Probability

MATH 105

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About

The Probability Module is a set of online resources that support students in acquiring knowledge and skills in working with probability for MATH 105 at The University of British Columbia [1], Vancouver. The module itself has no formal instructor. UBC students who have questions regarding content on this site are encouraged to contact their instructor or TA.

Topics Covered in the Module

The topics in the module are organized into three "lessons".

1. Discrete Random Variables
   - the mean of a discrete random variable
   - the variance of a discrete random variable
   - formulas for the expected value and variance of a discrete random variable
   - the probability density histogram of a discrete random variable
2. Continuous Random Variables
   - the probability density function (PDF) for a continuous random variable
   - the beta PDF
   - the cumulative distribution function
3. Expected Value and Variance
   - the expected value of a continuous random variable
   - the variance of a continuous random variable

Learning Objectives

Objective 1. [Procedural/Conceptual] Be able to calculate and interpret probabilities of events for discrete and continuous random variables given either a density function (pdf) or a distribution function (cdf).

Objective 2. [Procedural] Be able to express a pdf in a table or as an explicit function given sufficient information about the random variable.

Objective 3. [Procedural] Be able to verify that a given function is a pdf or cdf; be able to find multiplicative constants that make a given function a pdf or cdf.

Objective 4. [Conceptual] Be able to distinguish between discrete and continuous random variables given a simple experiment, pdf or cdf.

Objective 5. [Conceptual/Procedural] Be able to define and interpret the CDF of a random variable in terms of probabilities and to know basic properties of the cdf. Be able to find a cdf from a pdf.

Objective 6. [Conceptual/Procedural] Be able to relate the area under a pdf to the probability of an event for a particular random variable and to know basic properties of the pdf. Be able to find a pdf from a cdf.

Objective 7. [Procedural/Conceptual] Be able to define, calculate and interpret the expectation, variance and standard deviation of a random variable (discrete and continuous).

Discrete vs Continuous Random Variables

In MATH 105, we deal with

- discrete random variables, and
- continuous random variables.

These topics and their definitions will be explored in this module, and we will briefly explore discrete random variables first. Our introduction of discrete random variables is only to introduce the notions of probability, mean and variance, that we will use when discussing continuous random variables.

Throughout this module, concepts are introduced assuming no prior knowledge of probability. However, there are concepts that are discussed with an assumption of prior knowledge of integration. Essentially, probability is included in the MATH 105 curriculum as an application of integration.

References

[1] http://ubc.ca/

Lesson 1 DRV

What This Lesson Covers

In In Lesson 1 we introduce

- discrete random variables
- the definition of a probability density function (PDF)
- the mean and variance of a discrete PDF
- the cumulative distribution function (CDF)

Learning Outcomes

After reading this lesson and completing a sufficient number of problems, students should be able to meet the first two learning objectives:

- Objective 1. [Procedural/Conceptual] Be able to calculate and interpret probabilities of events for discrete and continuous random variables given either a density function (PDF) or a distribution function (CDF).
- Objective 2. [Procedural] Be able to express a pdf in a table or as an explicit function given sufficient information about the random variable.

Simply reading the content in this lesson will not be sufficient: students will need to complete a sufficient number or WeBWorK and paper-based problems in order to prepare themselves for the final exam.

1.01 Discrete Random Variables

In many areas of science we are interested in quantifying the **probability** that a certain outcome in an experiment occurs. To quantify the probability that an event occurs, we use a number between 0 and 1 that represents how likely that event occurs. A probability of 0 implies that the outcome *cannot* occur, whereas a probability of 1 implies that the outcome *must* occur. Any value in the interval (0, 1) means that the outcome only occurs some of the time.

When there is a discrete list of probabilities that can occur, we use the notation p_k to denote the probability that event k will occur.

Discrete Probability Rules
In discrete probability, 1. Probabilities are numbers between 0 and 1: $0 \leq p_k \leq 1$ for all k 2. The sum of all probabilities for a given experiment is equal to one: $\sum_k p = 1$

Example: Tossing a Fair Coin Once

If we toss a coin into the air, there are only two possible outcomes: it will land as either "heads" (H) or "tails" (T). If the tossed coin is a "fair" coin, it is equally likely that the coin will land as tails or heads. In other words, there is a 50% chance that the coin will land heads, and a 50% chance that the coin will land as tails.

Using our notation for probability, we can assign

- p_1 to be the probability that the tossed coin will land as heads
- p_2 to be the probability that the tossed coin will land as tails

Because there are two outcomes that are equally likely, we assign the probability of 0.5 to each of them.

- $p_1 = 0.5$
- $p_2 = 0.5$

As required, the sum of the probabilities equals 1, and each probability is a number in the interval [0, 1].

Example: Tossing a Fair Coin Twice

Similarly, if we toss a fair coin two times, there are four possible outcomes. Each outcome is a sequence of heads (H) or tails (T):

- HH
- HT
- TH
- TT

Using our notation for probability, we can assign

- p_1 to be the probability that the outcome will be HH
- p_2 to be the probability that the outcome will be HT
- p_3 to be the probability that the outcome will be TH
- p_4 to be the probability that the outcome will be TT

Because the coin is fair, each outcome is equally likely to occur. There are 4 possible outcomes, so we assign each outcome a probability of 1/4 = 0.25. That is, $p_1 = p_2 = p_3 = p_4 = 0.25$.

Again, all of our probabilities sum to 1, and each probability is a number on the interval [0, 1].

1.02 Notation

Random Variables and their Observed Values

We use uppercase letters to denote random variables, and lowercase letters to denote particular values that our random variables can have.

For example, consider a six sided die, pictured below.

We could let X be any one of the six possible values that *could* be observed on the upper face after a die is rolled, and we can let x denote the observed value *after* the die is rolled. X is a random variable, whereas x is not a random variable.

Probabilities of Discrete Random Variables

On the previous page, we used the notation p_1 to denote the probability that a tossed coin would land as "heads". In general, we can use the notation p_k to denote the probability that a random variable were equal to a particular value.

Sometimes it is more convenient to use the notation $\Pr(X = x)$ to denote the probability that a given random variable X is equal to x. Similarly, $\Pr(X \leq x)$ would denote the probability that the random variable X is less than or equal to the value x.

The Pr Notation
$\Pr(a \leq X \leq b)$ denotes the probability that the random variable X lies between values a and b.

6-Sided Die Example

Using our 6-sided dice example above, we could let p_k denotes the probability that a tossed die lands with the number *k* facing upwards. Then the proabability that *X* is equal to 5 can be denoted as:

$$p_5 = \Pr(X = 5) = \frac{1}{6}$$

1.03 The Discrete PDF

Usually we are interested in experiments where there is more than one outcome, each having a different probability. The **probability distribution** of a discrete random variable provides the all of the probabilities p_k for every possible value of *k*.

Discrete Probability Distribution
The discrete probability distribution can be represented in a table, graph, or formula, and provides the probabilities p_k for all possible values of *k*.

Example: Different Coloured Balls

The variable *X* does not have to represent a numerical value. Suppose that a box contains 10 balls:

- 5 of the balls are red
- 2 of the balls are green
- 2 of the balls are blue
- 1 ball is yellow

Suppose we take one ball out of the box. Let *X* could be the random variable that represents the colour of the ball. As 5 of the balls are red, and there are 10 balls, the *probability* that a red ball is drawn from the box is 5/10 = 1/2.

Similarly, there are 2 green balls, the probability that *X* is green is 2/10. Similar calculations for the other colours yields the probability distribution function, that is given by the following table.

