Helium burning starts in a shell
- As the temperature increases, pressure is constant so no decrease in density → no thermostat.
- Helium burning rate increases exponentially on a timescale of a week
  ↳ ~~[illegible]~~ nuclear luminosity can reach $10^9 L_\odot$
- core heats up sufficiently to lift degeneracy and burning moves to core and star goes to horizontal branch.

· Mass Loss during stellar evolution (low mass stars)

- on the RGB the luminosity gets very large expect some mass loss, so $M_{HB} < M_{MS}$



horizontal branch

asymptotic giant branch

- on AGB all the envelope is expelled (mass loss)

  ↳ $0.53 M_\odot$ C,O white dwarf

- key things during He-burning on AGB

  convective overshoot brings hydrogen into helium burning region

  $$^{12}C(p,\gamma)^{13}N(e^+\nu)^{13}C(\alpha,n)^{16}O$$

  ↓ slower process neutron capture

How do we know that we are right?

How can we look inside stars?

(1) Neutrinos: remember that the Sun apparently produce 1/3 the expected number.

How could we be so sure?

(2) Oscillations: helioseismology and with MOST, COROT and now Kepler we have data on lots of stars too.

- First let's talk about radial oscillations and think of the properties of the star and as a function of the mass enclosed.

$$P(m,t) = P_0(m) + P_1(m,t) = P_0(m)\left[1 + p(m)e^{i\omega t}\right]$$

$$r(m,t) = r_0(m) + r_1(m,t) = r_0(m)\left[1 + x(m)e^{i\omega t}\right]$$

$$\rho(m,t) = \rho_0(m) + \rho_1(m,t) = \rho_0(m)\left[1 + d(m)e^{i\omega t}\right]$$

These are Lagrangian coordinates so

Euler:

$$\ddot{r} = -\frac{Gm}{r^2} - 4\pi r^2 \frac{\partial P}{\partial m}$$

linearising:

$$\frac{P_0}{\rho_0}\frac{\partial p}{\partial r_0} = \omega^2 r_0 x + g_0 (p + 4x)$$

continuity

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}$$

linearising:

$$r_0 \frac{\partial x}{\partial r_0} = -3x - d$$

Now connect p and d as an adiabatic perturbation so

$$p = \gamma_{ad}\, d$$

and rewrite the continuity as

$$p = -3\gamma_{ad}\, x - \gamma_{ad}\, r_0 \frac{\partial x}{\partial r_0}$$

$$\frac{\partial p}{\partial r_0} = \cancel{-3\gamma_{ad}\frac{\partial x}{\partial r_0}}$$

$$-3\frac{\partial}{\partial r_0}(\gamma_{ad}\, x) - \frac{\partial}{\partial r_0}\left(\gamma_{ad}\, r_0 \frac{\partial x}{\partial r_0}\right)$$

Substituting this and the continuity equation into the Euler equation yields

$$\frac{\partial}{\partial r_0}\left(\gamma_{ad}\frac{\partial x}{\partial r_0}\right) + \frac{4}{r_0}\frac{\partial}{\partial r_0}(\gamma_{ad}x) - \frac{\rho_0 g_0}{P_0}\gamma_{ad}\frac{\partial x}{\partial r_0}$$

$$+ \frac{\rho_0}{P_0}\left[\frac{g_0}{r_0}(4-3\gamma_{ad}) + \omega^2\right]x = 0$$

Boundary conditions $x=0$ at $r_0=0$

At the surface take $p=0$; scale so
$x(M)=1$, $p(M)=0$

We have an eigenvalue equation
for ω^2

Using $\frac{\partial P}{\partial r} = -g\rho$ we can rearrange
to

$$\hat{\mathcal{L}}x \equiv -\frac{1}{\rho_0 r_0^4}\frac{\partial}{\partial r_0}\left(\gamma_{ad}P_0 r_0^4\frac{\partial x}{\partial r_0}\right) - \frac{1}{r_0\rho_0}\left[\frac{\partial}{\partial r_0}\left[(3\gamma_{ad}-4)P_0\right]\right]x$$

$$= \omega^2 x$$

Sturm-Liouville problem
ω^2_n ~~~ with $\omega_0^2 < \omega_1^2 \ldots$ as $n \to \infty$
all are real

