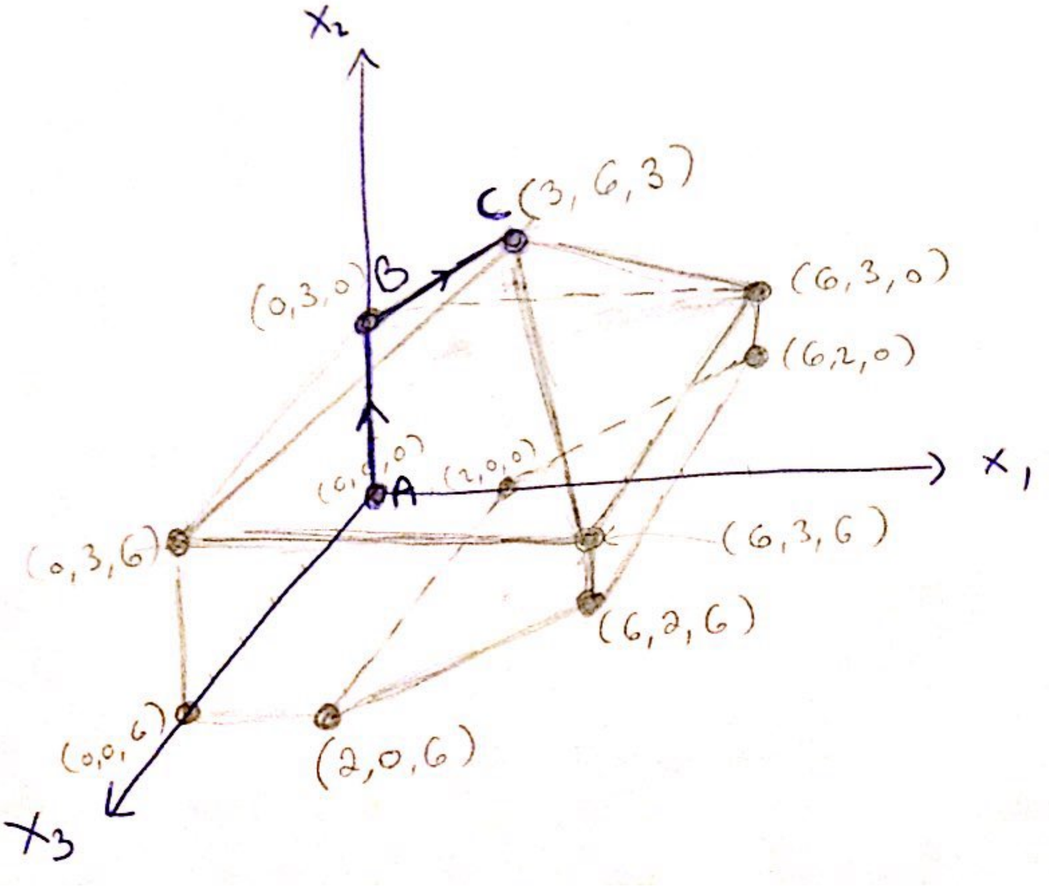
The simplex method cont. (geometry)

__Ex__ maximize $z = x_1 + 3x_2 + x_3 + 4$

s.t.

$$-x_1 + x_2 \le 3$$
$$x_1 + x_2 \le 9$$
$$x_1 \le 6$$
$$x_1 - 2x_2 \le 2$$
$$x_3 \le 6$$
$$x_2 - x_3 \le 3$$
$$x_2 + x_3 \le 9.$$

Here's a sketch of the domain:



A face corresponds to one constraint, and so to one variable equaling zero.

An edge is an intersection of two faces, and corresponds to two variables equaling zero (two constraints satisfyed at once)

Vertices correspond to three variables equaling zero, or to three non-basic variables.

So, a vertex corresponds to a dictionary.

A vertex is called a <u>basic solution</u> in the textbook.

