Practice Problems in Probability

1 Easy and Medium Difficulty Problems

Problem 1. Suppose we flip a fair coin once and observe either T for "tails" or H for "heads." Let X_1 denote the random variable that equals 0 when we observe tails and equals 1 when we observe heads. (This is called a *Bernoulli* random variable.)

(a) Make a table of the pdf of X_1 and calculate $\mathbb{E}(X_1)$ and $\mathrm{Var}(X_1)$.

(b) Now suppose that we flip a fair coin twice. We will observe one of four events: TT, TH, HT, or HH. Let X_2 be the random variable that counts the number of heads we observe. (So X_2 can take the values 0, 1, or 2.) Redo part (a) for this new random variable.

(c) Let X_3 be the random variable that counts the number of heads we observe after three successive flips of a fair coin. Redo part (a) for this new random variable.

(d) Do you notice any patterns in your calculations? Make a guess for the pdf table, the expectation and the variance of X_4, the random variable that counts the number of heads we observe after four successive flips of a fair coin, and then verify your guess by direct computation.

Problem 2. In general, for any positive integer n, the random variable defined in Problem 1 is called a *binomial* random variable. The pdf of the random variable is given by

$$p_k = \frac{n!}{k!(n-k)!}\left(\frac{1}{2}\right)^n,$$

where k is the number of heads (or tails) in n successive and independent flips of a fair coin. (Recall that the *factorial* notation denotes a product of integers: $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$.)

(a) If $n = 10$, calculate p_5. Interpret this probability.

(b) If $n = 100$, calculate p_1 and p_{99}. Interpret this result.

Problem 3. Recall the example of rolling a six-sided die. This is an example of a *discrete uniform* random variable, so named because the probability of observing each distinct outcome is the same, or uniform, for all outcomes. Let Y be the discrete uniform random variable that equals the face-value after a roll of an *eight*-sided die. (The die has eight faces, each with a number 1 through 8.) Calculate $\mathbb{E}(Y)$, $\mathrm{Var}(Y)$, and $\mathrm{StdDev}(Y)$.

Problem 4. Consider a slightly different coin tossing experiment. Suppose we toss a fair coin and continue to toss it until we first observe "heads". If we let Y denote the random variable that counts the number of tosses until we first observe heads, then we see the possible values of Y are $1,\ 2,\ 3, \ldots$. (This is an example of a *geometric* random variable.)

(a) Construct a pdf table for Y. Write down a formula for $\Pr(Y = y)$ for any positive integer y.

(b) Construct a cdf table for Y. Use this to deduce that $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = 1$.

Problem 5. In a manufacturing process, suppose that the probability that we produce a defective item is $\frac{1}{100}$. Let us observe the items on the production line until we find the first defective item. Denote this random variable by X and observe that X is a geometric random variable as in Problem 4.

(a) Construct a pdf table for X and compute $\Pr(X \leq 5)$. What is $\Pr(X = k)$ for any positive integer k?

(b) Write down the cdf for X; that is, write down a formula for $\Pr(X \leq k)$ for any positive integer k. Use this to show explicitly that $\lim_{k\to\infty} \Pr(X \leq k) = 1$.

(c) Calculate $\mathbb{E}(X)$.

Problem 6. Given the following table:

x	$\Pr(X \leq x)$
$(-\infty, 1)$	0
$[1, 3)$	$1/4$
$[3, 4)$	$3/7$
$[4, 7)$	$2/3$
$[7, 8)$	$7/8$
$[8, \infty)$	1

(a) Graph the cdf and pdf of X.

(b) Find a constant c such that $\Pr(X \le c) = \frac{7}{8}$.

(c) Find a constant c such that $\Pr(X < c) = \frac{2}{3}$.

(d) Find a constant c such that $\Pr(X > c) = \frac{4}{7}$.

(e) Find constants c_1, c_2 such that $\Pr(c_1 \le X < c_2) = \frac{3}{4}$.

Problem 7. Using properties of sums, show that $\mathrm{Var}(X) = \mathbb{E}(X) - [\mathbb{E}(X)]^2$, for any discrete random variable X.

Problem 8. Repeat Problem 7 for any continuous random variable X, using properties of integrals.

Problem 9. Are the following functions pdf's?

(a) $f(x) = \begin{cases} 12x^2(x-1) & \text{if } 0 < x < 1 \\ 0 & \text{if } \textit{elsewhere} \end{cases}$

(b) $f(x) = \begin{cases} 1 - |x| & \text{if } |x| \le 1 \\ 0 & \text{if } \textit{elsewhere} \end{cases}$

(c) $f(x) = \begin{cases} \frac{\pi}{2}\cos(\pi x) & \text{if } |x| < \frac{1}{2} \\ 0 & \text{if } \textit{elsewhere} \end{cases}$

(d) $f(x) = \begin{cases} \frac{1}{2} & \text{if } |x| \le 2 \\ 0 & \text{if } \textit{elsewhere} \end{cases}$

Problem 10. Are the following functions cdf's?

(a) $F(x) = \begin{cases} 0 & \text{if } x < -\frac{\pi}{2} \\ \frac{\sin(x-\pi/2)+1}{2} & \text{if } |x| \le \frac{\pi}{2} \\ 1 & \text{if } x > \frac{\pi}{2} \end{cases}$

(b) $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1-\cos(x)}{2} & \text{if } 0 \le x \le \pi \\ 1 & \text{if } x > \pi \end{cases}$

(c) $F(x) = \begin{cases} 0 & \text{if } x < -1 \\ 1 + x & \text{if } -1 \le x < 0 \\ x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$

(d) $F(x) = \arctan(x) + \frac{\pi}{2}$

Problem 11. Show that $e^{-x}e^{-e^{-x}}$ on $x \in \mathbb{R}$ is a pdf.

