Serre's conditions

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Serre's conditions R_n and S_m are stated below.

Definition 1 Let R be a Noetherian ring and n a non-negative integer. The ring R is said to satisfy Serre's condition R_n , if for all prime ideals $\mathcal{P}$ of height at most n, the localised ring $R_{\mathcal{P}}$ is a regular local ring.

The ring R is said to satisfy Serre's condition S_m if for all prime ideals $\mathcal{P}$ of R, the depth of $R_{\mathcal{P}}$ is at least $\min\{m, ht\mathcal{P}\}$.

Note that every zero dimensional ring satisfies S_m for every m.

Theorem 1. A Noetherian ring A is reduced if and only if it satisfies Serre's conditions R_0 and S_1..

Proof. A ring is reduced if its nilradical is zero, and the nilradical is the intersection of all the prime ideals in R. Condition $S_!$ is equivalent to saying that every associated prime is minimal for A. This follows from the theory of associated primes. We have the following facts: (i) The set of associated primes behaves well with respect to localisation. In other words,

$$Ass(S^{-1}A) = Ass(A) \cap \operatorname{Spec}(S^{-1}A).$$

In particulat,

$$\mathcal{P} \in Ass(A) \Leftrightarrow \mathcal{P}A_{\mathcal{P}} \in Ass(A_{\mathcal{P}}).$$

(ii) If $(A.m)$ is local, then $depth(A) = 0$ if and only if $m \in Ass(A)$. In particular,

$$\mathcal{P}A_{\mathcal{P}} \in Ass(A_{\mathcal{P}}) \Longleftrightarrow depth(A_{\mathcal{P}}) = 0.$$

Thus, if A satisfies S_1, then for any associated prime $\mathcal{P}$, we have

$$0 = \ depth(A_{\mathcal{P}} \geq \ min(1, dimA_{\mathcal{P}}) \Rightarrow dim\, A_{\mathcal{P}} = 0,$$

so $\mathcal{P}$ is minimal. Conversely, if $\mathcal{P}$ is nonminimal, then $\mathcal{P}$ is not in $Ass(A)$, so $depth(A_{\mathcal{P}}) \geq 1$, hence A satisfies S_1.

Now suppose that A is reduced. Then $A_{\mathcal{P}}$ is a field for every minimal prime ideal $\mathcal{P}$ of A. Consider the local ring $\mathcal{P}A_{\mathcal{P}}$, which is the maximal ideal in the local ring $A_{\mathcal{P}}$. This maximal ideal is in fact the radical of $A_{\mathcal{P}}$ since dimension of $A_{\mathcal{P}} = 0$. But A was reduced, and this property is preserved under localisation, so $A_{\mathcal{P}}$ is reduced. This implies that $\mathcal{P}A_{\mathcal{P}} = 0$, hence $A_{\mathcal{P}}$ is a field. Now A is reduced, so the intersection of all prime ideals of A is zero, and in particular, the intersection of all minimal prime ideals is zero. This gives us R_0.

Conversely, suppose A satisfies R_0 and S_1. By the above discussion $S_!$ implies that $Ass(A)$ consists only of minimal primes i of $Spec(A)$. Further, by condition R_0, for each $\mathcal{P}$ in $Ass(A)$, we have $A_{mathcalP}$ is a field. We proceed to show that A is reduced. By the theory of primary decomposition, we get an injection

$$A \hookrightarrow \prod A_{\mathcal{P}}$$

where the product on the right is taken over the set of prime ideals in $Ass(A)$. But each localisation $A_{\mathcal{P}}$ on the right is a field, as argued above, and hence the right hand side is reduced which implies that A is reduced. ■

Theorem 2. A Noetherian ring is normal if and only if it satisfies Serre's conditions R_1 and S_2..

Proof. Suppose A is a normal. domain. Then any localisation $A_{\mathcal{P}}$ of A at a prime ideal $\mathcal{P}$ of A is also normal. If $\mathcal{P} = 0$, then $A_{\mathcal{P}}$ is the quotient field and hence (R_1) and (S_2) both hold. If $\mathcal{P}$ is a prime of height one, then since A is normal the localisation $A_{\mathcal{P}}$ is a DVR, hence a regular local ring and condition (R_1) is satisfied. Further, note that the depth of any DVR is one as the maximal ideal is principal, so (S_2) is satisfied.

We may therefore assume that $\mathcal{P}$ is a prime of height at least 2. We must show that condition (S_2) holds, which amounts to showing that depth $(A_{\mathcal{P}}) \geq 2$. Replacing $A_{\mathcal{P}}$ by A, we may assume that A is local with maximal ideal $\mathcal{P}$ of height at least 2, so that dimension of A is at least 2. We must show that $depth(A) \geq 2$. As A is a domain, clearly $\text{depth}(A) \geq 1$. Pick a nonzerodivisor f of A and consider the ring A/f. We first make the following definition. Given an ideal I of A a prime ideal P of A is called a prime divisior of I if P is an associated prime of A/I, i.e. $P \in Ass(A/I)$. By the theory of associated primes, this means that there exists an element $x \in A \setminus I$ such that

$$P = \{p \in A |\, px \in I\}.$$

The proof is now a consequence of the following proposition.

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Proposition 3. Let R be an integral domain. Then R satisifes (S_2 if and only if every prime divisor of a nonzero principal ideal in R has height one.

Proof. Suppose R satisfies (S_2) and let $\mathfrak{q}$ be a prime divisor of the principal ideal (a) for some nonzero $a \in A$. Thus $a \in \mathfrak{q}$, and there exists $x \in R$ such that $\mathfrak{q}x \subset (a)$. Note that $x \notin (a)$, for if $x \in (a)$, this would imply $\mathfrak{q} = R$. Therefore $\{a\}$ is a maximal regular sequence in $R_{\mathfrak{q}}$ because any other element of $\mathfrak{q}R_{\mathfrak{q}}$ will be a zero divisor of $R_{\mathfrak{q}}/(a)$. Hence we have

$depth(R_{\mathfrak{q}}) = 1$. Since R satisfies (S_2), we have $ht(\mathfrak{q}) \leq 1$. But R is an integral domain and $\mathfrak{q} \neq 0$, hence height$(\mathfrak{q}) = 1$.

Conversely, suppose that every prime divisor of a nonzero principal ideal in R has height 1. Fix such a prime divisor $\mathfrak{q}$ in R. Then, by hypothesis, height$(\mathfrak{q}) = 1$. But the local ring $R_{\mathfrak{q}}$ is a domain as R is a domain, hence depth $R_{\mathfrak{q}} \geq 1$. Thus for such a prime divisor, we have $depth(R_{\mathfrak{q}}) \geq min\{ht(\mathfrak{q}, 2\}$. To complete the proof, we take a prime ideal $\mathfrak{q}$ in R which is not a prime divisor of a principal ideal, If $\mathfrak{q} \supset (a)$, and $x \notin (a)$, we have $\mathfrak{q}x \not\subset (a)$. This implies that $\{a, x\}$ is a regular sequence in $R_{\mathfrak{q}}$ and depth of $R_{\mathfrak{q}} \geq 2$, hence R satisfies S_2.■■

For more details on Associated primes and primary decomposition, please consult Atiyah-MacDonald's book.