2 (a) The initial dictionary for this LP is:

$$\begin{array}{l} z = \quad 4x_1 + x_2 - x_3 \\ \hline x_4 = 3 + 4x_1 - x_2 - 2x_3 \\ x_5 = 3 + x_1 + 3x_2 - 2x_3 \\ x_6 = -1 - x_1 - x_2 + x_3 \end{array}$$

Since $b = \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix} \not\geq 0$, this dictionary is <u>not primal feasible</u>.

Since $c = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} \not\leq 0$, this " " <u>not dual feasible</u>.

(b) We will device an auxiliary LP by changing the objective function (only) of the original LP so that the aux LP ~~is dual~~ has an ~~dual~~ dual feasible initial dictionary.

$\omega = c^T \cdot x$ with any $c \leq 0$ would do.

For example : $\omega = -x_1$, $\omega = -x_2$, $\omega = -x_3$

$\omega = -15x_1 - 20x_2 - x_3$, ...

(c) <u>Phase I (alternative)</u>

<u>LP-aux</u> has the initial dictionary:

$$\begin{array}{l} \omega = \quad -x_1 - x_2 - x_3 \\ \hline x_4 = 3 + 4x_1 - x_2 - 2x_3 \\ x_5 = 3 + x_1 + 3x_2 - 2x_3 \\ x_6 = -1 - x_1 - x_2 + x_3 \end{array}$$

$\rightsquigarrow$ dual feasible

<u>Leaving var</u>: <u>x_6</u> (as $-1 < 0$)

Entering var: $[-1,-1,-1] + s\,[-1,-1,1] \le 0$

$$\Leftrightarrow \quad \begin{array}{ccc} x_1 & x_2 & x_3 \\ \end{array}$$
$$[-1-s,\ -1-s,\ -1+s] \le 0$$

$$\Downarrow$$
$$s \le 1$$
$$\Downarrow$$

x_3 enters

Update the dictionary:

$$\begin{aligned} w &= -1 - 2x_1 - 2x_2 - x_6 \\ \hline x_4 &= 1 + 2x_1 - 3x_2 - 2x_6 \\ x_5 &= 1 - x_1 + x_2 - 2x_6 \\ x_3 &= 1 + x_1 + x_2 + x_6 \end{aligned}$$

This dictionary is optimal for the LP-aux.

(The optimal sol. of the LP-aux is not relevant for our purposes).

(d) We now use the optimal dictionary of LP-aux for our original problem after updating the original objective function (in terms of the non-basic variables x_1, x_2, x_6):

$$z = 4x_1 + x_2 - 2x_3 = 4x_1 + x_2 - 2(1 + x_1 + x_2 + x_6)$$
$$= -2 + 2x_1 - x_2 - 2x_6$$

Then: Phase II:

$$\left.\begin{aligned} z &= -2 + 2x_1 - x_2 - 2x_6 \\ \hline x_4 &= 1 + 2x_1 - 3x_2 - 2x_6 \\ x_5 &= 1 - x_1 + x_2 - 2x_6 \\ x_3 &= 1 + x_1 + x_2 + x_6 \end{aligned}\right\}$$

primal feasible! x_1 enters, x_5 leaves

$$\begin{array}{rl}
z = & x_2 - 2x_5 - 6x_6 \\
\hline
x_4 = & 3 - x_2 - 2x_5 - 6x_6 \\
x_1 = & 1 + x_2 - x_5 - 2x_6 \\
x_3 = & 2 + 2x_2 - x_5 - x_6
\end{array}$$

x_2 enters; x_4 leaves

$$\begin{array}{rl}
z = & 3 - x_4 - 4x_5 - 12x_6 \\
\hline
x_2 = & 3 - x_4 - 2x_5 - 6x_6 \\
x_1 = & 4 - x_4 - 3x_5 - 8x_6 \\
x_3 = & 8 - 2x_4 - 5x_5 - 13x_6
\end{array}$$

optimal!

Optimal solution of the original LP: $x_1^* = 4$; $x_2^* = 3$; $x_3^* = 8$;
$(x_5^* = x_6^* = x_4^* = 0)$

Optimal value of the objective: $z^* = 3$