Ball Colour	Probability
red	5/10
green	2/10
blue	2/10
yellow	1/10

Example: A Six-Sided Die

Consider again the experiment of rolling a six-sided die. A six-sided die can land on any of its six faces, so that a single experiment has six possible outcomes.

For a "fair die", we anticipate getting each of the results with an equal probability, i.e. - if we were to repeat the same experiment many many times, we would expect that, on average, the six possible events would occur with similar frequencies (we say that the events are **random and unbiased** for fair dice).

There are six possible outcomes. We can let

- x_1 be the outcome that a 1 is rolled,
- x_2 be the outcome that a 2 is rolled,

and so on up to x_6. The probability distribution function could be given by the following table.

Outcome	Probability
x_1	1/6
x_2	1/6
x_3	1/6
x_4	1/6
x_5	1/6
x_6	1/6

The PDF could also be given by the equation $p_k = 1/6$, for $k = 1, 2, 3, \ldots, 6$.

1.04 The Expected Value of a Discrete PDF

An important quantity related to a given random variable is its **expected value**.

Definition: Expected Value of a Discrete Probability Distribution
The expected value, $E(X)$, of a probability distribution for a random variable X is calculated by summing the probabilities of each possibile outcome by the corresponding outcome $E(X) = \sum_{k=1}^{N} x_k p_k$ where N is the number of possible outcomes.

Note the following:

- Do not confuse the *expected* value with the *average* value of a set of observations: they are two different concepts. The expected value is calculated as a weighted sum of probabilities, whereas the average value is a weighted sum of frequencies (more on this distinction later).
- The expected value equation requires numerical values for the x_k. So if the outcome for an experiment is either "heads" or "tails", we could calculate the expected value if we assign heads and tails numerical values (0 and 1, for example).

Example: Grade Distributions

Suppose that in a class of 10 people the grades on a test are given by 30, 30, 30, 60, 60, 80, 80, 80, 90, 100. Suppose a test is drawn from the pile at random.

1. Calculate the probability distribution for the randomly drawn test score.
2. Calculate the expected value value of the probability distribution

Solution

Part 1)

Looking at the test scores, we see that out of 10 grades,

- the grade 30 occurred 3 out of 10 times
- the grade 60 occurred 2 out of 10 times
- the grade 80 occurred 3 out of 10 times
- the grade 90 occurred 1 out of 10 times
- the grade 100 occurred 1 out of 10 times

Therefore,

- the probability that a randomly drawn test has a score of 30 is 3/10
- the probability that a randomly drawn test has a score of 60 is 2/10
- the probability that a randomly drawn test has a score of 80 is 3/10
- the probability that a randomly drawn test has a score of 90 is 1/10
- the probability that a randomly drawn test has a score of 100 is 1/10

The probability distribution can be presented in the following table.

Grade	p_k

30	3/10
60	2/10
80	3/10
90	1/10
100	1/10

Part 2)

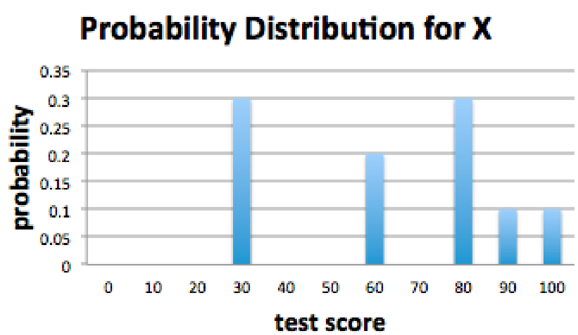
The expected value of the probability distribution is given by the sum

$$\begin{aligned} E(X) &= \sum_{k=1}^{N} x_k p_k \\ &= 30\frac{3}{10} + 60\frac{2}{10} + 80\frac{3}{10} + 90\frac{1}{10} + 100\frac{1}{10} \\ &= 9 + 12 + 24 + 9 + 10 \\ &= 64 \end{aligned}$$

Discussion

The PDF Could Be Presented as a Graph

The PDF was listed in a table, but an equivalent representation would be in a graph that plots the possible outcomes on the horizontal axis, and the probabilities on the vertical axis.



We have added points where the probability is zero (test scores of 0, 10, 20, 40, 50, 70). It isn't necessary to have these points displayed, but for other problems, having these points on a graph of a PDF can add clarity.

The Expected Value and the Average

A common source of confusion is the difference between the expected value (defined above), and the average value of a set of numbers. Whereas the expected value is a number calculated from probabilities for every x, the average value is calculated using the number of times each x occurred. Another way to say this is that the expected value is a weighted average, where each term is weighted by the *probability* of observing its value.

In our example, the average value is the sum the test scores multiplied by the number of times each score occurred:

$$\begin{aligned}\text{average test score} &= \frac{30\cdot 3+60\cdot 2+80\cdot 3+90\cdot 1+100\cdot 1}{10}\\ &= \frac{90+120+240+90+100}{10}\\ &= 64\end{aligned}$$

The expected value and the average have the same numerical value for this particular example, but describe two different concepts.

1.05 Variance and Standard Deviation

Another important quantity related to a given random variable is its **variance**. The variance is a numerical description of the width of a probability distribution.

Definition: Variance and Standard Deviation of a Discrete Random Variable
The variance, Var(X), of a probability distribution for a random variable X is calculated with the following formula $\mathrm{Var}(X) = \sum_{k=1}^{N} \Big(x_k - E(X)\Big)^2 p_k$ The integer N is the number of possible outcomes. The **standard deviation**, σ, is the square root of the variance: $\sigma(X) = \sqrt{\mathrm{Var}(X)}$

Observe that the variance of a distribution is always positive (p_k is non-negative, and the square of a number is never negative).

Students in MATH 105 are expected to memorize the formulas for variance and standard deviation.

Example: Grade Distributions

Using the grade distribution example in the previous page, calculate the variance of the probability distribution.

Solution

The variance of the probability distribution is given by

$$\begin{aligned}\mathrm{Var}(X) &= \sum_{k=1}^{N}(x_k - E(X))^2 p_k\\ &= (30-64)^2\frac{3}{10} + (60-64)^2\frac{2}{10} + (80-64)^2\frac{3}{10} + (90-64)^2\frac{1}{10} + (100-64)^2\frac{1}{10}\\ &= 744.\overline{44}\end{aligned}$$

1.06 The Cumulative Distribution Function

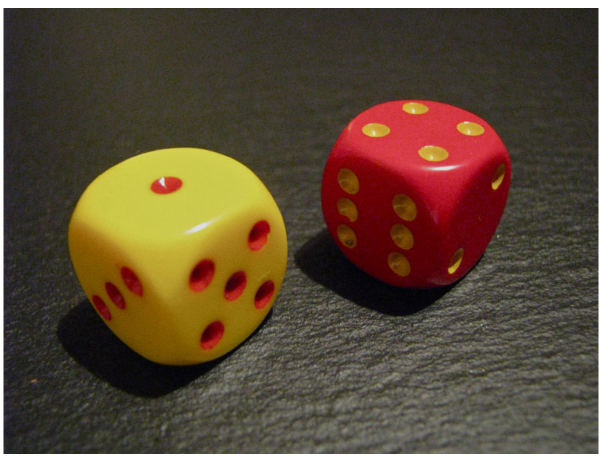
Given a probability distribution, we can calculate a **cumulative distribution function** as follows.

