Corollary 6.4: If we have a set of $\{M_1, \cdots, M_n\}$ Noetherian A-modules. so is $\bigoplus_{i=1}^n M_i$.

proof: ~~Apply induct~~
Base case:

$$0 \longrightarrow M_2 \longmapsto M_1 \oplus M_2 \longrightarrow M_1 \longmapsto 0$$

$M_1 \oplus M_2$ Noetherian

Apply induction to the exact seq.

$$0 \longrightarrow M_n \longmapsto \bigoplus_{i=1}^n M_i \longmapsto \bigoplus_{i=1}^{n-1} M_i \longrightarrow 0$$

Prop 6.5. Let A be a Noetherian ring. M f.g. A-mod. Then M is Noetherian ~~ring~~ A-module.

proof. we have the homomorphism. Let $\{x_1, \cdots, x_n\}$ be set of generators of M.

$$\varphi: A^n \longrightarrow M$$
$$(a_1, \cdots, a_n) \longrightarrow \sum_{i=1}^n a_i x_i$$

φ is homo between A-mod, $M \cong A^n / \ker\varphi$.

by Cor 6.4. A^n is Noetherian A-module.

by Prop 6.3 and the exact seq. $0 \longrightarrow \ker\varphi \longmapsto A^n \longrightarrow A^n/\ker\varphi \longmapsto 0$

$\Rightarrow$ $\underbrace{A^n/\ker\varphi}_{\cong M}$ is Noetherian A-module.

and we're done :).

Prop 7.2: $A \underset{\text{subring}}{\subseteq} B$, Suppose A is Noetherian and that B is f.g. as A-mod. B is a Noetherian ring.

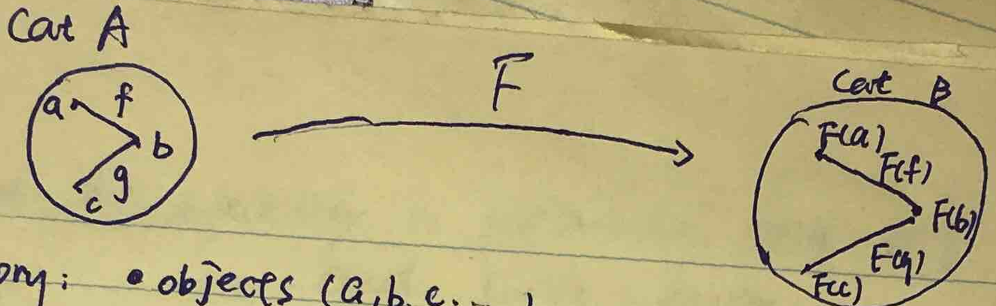
proof: by prop 6.5. B is Noetherian A-module.

$\Leftrightarrow$ every ideal of B is f.g. A-modules

$\Leftrightarrow$ every ideal of B is f.g. B-modules
(multiplication work for all of B)

$\Leftrightarrow$ B is Noetherian B-module (with generator 1)

$\Leftrightarrow$ B is a Noetherian ring.



Category: • objects $(a, b, c \ldots)$
• morphism between objects $(f, g \ldots)$
• identity morphism
• composition of morphisms satisfying associativity.

Briefly, the Functor F is the morphism between 2 different categories, which preserves structures of Categories.
• objects $\{a, b, c\}$ $\longrightarrow$ objects $\{F(a), F(b), F(c), \cdots\}$
• morphisms in Cat A are mapped to morphisms in Cat B, and laws of composition preserved.

So when we have ~~a~~ the useful tool s.e.s. it make sense to apply some functor to it, (that is apply some functor to the Cat of modules, apply it to the objects M_i, and morphism i, p.)

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r, hook, "i"] \arrow[d] & M_2 \arrow[r, "p"] & M_3 \arrow[r] & 0 \\ & F(M_1) \arrow[r, "F(i)"] & F(M_2) \arrow[r, "F(p)"] & F(M_3) \end{tikzcd}$$

then the very natural question to ask is that after apply the functor do we still get the exact sequence? or what kind of Functor preserve the exactness ?

We know the <u>Fact from Class:</u>
M a R-mod, $\mathrm{Hom}_R(M, -)$ is left exact.
i.e. if we have s.e.s. $0 \to M_1 \xrightarrow{i} M_2 \xrightarrow{p} M_3 \to 0$

$$0 \to \mathrm{Hom}(M, M_1) \xrightarrow{i_*} \mathrm{Hom}(M, M_2) \xrightarrow{p_*} \mathrm{Hom}(M, M_3)$$

$$f \xmapsto{i_*(f)} i \circ f$$

$$g \xmapsto{p_*(f) = p \circ f} p \circ g$$

where i_*, p_* are induced maps.

projective.

then the injectivity is preserved ($\ker i_* = \{0\}$)
and $\mathrm{Im}\, i_* = \ker p_*$

<u>Check</u>: (1) $\ker i_* = \{0\}$

If $f \in \ker i_*$, $f: M \to M_1$, $i_*(f) = 0 \overset{\text{by def}}{\Longrightarrow} i \circ f(x) = 0$ for $\forall x \in M$.
as i injective. it must be $f(x) = 0$ $\forall x \in M$.