The first dictionary corresponds to the origin (the vertex A in the figure):

$$
\begin{aligned}
x_4 &= 3 + x_1 - x_2 \\
x_5 &= 9 - x_1 - x_2 \\
x_6 &= 6 - x_1 \\
x_7 &= 2 - x_1 + 2x_2 \\
x_8 &= 6 \qquad\qquad\quad - x_3 \\
x_9 &= 3 \qquad\quad - x_2 + x_3 \\
x_{10} &= 9 \qquad\quad - x_2 - x_3 \\
\hline
z &= 4 + x_1 + 3x_2 + x_3
\end{aligned}
$$

We first pivot with x_2 entering the basis, and x_4 leaving it:

$$
\begin{aligned}
x_2 &= 3 + x_1 \phantom{{}- x_3} - x_4 \\
x_5 &= 6 - 2x_1 \phantom{{}- x_3} + x_4 \\
x_6 &= 6 - x_1 \\
x_7 &= 8 + x_1 \phantom{{}- x_3} - 2x_4 \\
x_8 &= 6 \phantom{{}+ x_1} - x_3 \\
x_9 &= - x_1 + x_3 + x_4 \\
x_{10} &= 6 - x_1 - x_3 + x_4 \\
\hline
z &= 13 + 4x_1 + x_3 - 3x_4.
\end{aligned}
$$

* Note we must have a negative coefficient in the objective function.

This dictionary corresponds to the point $(0, 3, 0)$.

Next, x_1 will enter the basis, and x_9 will leave.

$$x_1 = x_3 + x_4 - x_9$$
$$x_2 = 3 + x_3 - x_9$$
$$x_5 = 6 - 2x_3 - x_4 + 2x_9$$
$$x_6 = 6 - x_3 - x_4 + x_9$$
$$x_7 = 8 + x_3 - x_4 - x_9$$
$$x_8 = 6 - x_3$$
$$x_{10} = 6 - 2x_3 + x_9$$

$$z = 13 + 5x_3 + x_4 - 4x_9$$

This *still* corresponds to the point $(0,3,0)\ldots$

(Note z did not increase!)

Next, we pivot so that x_3 ~~leaves~~ enters the basis, and x_5 ~~enters~~ leaves.

$$x_3 = 3 - \tfrac{1}{2}x_4 + x_9 - \tfrac{1}{2}x_5$$
$$x_1 = 3 + \tfrac{1}{2}x_4 - \tfrac{1}{2}x_5$$
$$x_2 = 6 - \tfrac{1}{2}x_4 - \tfrac{1}{2}x_5$$
$$x_6 = 3 - \tfrac{1}{2}x_4 + \tfrac{1}{2}x_5$$
$$x_7 = 11 - 1\tfrac{1}{2}x_4 - \tfrac{1}{2}x_5$$
$$x_8 = 3 + \tfrac{1}{2}x_4 - x_9 + \tfrac{1}{2}x_5$$
$$x_{10} = + x_4 - x_9 + x_5$$

$$z = 28 - 1\tfrac{1}{2}x_4 + x_9 - 2\tfrac{1}{2}x_5$$

Point: $(3,6,3)$ = Vertex C

Next, x_9 enters the basis and x_{10} leaves:

$$x_9 = x_4 + x_5 - x_{10}$$
$$x_3 = 3 + \tfrac{1}{2}x_4 + \tfrac{1}{2}x_5 - x_{10}$$
$$x_1 = 3 + \tfrac{1}{2}x_4 - \tfrac{1}{2}x_5$$
$$x_2 = 6 - \tfrac{1}{2}x_4 - \tfrac{1}{2}x_5$$
$$x_6 = 3 - \tfrac{1}{2}x_4 + \tfrac{1}{2}x_5$$
$$x_7 = 11 - 1\tfrac{1}{2}x_4 - \tfrac{1}{2}x_5$$
$$x_8 = 3 - \tfrac{1}{2}x_4 - \tfrac{1}{2}x_5 + x_{10}$$

$$z = 28 - \tfrac{1}{2}x_4 - 1\tfrac{1}{2}x_5 - x_{10}$$

So our solution really was optimal!

Some conclusions:

1. The simplex method <u>can</u> stall. (take a step, and maybe many, that do not increase the objective function)
2. The real reason for this is the existence of vertices that lie on more

than three (or in general n) faces,
and ~~this~~ is called degeneracy.
3. This could result in cycling,
where the ~~sy~~ simplex method cycles
forever and doesn't terminate.