Definition: Cumulative Distribution of a Discrete Probability Distribution
The cumulative distribution function for a random variable X is denoted by $F(x_n)$, and is defined as $F(x) = \Pr(X \leq x_n)$. For a discrete probability distribution, the cumulative distribution function is calculated using the formula $F(x_n) = \sum_{k=1}^{n} p_k$ where $n \leq N$, and N is the number of possible outcomes.

In other words, the cumulative distribution function for any random variable gives the probability that the random variable X is less than some number x_n.

Example: Rolling Two Dice

Suppose that we have two fair six-sided dice, each face displays one number from 1 to 6. One of the die is yellow, the other is red, as in the image below.



We roll both dice at the same time and add the two numbers that are shown on the sides facing upwards.

Let X be the discrete random variable for the possible sums.

1. How many possible outcomes are there?
2. What is the probability distribution for X?
3. What is the probability that X is less than or equal to 6?

Solution

Part 1)

The simplest way to determine the number of possible outcomes is to count them. If a 1 is rolled on the yellow die, there are six possible outcomes for the red die (1, 2, 3, 4, 5, or 6). Likewise, if the yellow die is 2, there are another 6 outcomes from the red die.

There are 6 possible outcomes for the yellow die, so there are 6 × 6 = 36 different outcomes.

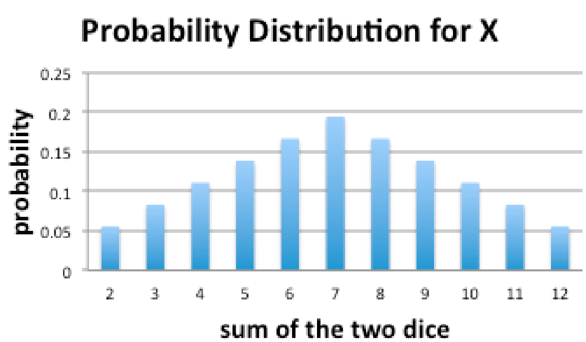
Part 2)

To construct the probability distribution for X, first consider the probability that the sum of the dice equals 2. There is only one way that this can happen: both dice must roll a 1. There are 36 possible outcomes, so the probability that the sum is equal to 2 is 1/36.

Let the Y represent the number rolled on the yellow die, and R be the number on the red die. The possible outcomes and probabilities for sums from 1 to 6 are listed in the table below.

Outcome (Y,R)	Sum	Probability
(1,1)	2	1/36
(1,2), (2,1)	3	2/36
(1,3), (2,2), (3,1)	4	3/36
(1,4), (2,3), (3,2), (4,1)	5	4/36
(1,5), (2,4), (3,4), (4,2), (5,1)	6	5/36

The probability distribution for all of the numbers from 2 to 12 are in the following graph.



Alternatively, if we let p_k equal the probability that the sum is equal to k, then the PDF can be expressed using the following formula:

$$p_k = \begin{cases} \frac{k-1}{36} & \text{if } k = 2,3,4,5,6,7 \\ \frac{13-k}{36} & \text{if } k = 8,9,10,11,12 \\ 0 & \text{else} \end{cases}$$

Part 3)

The probability that the sum is less than or equal to 6 can be expressed as Pr($X \leq 6$). Using the definition of the cumulative distribution function,

$$\Pr(X \leq 6) = F(6) = \sum_{k=1}^{6} p_k = \frac{1}{36} + \frac{2}{36} + \frac{3}{36} + \frac{4}{36} + \frac{5}{36} = \frac{15}{36} = \frac{5}{12}$$

In the above calculation, we assigned p_k to be the probability that the sum is k.

Intuitively, the probability that the sum is less than or equal to 6 is the probability that the sum is either a 2, 3, 4, 5, or 6. Adding up the probabilities for these sums yields the fraction 5/12.

1.07 Grade Distribution Example

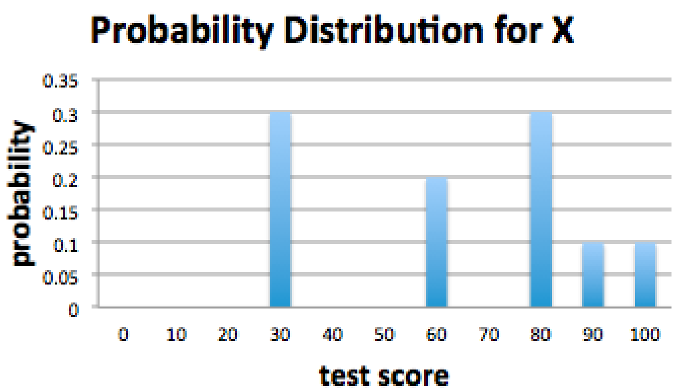
Consider the grade distribution example that we explored earlier: in a class of 10 people, grades on a test were 30, 30, 30, 60, 60, 80, 80, 80, 90, 100. A test is drawn from the pile at random.

1. Calculate the probability that a test drawn at random has a score less than or equal to 80.
2. Calculate the probability that a test drawn at random has a score less than or equal to x_n, where x_n = 0, 10, 20, 30, ... , 100.

Solution

Part 1)

Recall the probability distribution, calculated earlier:



Suppose we let p_k be the probability that the score of a randomly drawn test is $10k$. So, for example:

- p_0 is the probability that a randomly drawn test score is 0
- p_1 is the probability that a randomly drawn test score is 10
- p_2 is the probability that a randomly drawn test score is 20
- p_3 is the probability that a randomly drawn test score is 30

and so on. Values for each of these probabilities are given in the above bar graph.

Using our definition of the cumulative distribution function, $F(x_n)$,

$$\begin{aligned}
\Pr(X \leq 80) &= F(80) \\
&= \sum_{k=0}^{8} p_k \\
&= p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 \\
&= 0 + 0 + 0 + \frac{\mathbf{3}}{\mathbf{10}} + 0 + 0 + \frac{2}{10} + 0 + \frac{3}{10} \\
&= \frac{8}{10} \\
&= \frac{4}{5}
\end{aligned}$$

Colours in the above calculation were used to highlight non-zero probabilities.

Intuitively, if the grade is to be less than or equal to 80, than the grade could only be 30, 60, or 80. The probability that a randomly selected test has a grade of 30, 60, or 80 is the *sum* of the probabilities that the score is one of these possibilities. Looking at our distribution, this is the sum 0.3 + 0.2 + 0.3 = 0.8.

Part 2)

Using our definition of the cumulative distribution function, $F(x_n)$,

$$\Pr(X \leq 0) = F(0) = \sum_{k=0}^{0} p_k = p_0 = 0$$

Similarly, $F(0) = F(10) = F(20) = 0$. $F(30)$ is non-zero:

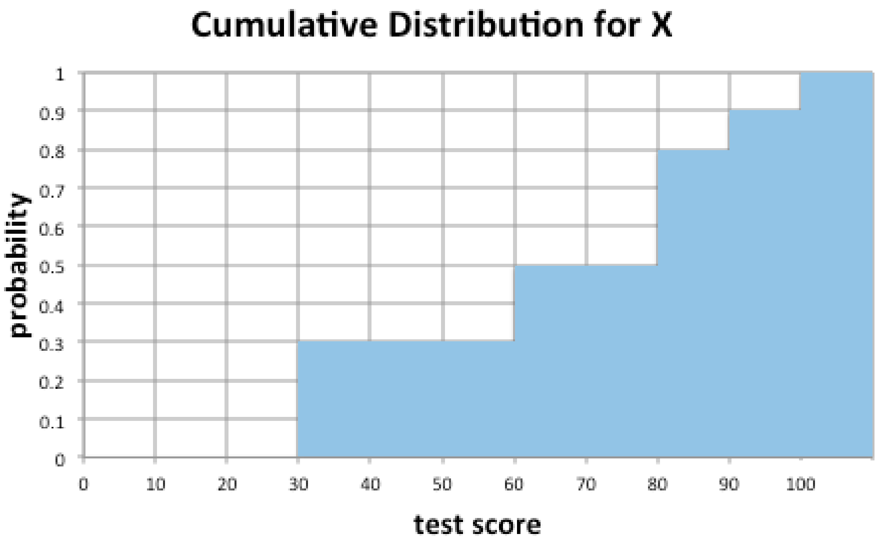
$$\Pr(X \leq 30) = F(30) = \sum_{k=0}^{3} p_k = 0 + 0 + 0 + \frac{3}{10}$$

However, $F(40)$ is equal to $F(30)$, because $p_4 = 0$.

