Math 110 Practice Exam #1

SOLUTIONS	002	
name (printed)	*section*	*student number*

I have read and understood the instructions below: ____________

signature

Instructions:

1. Calculators are not permitted.
2. There are 14 pages (including this cover page) in the test. **Justify every answer, and clearly show your work.** Unsupported answers will receive no credit.
3. You will be given **90 min** to write this test. Read over the test before you begin.
4. **Academic dishonesty:** Exposing your paper to another student, copying material from another student, or representing your work as that of another student constitutes academic dishonesty. Cases of academic dishonesty may lead to a zero grade in the test, a zero grade in the course, and other measures, such as suspension from this university.

Question	value	score
1	10	
2	4	
3	12	
4	12	
5	6	
6	4	
7	8	
8	4	
9	6	
10	4 bonus	
Total	**66**	

Question 1:[10 points] State whether each of the following statements is true. If it is, explain why. If it is not, explain why not (for example, by providing a counterexample).

(a)[2 marks]

$$\frac{d}{dx}\sec^2 x = \tan^2 x$$

FALSE $\frac{d}{dx}\sec^2 x = 2\sec x \cdot \frac{d}{dx}\sec x = 2\sec x \cdot \sec x \tan x$

$= 2\sec^2 x \tan x \neq \tan^2 x.$

(b)[2 marks] If $f(y)$ and $g(x)$ are differentiable,

$$\frac{d}{dx}(f \circ g)(x) = \frac{df}{dy} \circ \frac{dg}{dx}$$

FALSE: take $f(x) = g(x) = x^2$. Then $f' = g' = 2x$.

$(f \circ g)' = \frac{d}{dx}(x^2)^2 = 4x^3$. But $f' \circ g' = 2(2x) = 4x$.

(c)[2 marks] If f is continuous at a then so is $|f|$.

TRUE : f is continuous, and so is the abs. value fn.

Hence the composition $|f(x)|$ is continuous.

(d)[2 marks] If $f'(c) = 0$ and $f''(c) > 0$, then f has a local maximum at c.

FALSE: $f'(c) = 0$ and $f''(c) > 0 \Rightarrow f$ has a local

min at c.

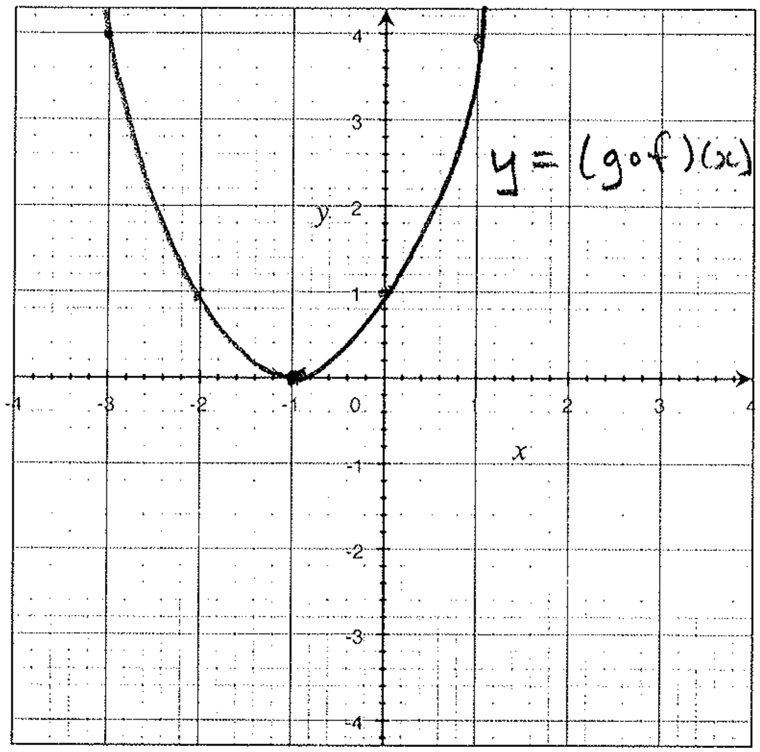
(e)[2 marks] The derivative of a continuous function is continuous.

