The University of British Columbia

Final Examination - June 23, 2015

Math 340

Closed book examination — Time: 2.5 hours

Last Name ________________ **First** __________ **Signature** ________________

Student Number ________

Special Instructions:

No books, notes, or calculators are allowed. The exams consists of 13 pages plus a formula sheet. Please show your work and explain your answers.

1		12
2		12
3		13
4		13
5		15
6		20
7		15
Total		100

Problem 1: Short answer questions (12 points)
Each question is worth 3 marks. Explain your answers.

(a) Alice and Betty are playing a matrix game, with the irreducible payoff matrix for Alice given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{for some real numbers } a, b, c, d.$$

Given the fact that this is a fair game, what is the value of Alice plays mixed?

The value is zero.

Otherwise, Alice plays mixed $= X \neq 0$, suppose $X > 0$. Betty plays mixed $= X$ from duality,

But for the game to be fair we should have

$X =$ Alice plays mixed $= -$(Betty plays mixed) $= -X$, which can happen only if $X = 0$.

(b) Suppose A is replaced by A^T. Can the value of Alice plays pure change?

It can.

Example - $A = \begin{pmatrix} 2 & 0 \\ -1 & -1 \end{pmatrix}$.

For A Alice chooses the first row

and Alice plays pure $= 0$,

For $A^T = \begin{pmatrix} 2 & -1 \\ 0 & -1 \end{pmatrix}$ Alice chooses ~~[illegible]~~ either

row and Alice plays pure $= -1$.

(c) Two professors at a famous university are arguing over who gets to teach Math 340 and who stays to teach Math 152. They both can teach both courses, and each get the same (top secret!) amount C for each of the courses. Formulate the problem of assigning a prof. for each course while minimizing the cost as an integer/binary programming problem.

Decision variables: $X_{ij} = \begin{cases} 1 & \text{Prof } i \text{ teaches course } j \\ 0 & \text{otherwise} \end{cases}$

for $1 \le i,j \le 2$.

Minimize $z = \sum_{i,j=1}^{2} C X_{ij}$,

subject to

$\sum_{i=1}^{2} X_{ij} = 1$ (only one prof. per course)

$\sum_{j=1}^{2} X_{ij} = 1$ (only one course per prof.)

$X_{ij} \in \{0, 1\}$.

(d) Formulate the appropriate LP relaxation of the above problem.

Minimize $z = \sum_{i,j=1}^{2} C X_{ij}$,

s.t.

$\sum_{i=1}^{2} X_{ij} = 1 \quad j = 1, 2$

$\sum_{j=1}^{2} X_{ij} = 1 \quad i = 1, 2$

$0 \le X_{ij} \le 1 \quad 1 \le i,j \le 2$.

Problem 2 (12 points)

Consider the following LP:

$$\begin{array}{ll} \textit{maximize} & 3x_1 + 5x_2 + 2x_3 \\ \textit{subject to} & -2x_1 + 2x_2 + x_3 \leq 5, \\ & 3x_1 + \ \ x_2 - x_3 \leq 10, \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

1. [3 pts] What is the dual LP problem?
2. [3 pts] Find (directly, without using the simplex method) the best solution to the primal problem with $x_2 = 0$
3. [6 pts] Use complementary slackness to check whether the solution you found in ~~b~~ 2 is an optimal solution for the problem.

1. Dual problem -

$$\begin{array}{ll} \text{minimize} & 5y_1 + 10y_2 \\ \text{s.t.} & -2y_1 + 3y_2 \geq 3 \\ & 2y_1 + y_2 \geq 5 \\ & y_1 - y_2 \geq 2. \\ & y_1, y_2 \geq 0. \end{array}$$

2. $X_2 = 0 \Rightarrow -2x_1 + X_3 \leq 5, \quad 3X_1 - X_3 \leq 10$

and by adding these - $X_1 \leq 15$

So it is optimal to put $X_1 = 15$

and $X_3 = 35$.

thus the solution is $(15, 0, 35)$.

3. From complementary slackness,

since $x_1 > 0$ we have

$$-2y_1 + 3y_2 = 3$$

and since $x_3 > 0$

$$y_1 - y_2 = 2.$$

$\Rightarrow$ $y_2 = 7$ and $y_1 = 9$.

Check feasibility using 2nd constraint:

$$2 \cdot 9 + 7 \geq 5 \checkmark$$

So the solution is optimal.

