1 Solutions to assignment 2, due May 24

Problem 11.1 Let $a, b, c, d \in \mathbb{Z}$ with $a, c \neq 0$. Prove that if $a \mid b$ and $c \mid d$, then $ac \mid (ad + bc)$.

Solution: Since $a \mid b$, we have that there exists an integer k with the property that $ka = b$. Similarly, as $c \mid d$ we have that there is an integer ℓ such that $\ell c = d$. Thus we have that

$$\begin{aligned} ad + bc &= a(\ell c) + (ka)c \\ &= ac\ell + ack \\ &= ac(\ell + k). \end{aligned}$$

As $\ell + k$ is an integer we have then that $ac \mid ad + bc$ as desired.

Problem 11.2 Let $a, b, \in Z$ with $a \neq 0$. Prove that if $a \mid b$, then $a \mid (-b)$ and $(-a) \mid b$.

Solution: We are given that $a \mid b$ and thus that there is an integer k so that $ka = b$. From this we can conclude that $(-k)a = -b$, and that $(-k)(-a) = b$. As $-k$ is also an integer, the first of these yields that $a \mid (-b)$, and the second, that $(-a) \mid b$ as claimed.

Problem 11.3 Let $a, b, c \in \mathbb{Z}$ with $a, c, \neq 0$. Prove that if $ac \mid bc$, then $a \mid b$.

Solution: We are, as before, given that there is an integer k with the property that $k(ac) = bc$. Since $c \neq 0$, we can divide both sides by c ('cancelling' c, if you will), to yield that $ka = b$. But this means that $a \mid b$, as we wanted.

Problem 11.4 Prove that $3 \mid (n^3 - n)$ for every integer n.

Solution: We would first like to note that this is very similar to something we covered in class, where we showed that $3 \mid n(n + 1)(n + 2)$ for every integer n.

We will first note that $n^3 - n = n(n^2 - 1)$. We have three cases ($n = 3k$, $n = 3k + 1$, $n = 3k + 2$) to deal with.

(a) Case 1: $n = 3k$. In this case, $n^3 - n = 3k\big((3k)^2 - 1\big)$ which is clearly divisibly by 3.

(b) Case 2: $n = 3k + 1$. In this case we have

$$n^2 - 1 = (3k + 1)^2 - 1 = 9k^2 + 6k + 1 - 1 = 3(3k^2 + 2k)$$

and so $n(n^2 - 1) = 3(3k + 1)(3k^2 + 2k)$, and so is divisible by 3.

(c) Case 3: $n = 3k + 2$. Lastly, we have that

$$n^2 - 1 = (3k + 2)^2 - 1 = 9k^2 + 12k + 4 - 1 = 3(3k^2 + 4k + 1)$$

and so $n(n^2 - 1) = 3(3k + 2)(3k^2 + 4k + 1)$ which is also divisible by three.

As in each of the three cases (which are exhaustive i.e. every integer falls into one of those cases) we have that $n^3 - n$ is divisible by three, the result follows.

Problem 11.5 Prove that if $n = k^3 + 1 \geq 3$, where $k \in \mathbb{Z}$, then n is not prime.

Solution: As in the given hint, we note that $k^3 + 1 = (k + 1)(k^2 - k + 1)$. Thus we can always factor n as a product of two integers, and so it could only be prime if one of those integers was equal to 1. That is, the only way that it could be prime is if $k + 1 = 1$ or if $k^2 - k + 1 = 1$.

In the first case we would have $k = 0$ and so $n = 1$ which is not greater than 3. In the second case, $k^2 - k = 0$ which has $k = 0, 1$ as solutions. The latter case yields that $n = 1^3 + 1 = 2$ which is also less than three. Thus if $n > 3$, and $n = k^3 + 1$, n must be composite.

Problem 11.9 Prove that for every positive integer n, there exist n consecutive positive integers, each of which is composite.

Solution: A composite number is one for which we can write $n = ab$ for some $1 < a < n$, and $1 < b < n$. Equivalently, n is composite if there is some number $1 < a < n$ such that $a \mid n$.

Consider, as suggested, the numbers

$$2 + (n + 1)!, 3 + (n + 1)!, \ldots, n + (n + 1)!, (n + 1) + (n + 1)!.$$

Note that there are n of these numbers. The goal is to show that these are all composite.

As an example, consider $n = 4$. Then $(4 + 1)! = 5! = 120$, and so these numbers are

$$122, 123, 124, 125$$

The first is even, the second, divisible by three, the fourth is even, and the last divisible by 5. So they are all composite.

Let us turn this into a proof.

We first note that by definition, we always have that if $k \leq m$, then $k \mid m!$. Thus for all integers $2 \leq k \leq (n + 1)$, we have that $k \mid (n + 1)!$. We also have that for all such integers, $k \mid k$ trivially.

Lastly, we have that if $k \mid a$ and $k \mid b$, that $k \mid (a + b)$.

It now follows that

$$\begin{aligned} 2 &\mid 2+(n+1)! \\ 3 &\mid 3+(n+1)! \\ &\vdots \\ n &\mid n+(n+1)! \\ n+1 &\mid n+1+(n+1)! \end{aligned}$$

and so all of these numbers, having proper divisors, must be composite. Note that these numbers might not have any divisors in common!