Sample Midterm 1

Name:

Marks

Short-Answer Questions. Questions 1 and 2 are short-answer questions. Put your answer in the box provided. Full marks will be given for a correct answer placed in the box. Show your work also, for part marks. Each part is worth 3 marks, but not all parts are of equal difficulty. **Simplify your answers as much as possible in Questions 1 and 2.**

[9] **1.** Determine whether each of the following limits exists, and find the value if they do. If a limit below does not exist, determine whether it "equals" ∞, $-\infty$, or neither.

(a) [3] $\lim_{x \to -1} \frac{x^2 - x - 2}{x^2 - 1}$

$$= \lim_{x \to -1} \frac{\cancel{(x+1)}(x-2)}{\cancel{(x+1)}(x-1)}$$

$$= \lim_{x \to -1} \frac{x-2}{x-1}$$

$$= \frac{-1-2}{-1-1} = \frac{3}{2}$$

Answer $3/2$

(b) [3] $\lim_{t \to 1} \frac{\sqrt{t^2+8} - 3}{t - 1} \left(\frac{\sqrt{t^2+8}+3}{\sqrt{t^2+8}+3}\right)$

Answer $1/3$

$$= \lim_{t \to 1} \frac{t^2+8-9}{(t-1)(\sqrt{t^2+8}+3)}$$

$$= \lim_{t \to 1} \frac{\cancel{(t-1)}(t+1)}{\cancel{(t-1)}(\sqrt{t^2+8}+3)} = \frac{2}{\sqrt{9}+3} = \frac{1}{3}$$

(c) [3] $\lim_{x \to 0^-} \left(\frac{1}{x} - \frac{1}{|x|}\right)$

Answer 0

$$= \lim_{x \to 0^-} \frac{1}{x} - \frac{1}{x} = 0$$

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[12] **2.** (a) [3] Find $\frac{d}{dx}\left(\frac{x^4 + x^{7/2}}{x^2}\right)$. Remember (see the instructions above Question 1) that your answers must be completely simplified in Questions 1 and 2.

$$= \frac{(4x^3 + \frac{7}{2}x^{5/2})x^2 - 2x(x^4 + x^{7/2})}{(x^2)^2}$$

$$= \frac{4x^5 + \frac{7}{2}x^{9/2} - 2x^5 - 2x^{9/2}}{x^4}$$

$$= \frac{2x^5 + \frac{3}{2}x^{9/2}}{x^4} = 2x + \frac{3}{2}x^{1/2}$$

Answer: $2x + \frac{3}{2}x^{1/2}$

(b) [3] If $y = x^2 \cos x$, find the second derivative y''. Express your answer in the form $p(x)\sin x + q(x)\cos x$ where $p(x)$ and $q(x)$ are polynomials.

$y' = x^2 \sin x + 2x\cos x$

Answer: $p(x) = x^2,\ q(x) = 2x$

(c) [3] If $y = x\sin(\sqrt{x} + x)$, find y'.

$y' = x\cos(\sqrt{x} + x)\left(\frac{1}{2}x^{-1/2} + 1\right)$

Answer: $x\cos(\sqrt{x} + x)\left(\frac{1}{2}x^{-1/2} + 1\right)$

(d) [3] f is a function that satisfies $f'(e) = e$. Let $g(x) = f(e^{x^2})$. Find $g'(1)$.

$g'(x) = f'(e^{x^2})\, 2x e^{x^2}$

$g'(1) = f'(e)\, 2e = 2e^2$

Answer: $2e^2$

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Full-Solution Problems. In questions 3–7, justify your answers and **show all your work**. If a box is provided, write your final answer there. Unless otherwise indicated, **simplification of answers is not required in these questions.**

[7] **3.** Let $f(x) = \dfrac{\sqrt{x^2+1}}{3x+5}$.

(a) [4] Determine the horizontal asymptotes of the graph $y = f(x)$.