Other values of F are calculated in a similar way using the definition of the cumulative distribution function. The following table contains the cumulative distrbution values $F(x_k)$ for 0, 10, 20, 30, ... 100.

k	x_k	$F(x_k)$
0	0	0
1	10	0
2	20	0
3	30	0.3
4	40	0.3
5	50	0.3
6	60	0.5
7	70	0.5
8	80	0.8
9	90	0.9
10	100	1.0

The cumulative distribution function is provided in the graph below.



1.08 Lesson 1 Summary

The treatment of discrete random variables was intentionally brief: the emphasis in MATH 105 is on continuous random variables. That being said, the are important concepts that were introduced in Lesson 1 that are important.

Expected Value and Variance

Namely, the concepts of an **expected value** and **variance** are important and will be revisited again when we explore contionuous random variables. Students should have memorized their respective formulas

$$E(X) = \sum_{k=1}^{N} x_k p_k$$
$$\text{Var}(X) = \sum_{k=1}^{N} (x_k - E(X))^2 p_k$$

The expected value represents a "mean" value of an experiment, or the average outcome of an experiment repeated many times. The variance of an experiment is a number that describes the "width" of the PDF.

The CDF and the PDF

Another important concept is the relationship between the CDF and the PDF. Recall that the CDF, $F(x_n)$ is calculated by summing over values in the PDF.

$$F(x_n) = \sum_{k=1}^{n} p_k$$

This is an important concept. When we revisit this concept in the continuous case, we will have continuous analogues for the PDF, CDF, and the relationships between them.

Lesson 2 CRV

In the previous lesson, we were working with **discrete** random variables. In this lesson, we consider **continuous** random variables.

If for example X is the average height of a person in Canada, or X is the average temperature of Vancouver, then X can vary continuously. So, sometimes we need to have X represent a **continuous random variable**.

What This Lesson Covers

In this lesson we introduce

- continuous random variables
- the definition of a probability density function (PDF)
- properties of a PDF
- the beta PDF
- the definition of a cumulative distribution function

Learning Outcomes

After reading this lesson and completing a sufficient number of problems, students should be able to meet the remaining learning objectives:

- Objective 3. [Procedural] Be able to verify that a given function is a pdf or cdf; be able to find multiplicative constants that make a given function a pdf or cdf.
- Objective 4. [Conceptual] Be able to distinguish between discrete and continuous random variables given a simple experiment, pdf or cdf.
- Objective 5. [Conceptual/Procedural] Be able to define and interpret the CDF of a random variable in terms of probabilities and to know basic properties of the cdf. Be able to find a cdf from a pdf.
- Objective 6. [Conceptual/Procedural] Be able to relate the area under a pdf to the probability of an event for a particular random variable and to know basic properties of the pdf. Be able to find a pdf from a cdf.
- Objective 7. [Procedural/Conceptual] Be able to define, calculate and interpret the expectation, variance and standard deviation of a random variable (discrete and continuous).

Simply reading the content in this lesson will not be sufficient: students will need to complete a sufficient number or WeBWorK and paper-based problems in order to prepare themselves for the final exam.

2.01 The Cumulative Distribution Function

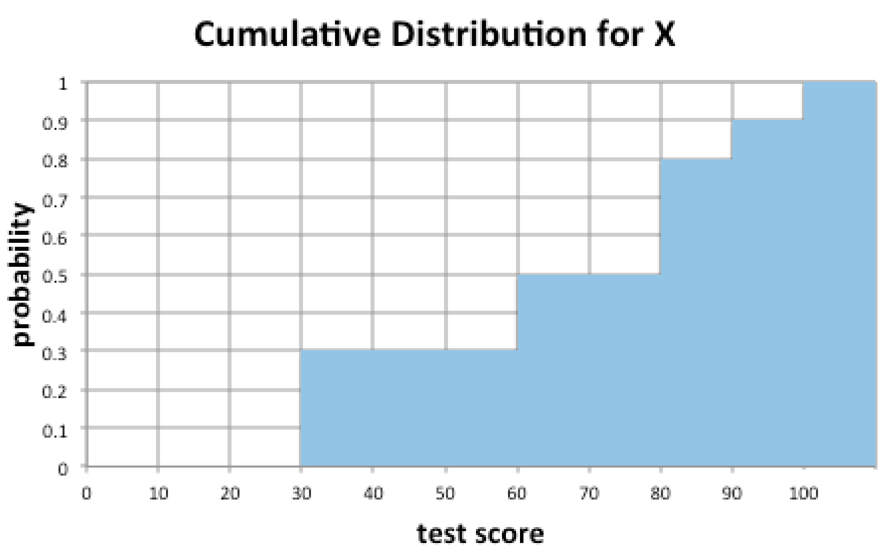
Recall the definition of a cumulative distribution function (CDF) for a random variable that was introduced in the previous lesson on discrete random variables.

Definition: The Cumulative Distribution Function
The **cumulative distribution function** for a random variable X is denoted by $F(x)$, and is a function defined as $F(x) = \Pr(X \leq x)$ The cumulative distribution function has the property that $0 \leq F(x) \leq 1$ for all values of x in the domain of F.

Defining the CDF as a function whose range lies in the closed interval [0,1] serves a practical purpose: negative probabilities and probabilities greater than one are meaningless. It is also useful to note that every random variable, discrete or continuous, has an associated CDF, and the above definition is the same definition for discrete random variables. But although the definition is the same, we will see that there is an important distinction between the two cases: a discrete CDF always increases in steps, whereas the continuous CDF is necessarily smooth.

Comparison to the Discrete Case

Recall the cumulative distribution function we had for the test scores example in the previous lesson. The cumulative distribution function was graphed at the end of the example.



Observe that

- from 0 to 30, F is constant because there are no test scores before 30
- from 30 to 60, F is constant because there are no scores between 30 and 60.

Essentially, the graph of *F* increases in "steps" at 30, 60, 80, 90, and 100. Cumulative distribution functions for *discrete* random variables are *always* step functions, because the cumulative distribution can only increase at a finite set of points. However, we will see that this is *not* true for the continuous case. For the continuous case, the cumulative distribution function is a smooth curve.

2.02 CDF Properties

CDFs share many of the same properties for both the continuous and discrete cases. In the following theorem, the three properties listed are common to both the discrete and continuous cases.

