Sample Midterm 1

Name:

Marks

Short-Answer Questions. Questions 1 and 2 are short-answer questions. Put your answer in the box provided. Full marks will be given for a correct answer placed in the box. Show your work also, for part marks. Each part is worth 3 marks, but not all parts are of equal difficulty. **Simplify your answers as much as possible in Questions 1 and 2.**

[9] **1.** Determine whether each of the following limits exists, and find the value if they do. If a limit below does not exist, determine whether it "equals" ∞, $-\infty$, or neither.

(a) [3] $\lim_{x \to -1} \frac{x^2 - x - 2}{x^2 - 1}$

$$= \lim_{x \to -1} \frac{(x-2)\cancel{(x+1)}}{(x-1)\cancel{(x+1)}}$$

$$= \frac{-3}{-2}$$

Answer $\frac{3}{2}$

(b) [3] $\lim_{t \to 1} \frac{\sqrt{t^2+8} - 3}{t - 1} \cdot \frac{\sqrt{t^2+8}+3}{\sqrt{t^2+8}+3}$

$$= \lim_{t \to 1} \frac{t^2+8-9}{(t-1)(\sqrt{t^2+8}+3)}$$

$$= \lim_{t \to 1} \frac{t^2-1}{(t-1)(\sqrt{t^2+8}+3)}$$

$$= \lim_{t \to 1} \frac{\cancel{(t-1)}(t+1)}{\cancel{(t-1)}(\sqrt{t^2+8}+3)}$$

$$= \frac{2}{\sqrt{9}+3} = \frac{2}{6}$$

Answer $\cancel{\frac{2}{6}}$ $\frac{1}{3}$

(c) [3] $\lim_{x \to 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right) = +\infty$

$$\frac{|x| - x}{x|x|}$$

Answer $+\infty$

Continued on page 3

[12] **2.** (a) [3] Find $\frac{d}{dx}\left(\frac{x^4 + x^{7/2}}{x^2}\right)$. Remember (see the instructions above Question 1) that your answers must be completely simplified in Questions 1 and 2.

Answer $2x + 2x^{1/2} - x^2 + x^{5/2}$

$$\frac{d}{dx} = \frac{(2x)(x^4 + x^{7/2}) - x^2(x^4 + x^{7/2})}{x^4}$$

$$= \frac{2x^5 + 2x^{9/2} - x^6 + x^{11/2}}{x^4} = 2x + 2x^{1/2} - x^2 + x^{5/2}$$

(b) [3] If $y = x^2 \cos x$, find the second derivative y''. Express your answer in the form $p(x)\sin x + q(x)\cos x$ where $p(x)$ and $q(x)$ are polynomials.

Answer $y'' = (2+x^2)\cos x - 4x\sin x$

$y' = 2x\cos x - x^2 \sin x$

$y'' = 2\cos x - 2x\sin x - 2x\sin x - x^2\cos x$

(c) [3] If $y = x\sin(\sqrt{x} + x)$, find y'.

Answer $y' = \sin(\sqrt{x} + x) + x\cos(\sqrt{x} + x)$

$y' = \sin(\sqrt{x} + x)$
$+ x\cos(\sqrt{x} + x)$.

(d) [3] f is a function that satisfies $f'(e) = e$. Let $g(x) = f(e^{x^2})$. Find $g'(1)$.

Answer $g'(1) = 1$

$\ln g(x) = \ln f(e^{x^2})$
$= f(x^2)$

Continued on page 4

Full-Solution Problems. In questions 3–7, justify your answers and **show all your work**. If a box is provided, write your final answer there. Unless otherwise indicated, **simplification of answers is not required in these questions.**

[7] **3.** Let $f(x) = \dfrac{\sqrt{x^2+1}}{3x+5}$.

(a) [4] Determine the horizontal asymptotes of the graph $y = f(x)$.