Let's suppose x, ω^2 are a solution to

$$\hat{\mathcal{L}}\, x = \omega^2$$

and integrate both sides from the centre to surface

$$\cancel{\int_0^M \hat{\mathcal{L}}\, x \, dx}$$

$$-\left(\gamma_{ad} P_0 r_0^4 \frac{\partial x}{\partial r_0}\right)\Bigg|_0^R - \int_0^R r_0^3 \frac{\partial}{\partial r_0}\left[\left[(3\gamma_{ad}-4)\right] P_0\right] x_0 \, dr_0$$

$$= \omega^2 \int_0^R x_0 \rho_0 r_0^4 \, dr_0$$

First term is zero so we have

$$\omega^2 = -\frac{\int_0^R r_0^3 \frac{\partial}{\partial r_0}\left[(3\gamma_{ad}-4) P_0\right] x_0 \, dr_0}{\int_0^R x_0 \rho_0 r_0^4 \, dr_0}$$

Suppose γ_{ad} is constant then

$$\omega^2 = (3\gamma_{ad}-4) \frac{\int_0^{R_0} r_0^3 \rho_0 g_0 x \, dr_0}{\int r_0^4 \rho_0 x \, dr}$$

if x_0 has no nodes positive

ω^2 has same sign as $(3\gamma_{ad}-4)$

What is the typical size of ω?

Let $\hat{r} = \frac{r_0}{R_0}$ $\hat{m} = \frac{m}{M}$ $\hat{\rho} = \frac{\rho}{\bar{\rho}}$

$$\omega^2 = (3\gamma_{ad} - 4) \frac{GM}{R^3} \frac{\int_0^1 \hat{r}' \hat{\rho} \hat{m} \, \cancel{x_0} \, d\hat{r}}{\int_0^1 \hat{r}^3 \hat{\rho} \, \cancel{x_0} \, d\hat{r}}$$

So for stars with a given typical γ_{ad} we have

$$\omega_0^2 \propto \bar{\rho} \quad \text{or} \quad P\sqrt{\bar{\rho}} = \text{constant}$$

Let's normalize with δ-Cephei
$M = 7M_\odot$, $R = 80R_\odot$ $P = 11$ days

$$\frac{P^2 M}{R^3} = \frac{121 \times 7}{512{,}000} = 1.65 \cdot 10^{-3} \frac{(\text{day})^2 M_\odot}{R_0^3}$$

WD: $M = 1M_\odot$ $R = 10^{-2}R_\odot$

$$P^2 \approx 1.65 \times 10^{-9} \, (\text{days})^2$$
$$P \approx 4 \cdot 10^{-5} \text{ day} = 3.5 \text{ seconds}$$

Supergiant $M = 20M_\odot$, $R = 10^3 R_\odot$

$$P^2 \approx 8 \cdot 10^4 \text{ days}^2 \quad P \approx 280 \text{ days}$$

What about high-order modes?

WKB approximation

$$\chi_0 = \sin\left(\pi \frac{n+1}{2} \hat{r}\right)$$

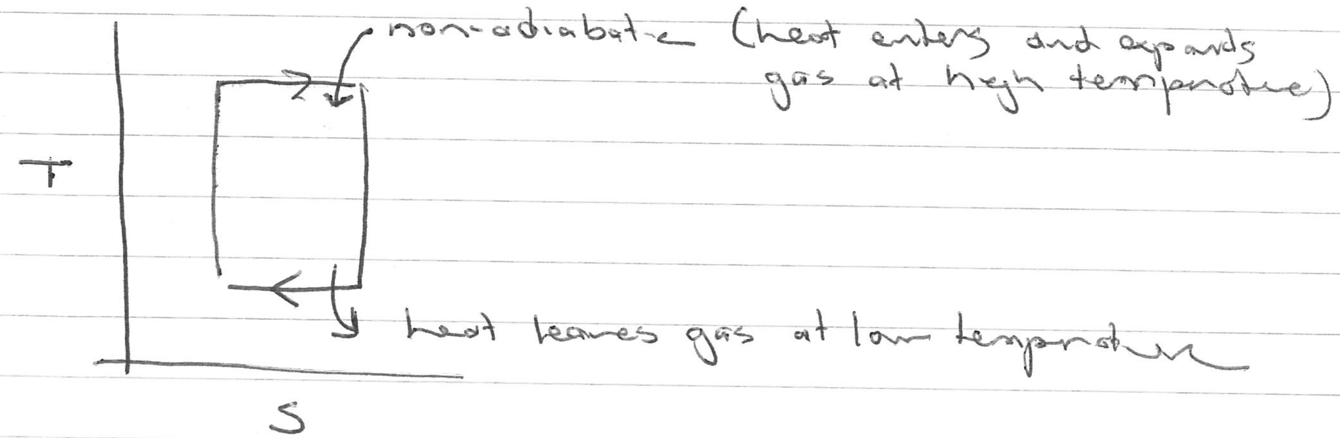
ω^2 is a ratio of two Fourier transforms of the unperturbed ★