Theorem: Properties of a Distribution Function
If $F(x)$ is a cumulative distribution function for the random variable X, then 1. $\lim_{x\to-\infty} F(x) = 0$ 2. $\lim_{x\to+\infty} F(x) = 1$ 3. $F(x)$ is a non-decreasing function of x

It may help the reader at this point to recall the definition of a "non-decreasing function": a function, $f(x)$, is a non-decreasing function if $f(x_1) \leq f(x_2)$ for all x_1 and x_2 in the domain of f, and $x_1 < x_2$.

Proofs for the first two property are beyond the scope of MATH 105. For the interested reader, a proof of the third property is below.

Proof of The Third Property

In general, for *any* interval of finite length Δx, there is a non-negative probability that X lies somewhere in that interval, no matter where our interval lies or how small we make Δx. Therefore, if we were to choose any interval $[x , x + \Delta x]$, the probability that our continuous random variable X lies inside of this interval,

$$\Pr(x \leq X \leq x + \Delta x)$$

must be non-negative:

$$0 \leq \Pr(x \leq X \leq x + \Delta x)$$

But this can be expressed as the difference

$$\begin{aligned} 0 &\leq \Pr(x \leq X \leq x + \Delta x) \\ &= \Pr(X \leq x + \Delta x) - \Pr(X \leq x) \\ &= F(x + \Delta x) - F(x) \end{aligned}$$

Rearranging yields, for all Δx,

$$F(x) \leq F(x + \Delta x)$$

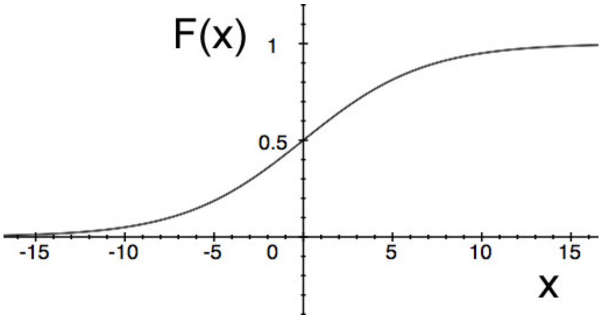
This is the definition of a non-decreasing function. Therefore, the CDF, which is defined as $F(x) = \Pr(X \leq x)$, is a non-decreasing function of x.

2.03 CDF Properties Example

Suppose that the outdoor temperature in Vancouver, in January, is given by the function

$$F(x) = \frac{1}{1+e^{-kx}},\ k>0$$

The cumulative distribution function *F* could look something like the plot below.



In the above plot, *X* can be the temperature at any given time, so that *x* could be an observed temperature.

Let's show that this function satisfies the properties of a CDF.

Property 1: Limit as *x* goes To - ∞

As *x* goes to negative infinity, F(x) goes to zero:

$$\lim_{x\to-\infty} F(x) = \lim_{x\to-\infty} \frac{1}{1+e^{-kx}} = 0$$

Therefore, the first property of cumulative distribution functions is maintained.

Property 2: Limit as *x* goes To + ∞

Likewise,

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \frac{1}{1+e^{-kx}} = 1$$

The second property of cumulative distribution functions is maintained.

Property 3: F is Non-Decreasing

A quick calculation yields

$$\frac{d}{dx}F(x) = \frac{ke^{x}}{(1+e^{-kx})^2}$$

which is certainly never negative. The third property of CDFs is maintained.

Discussion of this Example

What is Pr(*X* = *x*)?

A common point of confusion relates to the probability that a random variable is *equal* to a specific value: Pr(*X* = *x*). For example, we might ask: *what is the probability that the temperature in January is **exactly** 0°C?*

For a continuous random variable, this probability will always be zero, and we can provide one explanation here as to why this is the case.

Suppose we choose any interval [x , $x + \Delta x$]. The probability that the continuous random variable X lies inside of this interval is

$$\Pr(x \leq X \leq x + \Delta x)$$

which is the difference

$$\Pr(X \leq x + \Delta x) - \Pr(X \leq x)$$

If we take the limit as Δx goes to zero we obtain

$$\begin{aligned}\lim_{\Delta x \to 0} \Pr(x \leq X \leq x + \Delta x) &= \lim_{\Delta x \to 0} \Big(\Pr(X \leq x + \Delta x) - \Pr(X \leq x)\Big) \\ &= \Pr(X \leq x) - \Pr(X \leq x) \\ &= 0\end{aligned}$$

If this concept is confusing, fear not. We will revisit this again later when we have an interpretation of probability that uses integration.

2.04 Continuous Random Variables

The distinction between continuous and random variables can now be made more precise by using the CDF.

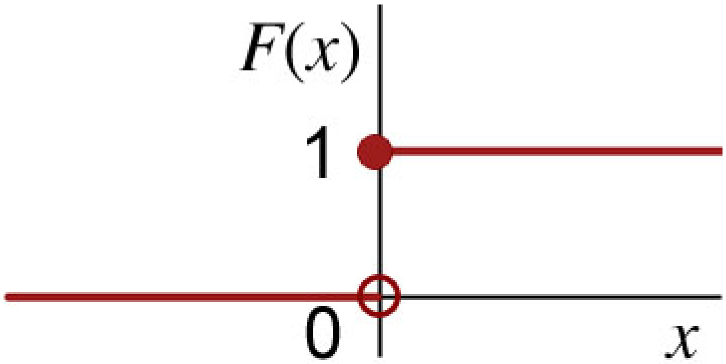
Definition: Continuous Random Variable
Let X denote a random variable with distribution function $F(x)$. The random variable X is a **continuous random variable** if its CDF, F, is continuous.

It is *by definition* that continuous CDFs of a random variable are CDFs of *continuous* random variables.

In our prior grade distribution example, the CDF was not continuous and so it corresponded to a discrete random variable that represented a grade. Whereas our previous temperature example, the CDF was continuous, the random variable was a continuous random variable that represented a temperature.

Example

Consider the function whose graph is given below.



This function cannot represent a cumulative distribution function for a *continuous* random variable because F is not continuous for all values of x. However, F could represent a cumulative distribution function, because the limit as x goes to negative and positive infinity are 0 and 1, respectively.

2.05 The PDF

The derivative of $F(x)$ is another important function in probability. Its derivative is the probability density function of the random variable X.

Definition: The Probability Density Function
The **probability density function** (PDF) for a continuous random variable X is defined as $f(x) = \frac{dF}{dx}$ wherever the derivative of $F(x)$ exists, and x is any real number.

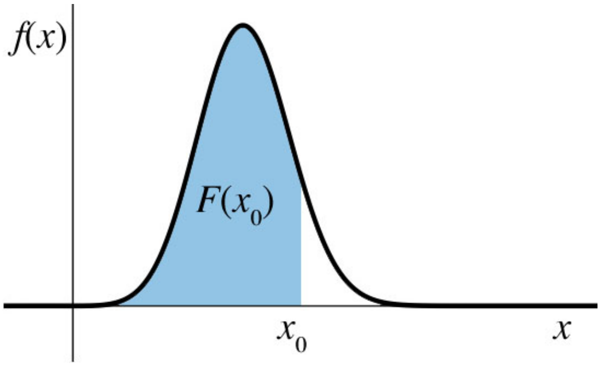
Properties of the PDF

The above definition allows us to derive the following properties.

Theorem: Properties of the Probability Density Function
If $f(x)$ is a **probability density function** for a continuous random variable X then 1) $F(b) = \Pr(X \leq b) = \int_{-\infty}^{b} f(x)dx$ 2) $f(x) \geq 0$ for any value of x 3) $\int_{-\infty}^{\infty} f(x)dx = 1$

The second property states that for a function to be a PDF, it must not go below the x-axis, and the third property states that the area between the function and the x-axis must be 1.

