[6] 1. **Definitions** Provide precise definitions of the following.

(a) Define what it means for a function $f : X \to Y$ to be injective.

A function is injective if, for all $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

(b) Define what it means for a function $f : X \to Y$ to be surjective.

A function is surjective if, for all $y \in Y$, there is at least one $x \in X$ s.t. $f(x) = y$

[Equivalently: for all $y \in Y$, $f^{-1}(y) \neq \emptyset$]

(c) Let A be a set. Give a definition of the power set of A.

The power set of A is the set of all subsets of A i.e.

$$P(A) = \{ S \mid S \subseteq A \}$$

[5] 2. Define $[n]$ to be the set $\{1, 2, \ldots, n\}$. For example, $[4] = \{1, 2, 3, 4\}$.

(a) [1 mark] Compute the number of injective functions $f : [2] \longrightarrow [3]$.

There are 6 of them:

$$\begin{bmatrix} f(1) = 1 \\ f(2) = 2 \end{bmatrix} \quad \begin{bmatrix} f(1) = 2 \\ f(2) = 1 \end{bmatrix} \quad \begin{bmatrix} f(1) = 3 \\ f(2) = 1 \end{bmatrix}$$

$$\begin{bmatrix} f(1) = 1 \\ f(2) = 3 \end{bmatrix} \quad \begin{bmatrix} f(1) = 2 \\ f(2) = 3 \end{bmatrix} \quad \begin{bmatrix} f(1) = 3 \\ f(2) = 2 \end{bmatrix}$$

(b) [2 marks] Compute the number of injective functions $f : [2] \longrightarrow [n]$.

Since there are n choices for $f(1)$, and for each of these there are $(n-1)$ remaining choices, this must be

$$n(n-1).$$

(c) [2 marks] Compute the number of injective functions $f : [m] \longrightarrow [n]$.

Similarly, if $m \leq n$, there are n choices for $f(1)$, $(n-1)$ for $f(2)$, $(n-2)$ for $f(3)$, ... $(n-(m-1))$ for $f(m)$ i.e

$$n(n-1)\cdots(n-m+1) = \frac{n!}{(n-m)!}$$

[5] 3. Show by induction that for any set A, and for any sets $B_1, \ldots, B_n$, that

$$A \cap (B_1 \cup \cdots \cup B_n) = (A \cap B_1) \cup \cdots \cup (A \cap B_n).$$

Hint: What is the base case?

The base case is $A \cap (B_1 \cup B_2) = (A \cap B_1) \cup (A \cap B_2)$.

This is the distributive law, which was shown earlier.

So assume that

$$A \cap (B_1 \cup \cdots \cup B_n) = (A \cap B_1) \cup \cdots \cup (A \cap B_n)$$

for some $n \in \mathbb{N}$, and consider

$$A \cap (B_1 \cup \cdots \cup B_n \cup B_{n+1}) = A \cap \left((B_1 \cup \cdots \cup B_n) \cup B_{n+1}\right)$$

$$= \left[A \cap (B_1 \cup \cdots \cup B_n)\right] \cup (A \cap B_{n+1}) \quad \text{by distributive law}$$

$$= \left[(A \cap B_1) \cup \cdots \cup (A \cap B_n)\right] \cup (A \cap B_{n+1}) \quad \text{by inductive hypothesis}$$

$$= (A \cap B_1) \cup \cdots \cup (A \cap B_{n+1})$$

as desired. So by induction, the theorem is proven.

$\square$

[5] 4. Show that for two sets A, B, that $|A| = |B|$ implies that $|\mathcal{P}(A)| = |\mathcal{P}(B)|$.

If $|A| = |B|$, then there exists a bijective $f: A \to B$. Similarly, there is $g: B \to A$ so that $f \circ g = id_B$ and $g \circ f = id_A$.

We want to define a bijective function from $P(A) \to P(B)$.

Define
$$\begin{aligned} \tilde{f}: P(A) &\longrightarrow P(B) \\ S &\longmapsto \{\underbrace{f(a)}_{\in B} \mid a \in S\} \end{aligned}$$
by

and analogously
$$\begin{aligned} \tilde{g}: P(B) &\longrightarrow P(A) \\ T &\longmapsto \{g(b) \mid b \in T\}. \end{aligned}$$
via

Then:
$$\tilde{f} \circ \tilde{g}(T) = \tilde{f}\big(\{g(b) \mid b \in T\}\big) = \{\underbrace{f \circ g(b)}_{= b} \mid b \in T\} = T$$

for any $T \in P(B)$.

Similarly,
$$\tilde{g} \circ \tilde{f}(S) = \tilde{g}\big(\{f(a) \mid a \in S\}\big) = \{g \circ f(a) \mid a \in S\} = S$$

for any $S \in P(A)$. As $\tilde{f}$ and $\tilde{g}$ are mutually inverse, they are bijective.

$\square$.

[5] 5. [4 marks] Prove or disprove that for every integer $n \geq 1$, that the Fibonacci numbers F_n and F_{n+2} are relatively prime.

We proceed by induction. The base case(s) $F_1 = 1$ $F_3 = 2$, $F_2 = 1$ $F_4 = 3$ are true.

So assume that this is true for some integer n, and assume that $d \mid F_{n+3}$ and $d \mid F_{n+1}$. Then: $d \mid \underbrace{F_{n+3} - F_{n+1}}_{=F_{n+2}}$, and so we also have that $d \mid \underbrace{F_{n+2} - F_{n+1}}_{=F_n}$. Thus $d \mid F_n$ and $d \mid F_{n+2}$.

By induction, $d = 1$, and so the theorem is proven. $\square$

[1 mark] Is the same true for F_n and F_{n+3}?

No. $F_3 = 2$ $F_6 = 8$

so ~~but~~ $\gcd(2, 8) = 2 > 1$.

[6] 6. A sequence is recursively defined by

$$a_n = 3a_{n-1} - 2a_{n-2}$$

with $a_0 = 0$ and $a_1 = 1$.

(a) [2 marks] Conjecture a form for the general term.

$$\left.\begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_2 &= 3 \\ a_3 &= 7 \\ a_4 &= 15 \end{aligned}\right\} \quad a_n = 2^n - 1$$

(b) [4 marks] Prove that your general form holds true.

It is shown to be true for the base case above.

Suppose that for some n, that $a_k = 2^k - 1$ for all $1 \le k < n$.

Then $a_n = 3a_{n-1} - 2a_{n-2}$

$= 3(2^{n-1} - 1) - 2(2^{n-2} - 1)$ by induction

$= 3 \cdot 2^{n-1} - 3 - 2 \cdot 2^{n-2} + 2$

$= 3 \cdot 2^{n-1} - 2^{n-1} - 1 = 2 \cdot 2^{n-1} - 1$

$= 2^n - 1$

So by induction, it is true for all $n \geq 0$.

The End