$\omega^2 \propto \left(\frac{n+1}{2}\right)^2$ from counting powers of $\hat{r}$

$$\omega \approx (n+1)\omega_0 \quad (\text{assuming } \omega_0^2 > 0)$$

Non-Adiabatic Effects

- for the modes to be sustained, they must not strictly be adiabatic



NB: $\kappa \propto \frac{\rho}{T^{3.5}}$ $\quad P \propto \rho^{\gamma} \quad T \propto \rho^{\gamma_{ad}-1}$

~~$\kappa \propto \rho$~~ $\quad \kappa \propto \cancel{T^{\frac{7\gamma-9}{2-2\gamma}}} \quad \kappa \propto T^{-\frac{7\gamma-9}{2\gamma-2}}$

$\boxed{\gamma < \frac{9}{7}}$

- An exception to the decrease of opacity with temperature is a partially ionized region
  - Compression can ionize the gas increasing the opacity
  - Heat trapped at high density
  - Heat released at low density
  - κ mechanism

H zone 10,000 - 15,000 K HI → HII; HeI → HeII

He zone 40,000 K HeII → HeIII

~~For Teff ≈ 7500 K these zones are too~~

- Different ionization zones yield different instability strips
- For the star to vary dramatically the fundamental or a low-n mode must be excited
  - Mira, ZZ Ceti: H-zone
  - RR Lyrae He-zone (Cepheids too) (Polaris)
  - β Cepheid Iron ionization (e.g. Spica)

Non-Radial Oscillations

$$\nabla^2 \phi = 4\pi G \rho$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

$$\rho \left(\frac{\partial}{\partial t} + v \cdot \nabla \right) v = -\nabla P - \rho \nabla \phi$$

Suppose the unperturbed star has

$$P(r),\ \rho(r),\ \phi(r),\ v(r)$$

$\vec{x}(\vec{r}, t)$ is the displacement of some fluid

$$\underbrace{\text{Lagrangian} \quad \delta\rho}\ = \rho' + \vec{x} \cdot \nabla \rho$$

ρ' — Eulerian perturbation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v \cdot \nabla \quad \text{(Lagrangian Derivative)}$$

$$\frac{\delta P}{P} = \gamma_{ad} \frac{\delta \rho}{\rho} \quad \text{(adiabatic perturbation)}$$

Euler

$$\rho \frac{\partial^2 \vec{x}}{\partial t} = -\nabla P - \rho \nabla \phi - \nabla P' - \rho \nabla \phi' - \rho' \nabla \phi$$

~~to~~ to first order also $\boxed{-\nabla P - \rho \nabla \phi = 0}$

<u>Continuity</u>

$$\rho' + \nabla \cdot (\rho \vec{x}) = 0 \quad \text{or} \quad \frac{\delta \rho}{\rho} = -\nabla \cdot \vec{x}$$

<u>Poisson</u> $\nabla^2 \Phi' = 4\pi G \rho'$

Let's ignore Φ' (Cowling approximation)

$$\vec{x}(\vec{r},t) = \vec{x}(\vec{r})\, e^{i\sigma t}$$

And put into Euler equation

$$\sigma^2 x_r = \frac{\partial}{\partial r}\left(\frac{P'}{\rho}\right) - \underbrace{\left(\frac{1}{\rho}\frac{d\rho}{dr} - \frac{1}{P\gamma_{ad}}\frac{dP}{dr}\right)}_{A_g}\frac{P\gamma_{ad}}{\rho}\nabla \cdot x$$

NB: $c_s^2 = \frac{P\gamma_{ad}}{\rho}$ $\quad A_g = \frac{1}{\gamma_{ad}}\frac{d\ln p}{dr}(\gamma_{ad} - \gamma_*)$

$$\sigma^2 x_\theta = \frac{\partial}{\partial \theta}\left[\frac{1}{r}\frac{P'}{\rho}\right] \qquad \sigma^2 x_\phi = \frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\left[\frac{1}{r}\frac{P'}{\rho}\right]$$

Use spherical harmonics

$$\vec{x} = \left[x_r(r)\hat{r} + x_t(r)\hat{\theta}\frac{\partial}{\partial\theta} + x_t(r)\hat{\phi}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right] Y_{lm}(\theta,\phi)$$

$$\frac{P'(\vec{r})}{\rho} = \frac{P(r)}{\rho} Y_{lm}(\theta,\phi) \qquad x_t = \frac{1}{\sigma^2}\frac{1}{r}\frac{P'(r)}{\rho}$$

Looking at the Euler equation we need

$$\nabla \cdot \vec{x} = \frac{1}{r^2}\frac{d}{dr}(r^2 x_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(x_\theta \sin\theta)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(x_\phi)$$