The first property is derived using the Fundamental Theorem of Calculus and from the definition of the PDF, which defines the PDF of a continuous random variable as the derivative of the CDF. The property tells us that the probability that $X \leq b$ is the integral of the PDF $f(x)$ over $(-\infty, b]$, and that the PDF $f(x)$ is a function that can be used to calculate this probability. Moreover this also gives us a helpful geometrical interpretation of a probability: the probability that a continuous random variable, X, is less than some value b, is equal to the area under the PDF $f(x)$ on the interval $(-\infty, b]$, as demonstrated in the following graph.



What is Pr(*X* = *x*)?

Let's now revisit this question that we have an interpretation of probabilities as integrals. Using the above definition of a PDF of a continuous random variable to find the probability that *X* was equal to some number *c*. Using the definition above, will **always** get zero:

$$\Pr(X = c) = \int_c^c f(x)dx = 0$$

In other words, if *X* is a continuous random variable, the probability that *X* is equal to a particular value will always be zero. This is different from the situation that we had with discrete random variables, where the probability that a particular event occurs can be non-zero.

2.06 A Simple PDF Example

Question

Let $f(x) = k(3x^2 + 1)$.

1. Find the value of *k* that makes the given function a PDF on the interval $0 \le x \le 2$.
2. Let *X* be a continuous random variable whose PDF is $f(x)$. Compute the probability that *X* is between 1 and 2.

Solution

Part 1)

$$\begin{aligned} 1 &= \int_0^2 f(x)dx \\ &= \int_0^2 k(3x^2 + 1)dx \\ &= k(\frac{3x^3}{3} + x)\Big|_0^2 dx \\ &= k(10) \end{aligned}$$

Therefore, $k = 1/10$.

Part 2)

Using our value for k from Part 1:

$$\Pr(1 \leq X \leq 2) = \int_1^2 \frac{3x^2+1}{10}dx = \frac{x^3+x}{10}\Big|_1^2 = 1 - 2/10 = 4/5$$

Therefore, $\Pr(1 \leq X \leq 2)$ is 4/5.

Problem Discussion

Getting Started

We are asked to find a value of k so that the given function is a probability density function. Since a probability density function **must** have unit area, we can write

$$1 = \int_0^2 f(x)dx$$

and solve for k.

2.07 The Beta PDF

Question

The proportion, p, of restaurants that make a profit in their first year of operation is given by the **beta probability density function**:

$$f(p) = \begin{cases} 12p(1-p)^2 & \text{if } 0 \leq x \leq 1, \\ 0 & \text{elsewhere.} \end{cases}$$

What is the probability that more than half of the restaurants will make a profit during their first year of operation?

Solution

Let P be the random variable representing the proportion of new restaurants opened this year that make a profit during their first year of operation. We are interested in knowing the probability that P is between 0.5 and 1, which can be calculated using the first property of the probability density function as follows:

$$\begin{aligned}\Pr(0.5 \leq P \leq 1) &= \int_{-\infty}^1 f(p)dp - \int_{-\infty}^{0.5} f(p)dp \\ &= \int_{0.5}^1 f(p)dp \\ &= \int_{0.5}^1 12p(1-p)^2dp \\ &= \int_{0.5}^1 12p - 24p^2 + 12p^3dp \\ &= 6p^2 - 8p^3 + 3p^4\Big|_{0.5}^1 \\ &= (6-8+3) - (1.5 - 1 + 0.1875) \\ &= 0.3125\end{aligned}$$

Therefore, $\Pr(0.5 \le P \le 1)$ is 0.3125.

Discussion of the Problem

The Beta Probability Density

For the purposes of MATH 105, the beta probability density function for a continuous random variable x has the generic form

$$f(x) = \begin{cases} kx^{a-1}(1-x)^{b-1} & \text{if } 0 \le x \le 1, \\ 0 & \text{elsewhere.} \end{cases}$$

The constants a and b are greater than zero, and the constant k is chosen so that f integrates to 1.

For the purposes of MATH 105, students are not expected to memorize the formula for the beta probability density, but may need to use this formula to complete assigned work.

2.08 CDF Example

Question

Consider the beta distribution function for X given in the last example,

$$f(x) = \begin{cases} kx^{a-1}(1-x)^{b-1} & \text{if } 0 \le x \le 1, \\ 0 & \text{elsewhere.} \end{cases}$$

If $a = 3$ and $b = 2$, find the associated cumulative distribution function $F(x)$ for $x \le 1$.

Solution

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt \\ &= \int_{0}^{x} 12t(1-t)^2 dt \\ &= 12\int_{0}^{x} (t - 2t^2 + t^3)dt \\ &= 12\Big(\frac{1}{2}t^2 - \frac{2}{3}t^3 + \frac{1}{4}t^4\Big)\Big|_0^x \\ &= x^2(6 - 8x + 3x^2) \end{aligned}$$

Discussion

Getting Started

Recall the first property of probability distribution functions:

$$F(b) = \Pr(X \leq b) = \int_{-\infty}^{b} f(x)dx$$

We are given $f(x)$ and need to find F, which can be found using the above integral.

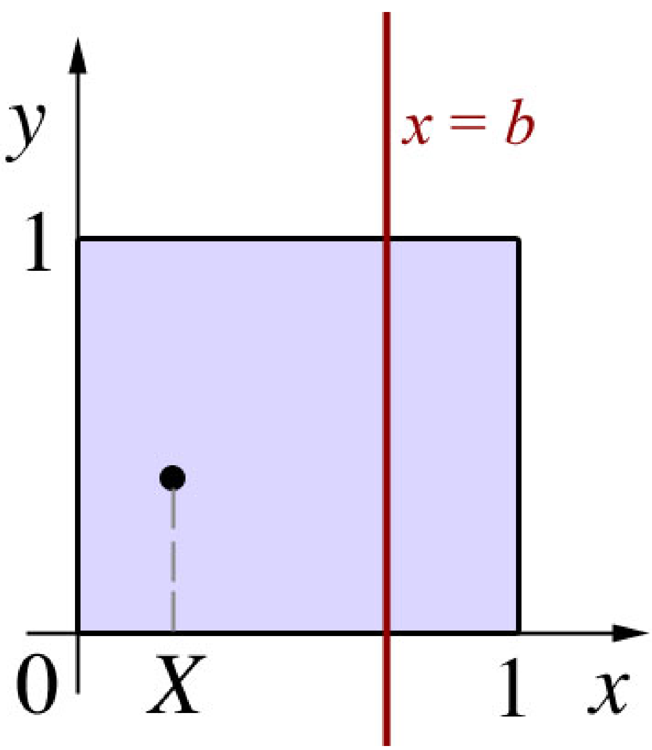
2.09 Additional PDF Example

Problem

Consider vertical line with equation $x = b$, where $0 \leq b \leq 1$, and the square bounded by the lines $x = 0$, $x = 1$, the x-axis and the line $y = 1$.

Suppose we select a random point inside the square. If we let X be the x-coordinate of this random point, what is the probability that X is in the interval $[0 , b]$?

An illustration of our problem is given in the figure below. Essentially, we are trying to find the probability that a randomly selected point inside the square lies to the **left** of the red line.



Consider Two Cases

Let's begin by considering two cases:

- Case 1: Find $\Pr(0 \leq X \leq b)$ if $b = 1$
- Case 2: Find $\Pr(0 \leq X \leq b)$ if b is in the interval [0, 1)

Case 1

If $b = 1$, then the random point must have an x-coordinate between 0 and b (because the **area** to the left of the red line is the entire area of the square, and our random point has to lie inside the square).