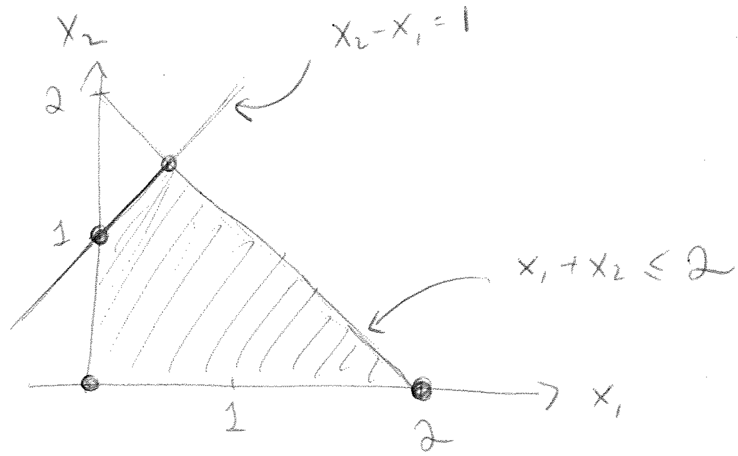
Problem 3 (13 points)

Consider the following integer problem:

$$\begin{array}{ll} \textit{maximize} & -x_1 + 3x_2 \\ \textit{subject to} & x_1 + x_2 \le 2, \\ & -x_1 + x_2 \le 1, \\ & x_1, x_2 \text{ integers}, \\ & x_1, x_2 \ge 0. \end{array}$$

1. [2 pts] Sketch the feasible domain for the LP relaxation of this problem.
2. [2 pts] What is the optimal solution for the LP relaxation?
3. [5 pts] What is an appropriate branching of the problem? Sketch the domains of the two subproblems.
4. [4 pts] What is the optimal solution to the original problem?

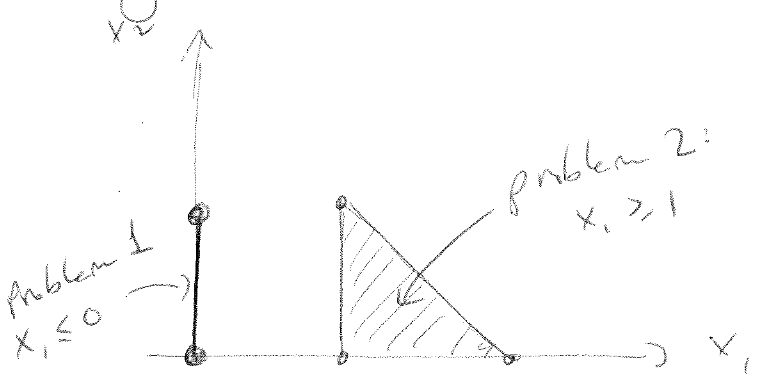
1.



2. $z(0,0) = 0$, $z(0,1) = 3$, $z = (\frac{1}{2}, 1\frac{1}{2}) = 4$, $z(0,2) = -2$

The optimal solution is $(\frac{1}{2}, 1\frac{1}{2})$.

3. Say $x_1 \le 0$ and $x_1 \ge 1$.



4. The optimal solution is the solution to problem 1: $(0,1)$, for which we have $z = 3$.

Problem 4 (13 points)

Consider the following dictionary obtained for a linear programming problem:

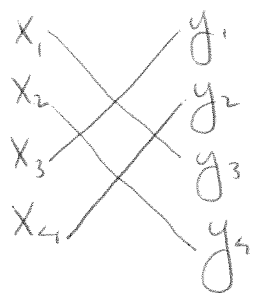
$$
\begin{array}{rcrcrcr}
x_1 & = & 1 & + & x_2 & - & x_3 \\
x_4 & = & -4 & - & x_2 & - & 2x_3 \\
\hline
z & = & 3 & - & 2x_2 & - & 3x_3
\end{array}
$$

1. [3 pts] Is this dictionary feasible (does it correspond to a feasible solution)?
2. [5 pts] Write down the dual dictionary. Is it feasible?
3. [5 pts] Can you get either an upper or lower bound for the optimal value of the objective function z using the dictionaries from part 1 and 2?

1. This dictionary isn't feasible as it corresponds to $x_4 = -4$.

2. The dual dictionary is feasible as the coefficients in the objective are negative:

$$
\begin{array}{rcl}
y_4 & = & 2 - y_3 + y_2 \\
y_1 & = & 3 + y_3 + 2y_2 \\
\hline
-\omega & = & -3 - y_3 + 4y_2
\end{array}
$$



3. We can get an ~~lower~~ upper bound from part (2): $z \leq 3$. Part 1 is infeasible and gives us no information.

Problem 5 (15 points)

Solve the following LP using the revised simplex method

$$\begin{array}{ll} \textit{maximize} & 2x_1 + 3x_2 + x_3 \\ \textit{subject to} & x_1 + x_2 + x_3 \leq 10, \\ & 2x_1 + x_2 - x_3 \leq 12, \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

What is the optimal solution and what is the optimal value of the objective function?
Note: You should not have to go through more than two iterations!

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & -1 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} x_4^* \\ x_5^* \end{pmatrix} = \begin{pmatrix} 10 \\ 12 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C = (2, 3, 1, 0, 0).$$

Step 1: $(y_1, y_2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (0, 0) \Rightarrow y = (0, 0).$

Step 2: $z_1 = 2 - (0,0)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2$

$z_2 = 3 - (0,0)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3$

$z_3 = 1 - (0,0)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$

So a_2 is the entering column.