Substituting ~~...~~ our $Y_{\ell m}$ guess for $\vec{x}$ gives

$$\nabla \cdot \xi = \frac{1}{r^2}\frac{d}{dr}(r^2 x_r) Y_{\ell m} - \frac{\ell(\ell+1)}{r} x_t Y_{\ell m}$$

Let's define some frequencies to make thing easier

Brunt-Väisälä frequency (buoyancy)

$$N^2 = -A_8 g \cancel{\left[\frac{d\ln\rho}{dr} - \frac{1}{\gamma_{ad}}\frac{d\ln P}{dr}\right]} = .$$

$$N^2 = -g\left[\frac{d\ln\rho}{dr} - \frac{1}{\gamma_{ad}}\frac{d\ln P}{dr}\right] = -\frac{g}{\gamma_{ad}}\frac{d\ln\rho}{dr}(\gamma_{ad} - \gamma^*)$$

Lamb Frequency

$$S_\ell^2 = \frac{\ell(\ell+1)}{r^2}\frac{\gamma_{ad} P}{\rho} = \frac{\ell(\ell+1)}{r^2} c_s^2 = k_t^2 c_s^2$$

Get two equations in the radial direction as the $Y_{\ell m}$ solve the angular part!

zonal
$m=0$

sectorial
$|m|=\ell$

$$r\frac{d\chi_r}{dr} = \left[\frac{k_t^2 g r}{S_\ell^2} - 2\right]\chi_r + r^2 k_t^2\left[1 - \frac{\sigma^2}{S_\ell^2}\right]\chi_t$$

$$r\frac{d\chi_t}{dr} = \left[1 - \frac{N^2}{\sigma^2}\right]\chi_r + \left[\frac{r}{g}N^2 - 1\right]\chi_t$$

Together we have a second-order linear ODE.

If we wanted to include $\phi' \Rightarrow 4^{th}$ order

Let's look at solutions $\chi_t, \chi_r \sim e^{ik_r r}$, (WKB)

$$k_r^2 = \frac{k_t^2}{\sigma^2 S_\ell^2}(\sigma^2 - N^2)(\sigma^2 - S_\ell^2)$$

If σ^2 is outside the range between N^2 and S_ℓ^2, then $k_r^2 > 0$ and we have standing waves

If σ^2 is between N^2 and S_ℓ^2, $k_r^2 < 0$, evanescent behavior!

Usually $N^2 < S_\ell^2$.

Take $\sigma^2 >> N^2, S_\ell^2$

$$k_r^2 \approx \frac{k_t^2}{\sigma_p^2 S_\ell^2}\sigma_p^4$$

$$\sigma_p^2 \approx \frac{S_\ell^2 k_r^2}{k_t^2} \approx \frac{k_t^2 k_r^2}{k_t^2} c_s^2 \approx k_r^2 c_s^2$$

We can do a bit better:

$$k_r^2 \approx \frac{k_t^2}{\sigma^2 S_\ell^2}\sigma^2(\sigma^2 - S_\ell^2)$$

$$k_r^2 \approx \frac{1}{c_s^2}(\sigma^2 - k_t^2 c_s^2)$$

$$\boxed{\sigma_p^2 \approx (k_r^2 + k_t^2) c_s^2}$$ (a sound wave) pressure mode

Take $\sigma^2 << N^2, S_\ell^2$

$$\boxed{\sigma_g^2 \approx \frac{k_t^2}{k_r^2 + k_t^2} N^2}$$ (gravity modes)

What does the flow look like?

$$\left|\frac{\chi_r}{\chi_t}\right| \sim \begin{cases} r k_r & \text{p-mode} \\ \dfrac{\ell(\ell+1)}{r k_r} & \text{g-mode} \end{cases}$$

What about the frequencies?

p-mode with $n \gg \ell$ so $k_r \gg k_t$

$$n \simeq \int_0^R k_r \, dr/\pi$$

$$\sigma_{p,n} \simeq n\pi \left[\int_0^R \frac{dr}{c_s}\right]^{-1} \quad \text{(equally spaced)}$$

For the g-modes we find

$$P_{g,n} = \frac{2\pi}{\sigma_g} \simeq n \frac{2\pi^2}{[\ell(\ell+1)]^{1/2}} \left[\int_0^R \frac{N}{r} dr\right]^{-1} = \frac{n P_0}{[\ell(\ell+1)]^{1/2}}$$

Period equally spaced

The Sun p-mode $f \sim 1-4$ mHz (five minute)

$\chi_r \sim 100$ km

g-modes evanescent in convection region $\sim$ mm

f-modes atmospheric g-modes