Therefore, the probability $\Pr(0 \leq X \leq b)$ is equal to 1 when $b = 1$ (remember that probabilities range from 0 to 1).

Case 2

The region to the left of the red line is a rectangle with area equal to b, and the probability that our random point inside the rectangle is proportional to the area of that rectangle (the larger the area of the rectangle, the larger the probability that the point is inside of it).

Thinking of probabilities in terms of areas, consider the following cases:

- if the probability that the point is between 0 and b were equal to 0.50, then the line would have to divide the square into two equal halves: $b = 0.5$
- if the probability that the point is between 0 and b were equal to 0.25, then the line would have to divide the square at 1/4: $b = 0.25$
- if the probability that the point is between 0 and b were equal to 0.10, then the line would have to divide the square at 1/10: $b = 0.1$

In general, interpreting the probability that the point is between 0 and b as the area between 0 and b and is equal to b. We see that this result also works for the case when $b = 1$.

Discussion

More General Problems

In this simple problem, we found that $\Pr(0 \leq X \leq b) = b$. If however we were not given a square, but instead some other shape, this problem could still be solved using the concept of areas, and so could be solved using an integral.

Indeed, the probability that the point is between 0 and b is the ratio of the area to the left of the line $x = b$, and the area of the square:

$$\Pr(0 \leq X \leq b) = \frac{\text{area to the left of line } x = b}{\text{area of square}} = \frac{b}{1} = b$$

What About the Case When $b = 0$?

You may have been wondering about the case when $b = 0$. If b were zero, would the probability be zero because the area between zero and b is zero, or would it be non-zero because the random point could be on the y-axis? In other words, what is $\Pr(X = 0)$?

Unlike discrete random variables, **the probability that a continuous random variable X has a particular value is always zero**. In MATH 105, we will always interpret probabilities of continuous random variables using areas, which means that $\Pr(A \leq X \leq A)$ will always be zero for any value of A.

However, for continuous random variables, we could compute the probability that X lies in an interval whose width is non-zero.

2.10 Expected Value, Variance, Standard Deviation

Analogous to the discrete case, we can define the expected value and variance of a continuous random variable.

Definition: Expected Value, Variance, and Standard Deviation of a Continuous Random Variable
The **expected value** of a random variable X is a number given by $E(X) = \int_{-\infty}^{\infty} x f(x) dx$ The **variance** of X is a number given by $\mathrm{Var}(X) = \int_{-\infty}^{\infty} \left(x - E(X)\right)^2 f(x) dx$ As in the discrete case, the **standard deviation**, σ, is the square root of the variance: $\sigma(X) = \sqrt{\mathrm{Var}(X)}$

The above definitions for a continuous random variable possess the same interpretation as that for the discrete case. The variance in both the continuous and discrete cases measure the "spread" of the possible values of X around its expected value. The connection between the expected value for both cases is more subtle.

Animation

The following animation encapsulates the concepts of the CDF, PDF, expected value, and standard deviation. When viewing the animation, it may help to remind you that

- the "mean" is another term for expected value
- the standard deviation is equal to the square root of the variance
- the integral of the PDF (upper plot) is the CDF (lower plot)

Connecting the CDF and the PDF [1] (requires the Wolfram "CDF Player")

Simple Example

Calculate the variance of X whose PDF is given by

$$f(x) = \begin{cases} 2(1-x) & \text{if } 0 \le x \le 1, \\ 0 & \text{else} \end{cases}$$

Solution

The formula for variance of X requires the expected value

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_0^1 x\Big(2(1-x)\Big) dx \\ &= 2\int_0^1 \Big(x - x^2\Big) dx \\ &= 2\Big(\frac{x^2}{2} - \frac{x^3}{3}\Big)\Big|_0^1 \\ &= 1/3 \end{aligned}$$

Using this value, we can compute the variance of X as follows

$$\begin{aligned} \text{Var}(X) &= \int_{-\infty}^{\infty} \big(x - E(X)\big)^2 f(x) dx \\ &= \int_0^1 (x - 1/3)^2 2(1-x) dx \\ &= 2\int_0^1 \big(x^2 - \frac{2}{3}x + \frac{1}{9}\big)(1-x) dx \\ &= 2\int_0^1 \big(-x^3 + \frac{5}{3}x^2 - \frac{7}{9}x + \frac{1}{9}\big) dx \\ &= 2\big(-\frac{1}{4}x^4 + \frac{5}{9}x^3 - \frac{7}{18}x^2 + \frac{1}{9}x\big)\Big|_0^1 \\ &= 2\big(-\frac{1}{4} + \frac{5}{9} - \frac{7}{18} + \frac{1}{9}\big) \\ &= \frac{1}{18} \end{aligned}$$

Discussion of This Example

Checking that the Given Function is a PDF

If we were asked to verify that $f(x)$ is a PDF, we would only need to verify that it has unit area:

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^1 \Big(2(1-x)\Big)dx = 2\Big(x - \frac{x^2}{2}\Big)\Big|_0^1 = 1$$

Indeed, the given function has unit area and satisfies our only requirement for a PDF.

Is There An Easier Way of Calculating the Variance?

Yes! There is an alternative formula for variance of a continuous random variable that is less tedious than the above definition. In the following section we introduce this formula.

References

[1] http://demonstrations.wolfram.com/ConnectingTheCDFAndThePDF

2.11 Alternate Variance Formula

An following formula for the variance of a continuous random variable is often less tedious than its definition.

Theorem: Alternate Formula for the Variance of a Continuous Random Variable
The **variance** of a continuous random variable X with PDF $f(x)$ is a number given by $\mathrm{Var}(X) = \int_{-\infty}^{\infty} x^2 f(x)dx - E(X)^2$

Simple Example

Use the alternate formula for variance to calculate the variance the PDF given in the last example, which was

$$f(x) = \begin{cases} 2(1-x) & \text{if } 0 \leq x \leq 1, \\ 0 & \text{else} \end{cases}$$

Solution

Remembering that $E(X)$ was found to be 1/3, we may compute the variance of X as follows

$$\begin{aligned}
\text{Var}(X) &= \int_{-\infty}^{\infty} x^2 f(x)dx - E(X)^2 \\
&= \int_0^1 x^2\big(2(1-x)\big)dx - \frac{1}{9} \\
&= 2\int_0^1 \big(x^2 - x^3\big)dx - \frac{1}{9} \\
&= 2\big(\frac{1}{3}x^3 - \frac{1}{4}x^4\big)\Big|_0^1 - \frac{1}{9} \\
&= 2\big(\frac{1}{3} - \frac{1}{4}\big) - \frac{1}{9} \\
&= \frac{1}{18}
\end{aligned}$$

Discussion of This Example

Do MATH 105 Students Have to Memorize The Alternate Formula for Variance?

No. MATH 105 students do not have to memorize the alternate formula. Students need to memorize the definition of variance, but can use the alternate formula if they like on an exam.