FALSE: take $f(x) = |x|$. This is continuous,

but $f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$; it's not continuous

Question 2: [4 points] Let $f(x) = x + 1$, let $g(x) = x^2$, and let $h(x) = \frac{1}{x}$.

(a)[**2 points**] Sketch the graph of $g \circ f$. $(g \circ f)(x) = g(f(x)) = (x+1)^2$



(b)[**2 points**] Is $h \circ g$ an even function? Explain your reasoning.

$$(h \circ g)(-x) = h(g(-x)) = h((-x)^2) = h(x^2) = \frac{1}{x^2}$$

$$(h \circ g)(x) = h(g(x)) = h(x^2) = \frac{1}{x^2}$$

so $(h \circ g)(-x) = (h \circ g)(x)$; so $h \circ g$ is <u>even</u>.

Question 3: [12 points] Evaluate the following limits.

(a)[3 points] $\lim_{x\to 2} \frac{x-1}{x^2-1}$ $= \frac{2-1}{4-1} = \frac{1}{3}$

(b)[3 points] $\lim_{x\to\infty} \frac{6x^2+2x+1}{6x^2-3x+4}$ $= \lim_{x\to\infty} \frac{6+2/x+1/x^2}{6-3/x+4/x^2}$

$= \frac{6}{6} = 1$

(c)[3 points] $\lim_{x\to\infty} \frac{\sin x}{1+x^2}$

Since $-1 \le \sin x \le 1$,

$$\frac{-1}{1+x^2} \le \frac{\sin x}{1+x^2} \le \frac{1}{1+x^2}$$

Since $\lim_{x\to\infty} \frac{-1}{1+x^2} = 0$ and $\lim_{x\to\infty} \frac{1}{1+x^2} = 0$,

by the squeeze thm, $\lim_{x\to\infty} \frac{\sin x}{1+x^2} = 0$

(d)[3 points] $\lim_{x\to 0}$ $\frac{\sin x}{x^2} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{x}$

$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{x}$

$= 1 \cdot \lim_{x\to 0} \frac{1}{x} = \infty$

Question 4: [12 points] For each of the following curves, find y'. (You do not need to simplify your answer.)

(a)[3 points] $y = \dfrac{3x^2}{2 - x^3}$

$$y' = \frac{6x(2-x^3) - 3x^2(-3x^2)}{(2-x^3)^2}$$

$$= \frac{6x(2-x^3) + 9x^4}{(2-x^3)^2}$$

(b)[3 points] $x^2 - 4xy - y^2 = 4$

diff. implicitly

$$2x - 4(y + xy') - 2yy' = 0$$

$$2x - 4y = 2yy' + 4xy'$$

$$2x - 4y = y'(2y + 4x)$$

$$y' = \frac{2x - 4y}{2y + 4x}$$

(c)[3 points] $y = x^{\tan x}$

use logarithmic differentiation:

$$\ln y = \tan x \ln x.$$

$$\frac{1}{y} \cdot y' = \sec^2 x \cdot \ln x + \frac{\tan x}{x}$$

$$y' = y\left(\sec^2 x \ln x + \frac{\tan x}{x}\right)$$

$$= x^{\tan x}\left(\sec^2 x \ln x + \frac{\tan x}{x}\right)$$

(d)[3 points] $y = \ln(\sin(\tan(x)))$

$$y' = \frac{1}{\sin(\tan(x))} \cdot \cos(\tan(x)) \cdot \sec^2(x)$$

$$= \cot(\tan(x)) \cdot \sec^2(x)$$

Question 5: [6 points] Let $f(x) = x^2 + 3x$.