Step 3: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow d = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

Step 4: max t : $\begin{pmatrix} 10 \\ 12 \end{pmatrix} - t\begin{pmatrix} 1 \\ 1 \end{pmatrix} \geq 0 \Rightarrow t = 10.$

Leaving column is a_4.

Step 5: $\begin{pmatrix} x_2^* \\ x_5^* \end{pmatrix} = \begin{pmatrix} 10 \\ 2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

— Next iteration:

Step 1: $(y_1\ y_2)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = (3,0) \Rightarrow y = (3,0)$.

Step 2: $z_1 = 2 - (3,0)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = -1 < 0$

$z_3 = 1 - (3,0)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2 < 0$

$z_4 = 0 - (3,0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -3 < 0$

So last solution was optimal:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ 0 \\ 0 \\ 2 \end{pmatrix}.$$

$z = 2\cdot 0 + 3\cdot 10 + 0 = 30.$

Problem 6 (20 points)

A local shop can bake three types of energy bars from three available ingredients- puffed rice, almonds and chocolate, given in the following table:

	bar 1	bar 2	bar 3	availability
puffed rice	0	1	3	5
almonds	1	1	4	8
chocolate	1	2	6	14
$ profit	2	3	8	

Let x_i denote the amount of bar i to bake and let x_{3+i} denote the ith slack for $i = 1, 2, 3$. The final dictionary is:

$$
\begin{array}{rcrcrcrcr}
x_1 & = & 3 & - & x_3 & + & x_4 & - & x_5 \\
x_2 & = & 5 & - & x_3 & - & x_4 & & \\
x_6 & = & 1 & + & x_3 & + & x_4 & + & x_5 \\
\hline
z & = & 21 & - & 3x_3 & - & x_4 & - & 2x_5
\end{array}
$$

1. [2] How many units from each type of energy bar should the shop bake? What is the maximum net profit?
2. [4] What is x_B^* for the final dictionary?
3. [4] What is B for the final dictionary?
4. [10] Find the range of c_2, i.e., the net profit per unit for bar 2, such that the original optimal basis remains optimal.

1. They sould bake 3 bars of type 1 and 5 of type 2. They will make \$21.

2. $X_B^* = \begin{pmatrix} x_1^* \\ x_2^* \\ x_6^* \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$.

3. $B = (a_1 a_2 a_6) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$

4. Let's change c_2 from 3 to $3+\Delta$, and run the next iteration of the revised simplex:

Step 1: $(y_1\ y_2\ y_3)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} = (2, 3+\Delta, 0)$.

$y_3 = 0,\ y_2 = 2,\ y_1 = 1+\Delta$

$y = (1+\Delta, 2, 0)$.

Step 2:

$z_3 = 8 - (1+\Delta, 2, 0)\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} = -3(1+\Delta)$

$z_4 = 0 - (1+\Delta, 2, 0)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -1-\Delta$

$z_5 = 0 - (1+\Delta, 2, 0)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 2$.

So these stay negative as long as

$-1-\Delta \leq 0 \Rightarrow \Delta \geq -1$.

Problem 7 (15 points)

Consider the following LP:

$$\begin{array}{ll} \textit{maximize} & -2x_1 + 3x_2 \\ \textit{subject to} & x_2 \leq 2, \\ & -x_1 \leq -3, \\ & x_1, x_2 \geq 0. \end{array}$$

1. [5 pts] Taking advantage of the symmetry between the problem and its dual, compute step 1 of the first iteration of the path following algorithm, when starting from the point

$$\begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ t_1 \\ t_2 \end{pmatrix}.$$

2. [10 pts] Write down **explicitly** the system of four equations you get for the four variables $\Delta x_1, \Delta x_2, \Delta s_1, \Delta s_2$. You do not have to solve these equations.

1. $\gamma = x \cdot t + y \cdot s = 4+4+4+4 = 16.$

$$\mu = \frac{\gamma}{10(n+m)} = \frac{16}{10 \cdot 4} = \frac{4}{10} = 0.4$$

2. $\rho = b - Ax - s = \begin{pmatrix} 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} + \begin{pmatrix} \Delta s_1 \\ \Delta s_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$$

and $\begin{pmatrix} 2\Delta x_1 \\ 2\Delta x_2 \end{pmatrix} + \begin{pmatrix} 2\Delta s_1 \\ 2\Delta s_2 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.4 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

So:

$$\begin{cases} \Delta x_2 + \Delta s_1 = -2 \\ -\Delta x_1 + \Delta s_2 = -3 \\ 2\Delta x_1 + 2\Delta s_2 = -3.6 \\ 2\Delta x_2 + 2\Delta s_2 = -3.6 \end{cases}$$