Answer $y = \frac{1}{3}$

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{3x+5} = \lim_{x\to\infty} \frac{\frac{1}{x}\sqrt{x^2+1}}{\frac{1}{x}(3x+5)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^2}x^2+\frac{1}{x^2}}}{3+\frac{5}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{x^2}}}{3+\frac{5}{x}}$$

$$= \frac{\sqrt{1+0}}{3+0}$$

$$= \frac{\sqrt{1}}{3}$$

$$= \frac{1}{3}$$

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{3x+5} = \frac{1}{3}$$

by same argument

(b) [3] Determine the vertical asymptote(s) of the graph $y = f(x)$. For each vertical asymptote $x = a$, determine whether each of the one-sided limits "equals" ∞ or $-\infty$ as x approaches a.

$3x+5=0$

$x = -\frac{5}{3}$ asymptote

$$\lim_{x\to-\frac{5}{3}} \sqrt{x^2+1} = \sqrt{\frac{25}{9}+1}$$

$$\lim_{x\to-\frac{5}{3}} 3x+5 = 0$$

$$\lim_{x\to-\frac{5}{3}} \frac{\sqrt{x^2+1}}{3x+5} = \infty \qquad \text{looks like} \left(\frac{\sqrt{\frac{25}{9}+1}}{0}\right)$$

Continued on page 5

[6] 4. Let a and b be constants, and define

$$f(x) = \begin{cases} (x^2 + b) & \text{if } x < 1 \\ ax + b & \text{if } 1 \leq x \leq 2 \\ 5x - 3 & \text{if } x > 2 \end{cases}$$

Find the values of a and b for which f is continuous at $x = 1$ and $x = 2$. **Fully justify your answer.**

Answer
$a=1,\ b=6$

Need $x^2+b = ax+b$ at $x=1$

$ax+b = 5x-3$ at $x=2$

$1+b = a+b$

$2a+b = \cancel{10-3}\ 8$

$\boxed{a=1}$

$2(1)+b=8$

$\boxed{b=6}$

Continued on page 6

[4] **5.** Prove that the equation $x^3 - 3x + 1 = 0$ has at least two positive real solutions. Carefully cite any theorem you use, and justify why the theorem can be used.

$x^3 - 3x + 1$ is a continuous polynomial

x	$x^3 - 3x + 1$
0	1
1	$1 - 3 + 1 = -1$
2	$8 - 6 + 1 = 3$

← solution between 0 and 1 by IVT

← solution between 1 and 2 by IVT

[6] **6.** Find the point(s) on the curve $y = x^2 + 6$ such that the tangent line(s) to the curve at these point(s) pass through the point $(2, 1)$.

Answer

$y - 2 = 2(x - 1)$

$y' = 2x$

$m = y'(2) = 2(2) = 4$

$\cancel{y = mx}$

$y - y_1 = m(x - x_1)$

$y - 2 = 2(x - 1)$

$\cancel{y = mx +}$

$\cancel{y - y_1}$

$\cancel{y = (2x)}$

Continued on page 7

[6] 7. Use the definition of the derivative in the questions below. **No marks** will be given for any other method. In this and any other question, you may use the back of a page if necessary, but please indicate so.

(a) [3] Find

$$\frac{d}{dx}\left(\frac{1}{1-x}\right)$$

$$= \lim_{h \to 0} \frac{\frac{1}{1-(x+h)} - \frac{1}{1-x}}{h} = \lim_{h \to 0} \frac{1-x-(1-(x+h))}{h(1-(x+h))(1-x)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(1-(x+h))(1-x)}$$

$$= \lim_{h \to 0} \frac{1}{1+(x+h)(x)-(x+h)-x}$$

$$= \cancel{\lim_{h \to 0}} \frac{1}{1+x^2-x-x} = \frac{1}{1-2x+x^2}$$

(b) [3] Determine whether the function

$$f(x) = \begin{cases} x^3 \sin \frac{1}{x} & \text{if } x < 0 \\ 4x^2 - x & \text{if } x \geq 0 \end{cases}$$

is differentiable at $x = 0$.

$$\frac{d}{dx} x^3 \sin\frac{1}{x}\Big|_{x=0} = 3x^2 \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right)\Big|_{x=0} = \text{DNE}$$

$$\frac{d}{dx} 4x^2 - x\Big|_{x=0} = 8x\Big|_{x=0} = 0$$

Not differentiable at 0

The End