(a)[4 points] Use any of the limit definitions of derivative to find $f'(-4)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + 3(x+h) - (x^2 + 3x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{3x} + 3h - \cancel{x^2} - \cancel{3x}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h + 3)}{\cancel{h}} = \lim_{h \to 0} 2x + h + 3$$

$$= \boxed{2x + 3}$$

(b)[2 points] Find an equation of the tangent line to $f(x)$ at $x = -4$.

$$f'(-4) = -8+3 = \boxed{-5}$$

tangent line:

$$y = y_0 + m(x - x_0)$$

at $x = -4$, $y = (-4)^2 + 3(-4) = 16 - 12 = 4$

$m = -5$ (from part a))

$$\boxed{y = 4 - 5(x + 4)}$$

Question 6: [5 points] Sketch a function f satisfying each of the following conditions:

a) f has a vertical asymptote at $x = 1$.

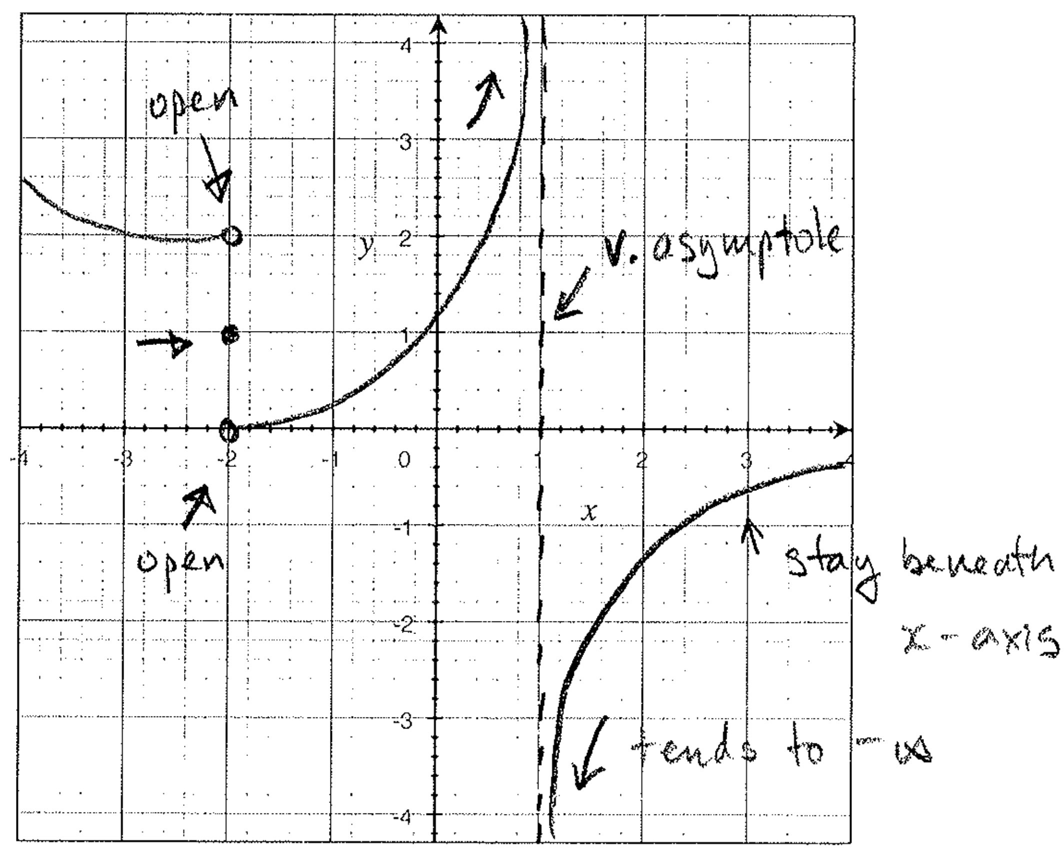
b) $f(x) < 0$ if $x > 1$ and $f(x) > 0$ if $x < 1$ and

c) $\lim_{x \to -2^+} f(x) = 0$

d) $\lim_{x \to -2^-} f(x) = 2$

e) $f(-2) = 1$

one mark each



Question 7: [8 points] Given $y = x^4 + 2x^3 - 3x^2 - 4x + 4$.