$\Rightarrow f$ is zero map $\Rightarrow \ker i_* = \{0\}$

(2) $\mathrm{Im}\, i_* \subseteq \ker p_*$, if $g \in \mathrm{Im}\, i_*$, $g: M \to M_2$, $\exists f \in \mathrm{Hom}(M, M_1)$, $g = i_*(f) = i \circ f$

$p_*(g) = p \circ g = p(\underbrace{i f}_{\in \ker p}) = 0$

(3) $\ker p_* \subseteq \mathrm{im}\, i_*$.

If $g \in \ker p_*$, $g: M \to M_2$, $\overset{p_*(g)=0}{pg(x) \equiv 0}$, $\forall x \in M$, $\Rightarrow g(x) \in \ker p = \mathrm{Im}\, i$.

for $\forall x \in M$.

$g(x) = i(a)$ for some $a \in M_1$, a is unique since i injective.
Hence, the function $f: M \to M_1$ given by $f(x) = a$ if $g(x) = i(a)$ is well-defined
It's easy to check that $f \in \mathrm{Hom}_R(M, M_1)$; that is, f is an R-hom by g is a R-hom.

$i_*(f) = if$. $\Rightarrow if(x) = i(a) = g(x)$, $\forall x \in M \Rightarrow i_*(f) = g$. $g \in \mathrm{Im}\, i_*$

We'll see the definition of a Projective module P leads directly to "$\mathrm{Hom}_R(P, -)$ is an exact functor"

<u>Def (Projective mod)</u> A left R-mod P is projective if whenever we have surjective R-hom of R-mods, $p: M_1 \to M_2$, and h is any map from P to M_2, $\exists$ lifting g, that the diagram commutes

$$\begin{tikzcd} & P \arrow[dl, dashed, "g"'] \arrow[d, "h"] \\ M_1 \arrow[r, "p"'] & M_2 \end{tikzcd}$$

i.e. $p \circ g = h$.

Prop 3.2. A left R-mod P is Surjective iff $Hom_R(P,-)$ is an exact functor.

proof: it suffices to prove that $Hom_R(P,-)$ is right exact.
i.e. it suffices to show p_* is ~~projective~~ surjective whenever p is projective

by def,

P projective:

$$\begin{tikzcd} & & & P \arrow[dl, dashed, "g"'] \arrow[d, "h"] & \\ 0 \arrow[r] & M_1 \arrow[r, "i"] & M_2 \arrow[r, two heads, "p"'] & M_3 \arrow[r] & 0 \end{tikzcd}$$

any $h: P \to M_3$, ∃ a lifting $g: P \to M_2$ s.th. $p \circ g = h$

$\Longleftrightarrow$ (induce $0 \to Hom_R(P, M_1) \xrightarrow{i_*} Hom(P, M_2) \xrightarrow{p_*} Hom_R(P, M_3)$

$\forall\, h \in Hom_R(P, M_3)$, $\exists\, g \in Hom(P, M_2)$ s.th. $pg = p_*(g) = h$.

$\Longleftrightarrow$ p_* is surjective.

(Finding ~~examples of~~ non-trivial examples of projective modules leads to algebraic k-theory.

Example of projective modules: F free left R-modules

$$\begin{tikzcd} & F \arrow[dl, dashed] \arrow[d, "h"] \\ M \arrow[r, two heads, "p"'] & M' \end{tikzcd}$$

p projective, h any map $h: F \to M'$

B be a basis of F, $h(b) \in M'$, p surjective

$h(b) = p(m_b)$, for some $m_b \in M$.

by axiom of choice: ∃ a function $u: B \to M$ $u(b) = m_b$

u from the Basis B to M can be lifted to $g: F \to M$.

$$\begin{tikzcd} F \arrow[dr, dashed, "g"] & \\ B \arrow[u, "i"] \arrow[r, "u"'] & M \end{tikzcd}$$

$\Rightarrow$ the diagram commute, F is projective.

Def: s.e.s $0 \to M_1 \xrightarrow{i} M_2 \xrightarrow{p} M_3 \to 0$ split.
if ∃ an inverse j s.th. $p \circ j = id_C$

p becomes a retract, i.e. M_3 is a summand of M_2.

$$M_2 \cong M_1 \oplus M_3$$
$$\cong M_1 \oplus M_2/M_1$$

Prop 3.3. A left R-module P is projective if and only if every s.e.s. $0 \to M_1 \xrightarrow{i} M_2 \xrightarrow{p} P \to 0$ splits.

Proof: if P is projective, ∃ $g: P \to \cancel{B} M_2$ with $1_P = p_*(f) = pj$.
that is, P is a retract of B. ~~[illegible]~~ ~~[illegible]~~.

$$\begin{tikzcd} & P \arrow[dl, "g"'] \arrow[d, "1_P"] & \\ M_2 \arrow[r, "p"'] & P \arrow[r] & 0 \end{tikzcd}$$

Conversely, assume that every s.e.s. ending with P splits.
Consider the diagram

$$\begin{tikzcd} F \arrow[r, shift left, "f"] \arrow[d, "g"'] & P \arrow[l, shift left, "j"] \arrow[r] \arrow[d, "h"] & 0 \\ M_1 \arrow[r, "p"'] & M_2 \arrow[r] & 0 \end{tikzcd}$$

there ∃ a Free R-mod F and a surjection $f: F \to P$, F is projective, ∃ $g_0: F \to M$, s.th. $p \circ g = \cancel{h\circ f}\, f \circ h$.
By assumption, ∃ $j: P \to F$ s.th. $h \circ j = I_P$
we have a commutative diagram, and so P is projective.