1 Convergence criteria

We have covered the following convergence criteria. In each case, we will assume that $(a_n)_{n\in\mathbb{N}}$ is a sequence of non-negative real numbers.

Theorem 1 (Comparison test) *1. Suppose that $(b_n)_{n\in\mathbb{N}}$ is a non-negative sequence such that $a_n \leq b_n$ for all n and so that $\sum_{n=1}^{\infty} b_n$ converges. Then $\sum_{n=1}^{\infty} a_n$ converges.*

2. Suppose that $(b_n)_{n\in\mathbb{N}}$ is a non-negative sequence such that $b_n \leq a_n$ for all n, and so that $\sum_{n=1}^{\infty} b_n$ diverges. Then $\sum_{n=1}^{\infty} a_n$ also diverges.

Theorem 2 (Ratio test) *1. Suppose that $\frac{a_{n+1}}{a_n} \leq \theta$ for all n and for some $0 < \theta < 1$. Then $\sum_{n=1}^{\infty} a_n$ converges.*

2. Suppose that $\frac{a_{n+1}}{a_n} \geq \theta$ for all n and for some $\theta \geq 1$. Then $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem 3 (Root test) *1. Suppose that there is some $0 \leq \theta < 1$ such that for every n we have $\sqrt[n]{a_n} \leq \theta$. Then $\sum_{n=1}^{\infty} a_n$ converges.*

2. Suppose that there is some $\theta \geq 1$ such that for every n we have $\sqrt[n]{a_n} \geq \theta$. Then $\sum_{n=1}^{\infty} a_n$ diverges

For each of these tests, we do not in fact require that the inequalities hold for *all* n, but that they hold *eventually*. That is in each case the phrase "for all n" can be replaced by "for all $n \geq N_0$ for some fixed N_0". This is in line with the general philosophy that for limits and convergence, all we care about is the eventual behaviour, not the behaviour that happens before any specified finite time.

There are also limit versions of both the root and ratio test. That is, it is enought to consider $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ or $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$ to test for convergence (and similarly for divergence).

The last test requires a slightly different setup. Suppose that $f : [1,\infty) \to [0,\infty)$ is a function. Recall that $\int_1^{\infty} f(x)dx$ is defined by

$$\int_1^{\infty} f(x)dx = \lim_{N\to\infty} \int_1^{N} f(x)dx.$$

In the case that this limit exists and is finite, we then say that the integral *converges*.

Theorem 4 (Integral test) *The series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the integral $\int_1^{\infty} f(x)dx$ converges.*

To use this test on a particular series $\sum_{n=1}^{\infty} a_n$ we seek to find a continuous function f defined on $[1,\infty)$ such that $f(n) = a_n$. For example, if $a_n = \frac{n}{2^n+3}$ then we might have $f(x) = \frac{x}{2^x+3}$.