2.12 Example

Problem

The length of time, *x*, needed by students in a particular course to complete a 1 hour exam is a random variable with a PDF given by

$$f(x) = \begin{cases} kx^2 + x & \text{if } 0 \le x \le 1, \\ 0 & \text{else} \end{cases}$$

For this PDF,

1. find *k*
2. find the cumulative distribution function (CDF)
3. graph the PDF and the CDF
4. use the CDF to find
   1. $\Pr(X \le 0)$
   2. $\Pr(X \le 1)$
   3. $\Pr(X \le 2)$
5. find the probability that that a randomly selected student will finish the exam in less than half an hour

Solution

Part 1

The given PDF must have unit area. Integrating the integral over negative infinite to positive infinity we have

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(x)dx \\ &= k\int_0^1 (x^2+x)dx \\ &= k\Big(\frac{x^3}{3}+\frac{x^2}{2}\Big)\Big|_0^1 \\ k &= \frac{6}{5} \end{aligned}$$

Part 2

The CDF is obtained by integrating the PDF from negative infinity to *x*. If *x* is in the interval (-∞,0), then *F* is zero, because *f* is zero on this interval.

If *x* is in the interval (∞, 0), then

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt \\ &= k\int_{-\infty}^{0} 0dt \\ &= 0 \end{aligned}$$

If, *x* is in the interval [0, 1], then

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt \\ &= \int_{-\infty}^{0} f(t)dt + \int_0^x f(t)dt \\ &= 0 + k\Big(\frac{x^3}{3}+\frac{x^2}{2}\Big) \\ &= k\Big(\frac{x^3}{3}+\frac{x^2}{2}\Big) \end{aligned}$$

If, *x* is in the interval (1, ∞) then

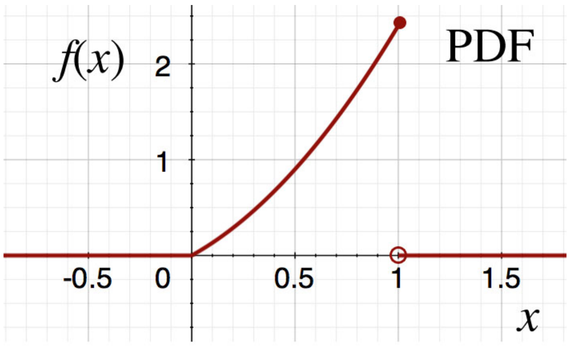
$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt \\ &= \int_{-\infty}^{0} f(t)dt + \int_0^1 f(t)dt + \int_1^x f(t)dt \\ &= 0 + k\Big(\frac{x^3}{3}+\frac{x^2}{2}\Big) + 0 \\ &= \frac{6}{5}\cdot\frac{5}{6} \\ &= 1 \end{aligned}$$

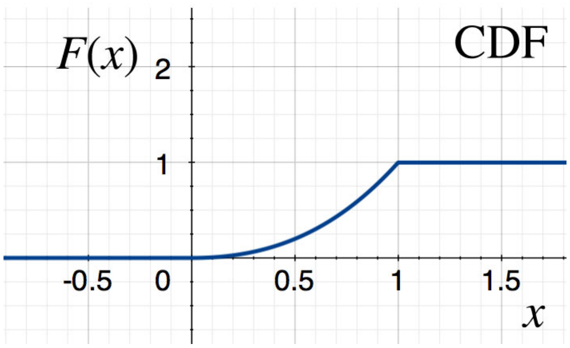
Note that the PDF *f* is equal to zero for *x* > 1. The CDF is therefore given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ k\left(\frac{x^3}{3} + \frac{x^2}{2}\right) & \text{if } 0 \le x \le 1, \\ 1 & \text{if } x > 1. \end{cases}$$

Part 3

The PDF and CDF are shown below.





Part 4

These probabilities can be calculated by using the CDF:

$$\Pr(X \le 0) = F(0) = k\left(\frac{0^3}{3} + \frac{0^2}{2}\right) = 0$$

$$\Pr(X \le 1) = F(1) = k\left(\frac{1^3}{3} + \frac{1^2}{2}\right) = \frac{5}{6}k = 1$$

$$\Pr(X \le 2) = 1$$

Part 5

The probability that a student will complete the exam in less than half an hour is Pr(X ≤ 0.5). This is given by $F(0.5)$,

$$k\Big(\frac{0.5^3}{3}+\frac{0.5^2}{2}\Big)=k\Big(\frac{1}{24}+\frac{1}{8}\Big)=\frac{6}{5}\cdot\frac{1}{6}=\frac{1}{5}$$

Discussion

Part 4 With the PDF

The probabilities in Part 4 could also have been calculated using the PDF. Pr(X ≤ 0) is given by the integral

$$\int_{-\infty}^{0} f(t)dt = 0$$

The integral is equal to zero because f is equal to zero on the interval (-∞, 0].

Pr(X ≤ 1) is given by the integral

$$\int_{-\infty}^{1} f(t)dt = \int_{-\infty}^{0} f(t)dt + \int_{0}^{1} f(t)dt = 0 + 1 = 1$$

The total area under f must be one, so the integral from 0 to 1 is equal to 1.

Pr(X ≤ 2) is given by the integral

$$\begin{aligned}\int_{-\infty}^{2} f(t)dt &= \int_{-\infty}^{0} f(t)dt + \int_{0}^{1} f(t)dt + \int_{1}^{2} f(t)dt \\ &= 0 + 1 + 0 \\ &= 1.\end{aligned}$$

2.13 Lesson 2 Summary

Relationship Between CDF and PDF

One of the key points to the lesson on probability is the relationship between the PDF and the CDF. For the continuous case, the two are related through a derivative:

$$f(x) = \frac{dF}{dx}$$

This is the *definition* of the PDF in the continuous case: the PDF, $f(x)$, is defined as the derivative of its associated CDF, $F(x)$. Using the fundamental theorem of calculus, we obtain another relationship: the two are also related through an integral:

$$F(b) = \int_{-\infty}^{b} f(x)dx$$

Calculating Probabilities

The CDF of a random variable, $F(x)$, is defined as

$$F(x) = \Pr(X \leq x)$$

Using the relationship between the CDF and the PDF, probabilities for continuous random variables can be computed in two different ways. Suppose we wish to calculate the probability that a continuous random variable X is between two values a and b. We could use the PDF and integrate to find this probability.

$$\Pr(a \leq X \leq b) = \int_{a}^{b} f(x)dx$$

Alternatively, if we have the CDF, we can evaluate the difference $F(b)$ - $F(a)$ to find this probability.

$$\Pr(a \leq X \leq b) = F(b) - F(a)$$

Either approach will yield the same result.

Learning Objectives

At this point, students may want to reflect on the learning objectives for the content on probability. As stated earlier, students should not feel that reading the content in this module is not sufficient for success in MATH 105: *after* completing a sufficient number of exercises, students should be able to:

Objective 1. [Procedural/Conceptual] Be able to calculate and interpret probabilities of events for discrete and continuous random variables given either a density function (pdf) or a distribution function (cdf).

Objective 2. [Procedural] Be able to express a pdf in a table or as an explicit function given sufficient information about the random variable.

Objective 3. [Procedural] Be able to verify that a given function is a pdf or cdf; be able to find multiplicative constants that make a given function a pdf or cdf.

Objective 4. [Conceptual] Be able to distinguish between discrete and continuous random variables given a simple experiment, pdf or cdf.

Objective 5. [Conceptual/Procedural] Be able to define and interpret the CDF of a random variable in terms of probabilities and to know basic properties of the cdf. Be able to find a cdf from a pdf.

Objective 6. [Conceptual/Procedural] Be able to relate the area under a pdf to the probability of an event for a particular random variable and to know basic properties of the pdf. Be able to find a pdf from a cdf.

Objective 7. [Procedural/Conceptual] Be able to define, calculate and interpret the expectation, variance and standard deviation of a random variable (discrete and continuous).

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