Answer: $y = \frac{1}{3}$

HA:

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{3x+5} \approx \lim_{x\to\infty} \frac{\cancel{x}}{3\cancel{x}} = \frac{1}{3}$$

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{3x+5} = -\frac{1}{3}$$

(b) [3] Determine the vertical asymptote(s) of the graph $y = f(x)$. For each vertical asymptote $x = a$, determine whether each of the one-sided limits "equals" ∞ or $-\infty$ as x approaches a.

$x = -\frac{5}{3}$ vert. asymptote

$$\lim_{x\to -5/3^+} \frac{\sqrt{x^2+1}}{3x+5} = \infty$$

$$\lim_{x\to -5/3^-} \frac{\sqrt{x^2+1}}{3x+5} = -\infty$$

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[6] **4.** Let a and b be constants, and define

$$f(x) = \begin{cases} (x^2 + b) & \text{if } x < 1 \\ ax + b & \text{if } 1 \leq x \leq 2 \\ 5x - 3 & \text{if } x > 2 \end{cases}$$

Find the values of a and b for which f is continuous at $x = 1$ and $x = 2$. **Fully** justify your answer.

Answer
$a = 1, b = 5$

$ax + b = 5x - 3$ at $x = 2$

$2a + b = 10 - 3 = 7$

$x^2 + b = ax + b$ at $x = 1$

$1 + \cancel{b} = a + \cancel{b} \Rightarrow a = 1$

$2 + b = 7 \Rightarrow b = 5$

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[4] **5.** Prove that the equation $x^3 - 3x + 1 = 0$ has at least two positive real solutions. Carefully cite any theorem you use, and justify why the theorem can be used.

$3x^2 - 3 = 0 \Rightarrow \pm 1$

Let $f(x) = x^3 - 3x + 1$

$f(-2) = -8 + 6 + 1 = -1 < 0$

$f(0) = 0 - 0 + 1 = 1 > 0$

$f(1) = 1 - 3 + 1 = -1 < 0$

By IVT, f has a root between -2 and 0 and 0 and 1

Since f is differentiable on $\mathbb{R}$

[6] **6.** Find the point(s) on the curve $y = x^2 + 6$ such that the tangent line(s) to the curve at these point(s) pass through the point $(2, 1)$.

Answer
$x_1 = -1, 5$

$y' = 2x$

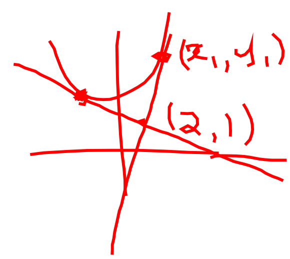
$(y - y_0) = m(x - x_0)$

$(y_1 - 1) = 2x_1(x_1 - 2)$

$x_1^2 + 6 - 1 = 2x_1^2 - 4x_1$

$0 = x_1^2 - 4x_1 - 5$

$= (x_1 - 5)(x_1 + 1)$



$y_1 = x_1^2 + 6$

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[6] **7.** Use the definition of the derivative in the questions below. **No marks** will be given for any other method. In this and any other question, you may use the back of a page if necessary, but please indicate so.

(a) [3] Find

$$\frac{d}{dx}\left(\underbrace{\frac{1}{1-x}}_{f(x)}\right)$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{1-(x+h)} - \frac{1}{1-x}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{1-(x+h)-(1-x)}}{h} = \lim_{h\to 0} \left(\frac{1}{\frac{h}{h}}\right) = \infty$$

(b) [3] Determine whether the function

$$f(x) = \begin{cases} x^3 \sin\frac{1}{x} & \text{if } x < 0 \\ 4x^2 - x & \text{if } x \geq 0 \end{cases}$$

is differentiable at $x = 0$.

$$\lim_{x\to 0} x^3 \sin\frac{1}{x} = 0 = \lim_{x\to 0} 4x^2 - x$$

deriv on left

$$x^3 \cos\frac{1}{x}\left(-\frac{1}{x^2}\right) + 3x^2 \sin\frac{1}{x} \xrightarrow{x\to 0} 0$$

deriv on right

$$4x - 1 \longrightarrow -1$$

$\neq$ not differentiable