Problem 12. What is the cdf of the density function $\frac{1}{\pi(1+x^2)}$?

Problem 13. Show that $p(x) = \frac{e^{-x}}{(1+e^{-x})^2}$ on $x \in \mathbb{R}$ is a pdf.

Problem 14. Show that $f(x) = \begin{cases} 1 - e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ is a cdf.

Problem 15. Find the constant k that makes the following functions pdf's.

(a) $p(x) = k\sin(x)$, $0 < x < \pi$

(b) $p(x) = kx^2(x-1)^2$, $0 < x < 1$

(c) $p(x) = kx(1-x)^3$, $0 < x < 1$

(d) $p(x) = k$, $-1 \leq x \leq 3$

(e) $p(x) = kx^3e^{-\frac{x}{2}}$, $x \geq 0$.

Problem 16. Compute the expectations, variances and standard deviations of the following random variables.

(a) Z is a *standard normal* random variable (the classic "bell curve"), given by the density $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, $x \in \mathbb{R}$.

(b) X is given by the pdf $p(x) = \frac{3}{4}(1-x^2)$ on $|x| \leq 1$.

Problem 17. Expectations (and variances) need not be finite. Let Y be a *Cauchy* random variable given by the pdf $p(x) = \frac{1}{\pi(1+x^2)}$, for $x \in \mathbb{R}$.

(a) Prove that $\mathbb{E}(X)$ does not exist (i.e. the expectation is infinite).

(b) Prove that $\mathbb{E}(X^2)$ does not exist.

Problem 18. Let $p(x) = \frac{1}{\beta}e^{-\frac{x}{\beta}}$, for $0 \leq x < \infty$, $\beta > 0$, an *exponential* random variable with parameter β.

(a) Show that this density integrates to 1.

(b) Calculate $\Pr(X \geq 1)$, $\mathbb{E}(X)$ and $\mathrm{StdDev}(X)$, for the geometric random variable X.

(c) Sketch the pdf of X for $\beta = \frac{1}{10}, \frac{1}{2}, 1, 5$.

(d) Find the cdf of X in general.

Problem 19. Show that, for any continuous random variable, $\Pr(X \leq x) = \Pr(x < x)$.

Problem 20. Prove that $\Pr(X \leq x) = 1 - \Pr(X > x)$ first for discrete random variables, then for continuous random variables.

Problem 21. Let X be a *Laplace* random variable given by the pdf $p(x) = \frac{1}{2}e^{-|x|}$, for $x \in \mathbb{R}$.

(a) Verify that $p(x)$ is in fact a valid pdf.

(b) Calculate $\Pr(|X| \geq 1)$, $\mathbb{E}(X)$, and $\mathrm{StdDev}(X)$.

(c) Sketch the pdf of X.

(d) Find an explicit formula for the cdf of X.

Problem 22. Let Y be a (continuous) *uniform* random variable on the interval $[a, b]$.

(a) Graph the pdf and cdf of Y.

(b) Calculate $\mathbb{E}(Y)$, $\mathrm{Var}(Y)$, and $\mathrm{StdDev}(Y)$.

Problem 23. Show that $\mathbb{E}(X) \leq \sqrt{\mathbb{E}(X^2)}$.

Problem 24. The Beta Distribution k Value

Consider a beta distribution function with $a = 3$ and $b = 2$. Calculate the value of k so that the distribution is a PDF.
Hint: the beta distribution is defined at:
http://blogs.ubc.ca/math105/continuous-random-variables/the-beta-pdf

Problem 25.

For the following functions, determine whether the function represents a cumulative distribution function for a random variable X, and if it does, whether X is continuous or discrete.

1.

$$F(x) = \frac{1}{1 - x^2}$$

2.

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } 0 \leq x \leq 1, \\ 1 & \text{if } x > 1. \end{cases}$$

3.

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ 0.5 & \text{if } 0 \leq x \leq 1, \\ 1 & \text{if } x > 1. \end{cases}$$

A Few Challenging Problems

Problem 26. A *Poisson* random variable X is given by the pdf

$$\Pr(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \text{ for } \lambda \geq 0, \ x = 0, 1, 2, \ldots.$$

(a) Use Taylor series to show that $\sum_{x=0}^{\infty} \Pr(X = x) = 1$.

(b) Use Taylor series to show that $\mathbb{E}(X) = \lambda$.

(c) Use Taylor series to show that $\mathrm{Var}(X) = \lambda$. (Hint: Compute $\mathbb{E}[X(X-1)]$.)

Problem 27. Technically, a function $f(x)$ is a cdf if and only if it is nondecreasing, $\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to\infty} f(x) = 1$ *and* if $f(x)$ is continuous from the right. For example, the function $f_1(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0 \end{cases}$ is a cdf, while the function $f_2(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0 \end{cases}$ is not a cdf. Considering these examples, why does this technical requirement make intuitive sense?

Problem 28. Recall that the average value of a function on the interval $[a, b]$ is given by $f_{avg}(a, b) = \frac{1}{b-a}\int_a^b f(x)dx$. Let X be a (continuous) uniform random variable on $[a, b]$. Define the new random variable $Y = f(X)$ for any strictly increasing function f. Show that $\mathbb{E}(Y) = f_{avg}(a, b)$. Interpret this result.