(a)[4 points] Find the intervals on which y is increasing and decreasing

$y' = 4x^3 + 6x^2 - 6x - 4$ 1 pt

Solve for $y' = 0$ (find. critical pts).

$0 = 4x^3 + 6x^2 - 6x - 4.$ 1 pt

$= (x-1)(4x^2+10x+4)$

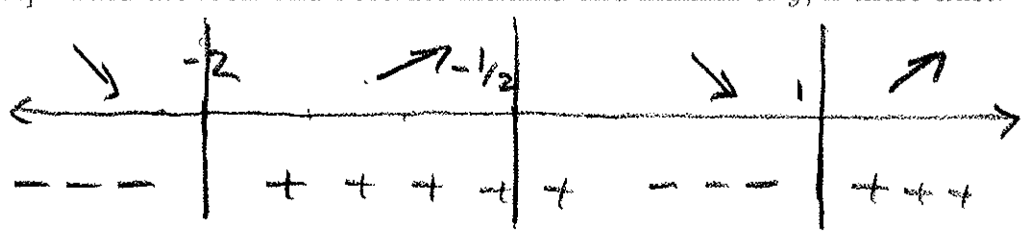
$= (x-1)(2x+4)(2x+1)$ 1 pt

$x = 1,\ x = \frac{-1}{2},\ x = -2$: crit. pts.

$$\begin{array}{r} 4x^2+10x+4 \\ x-1 \,\overline{)\, 4x^3+6x^2-6x-4} \\ \underline{4x^3-4x^2+10x} \\ 10x^2-6x-4 \\ \underline{10x^2-10x} \\ 4x-4 \end{array}$$

(b)[4 points] Find the local and absolute maxima and minima of y, if these exist.

1pt



$y'(3) < 0$
$y'(-1) > 0$
$y'(0) < 0$
$y'(2) > 0$

4pts

from this table, by 1st. deriv. test,

$\boxed{x = -2}$ and $\boxed{x = 1}$ correspond to $\boxed{\text{loc. min}}$.

$y(-2) = (-2)^4 + 2(-2)^3 - 3(-2)^2 - 4(-2) + 4 = 0$ abs min

$y(1) = 1 + 2 - 3 - 4 + 4 = 0$

$y(-\frac{1}{2})$ is a local max but not abs since

$y \to +\infty$ as $x \to \pm\infty$.

Question 8: [4 points] Prove that $x + 2\cos x = 0$ has at least one solution. Mention any theorems you use.

Let $f(x) = x + 2\cos x$.

note $f(0) = 0 + 2\cos 0 = 2 > 0$

$$f\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} + \cancelto{0}{2\cos\frac{-\pi}{2}}$$

$$= -\frac{\pi}{2} < 0.$$

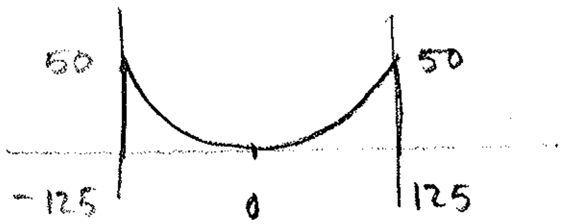
f is continuous, hence by the <u>intermediate value theorem</u>, there is a number c, $-\frac{\pi}{2} < c < 0$, so that $f(c) = 0$, since 0 lies between $-\frac{\pi}{2}$ and 2.

Since $f(c) = 0$, c is a solution of

$$x + 2\cos x = 0.$$

Question 9: [6 points] The cable of a suspension bridge is attached to supporting pillars 250 metres apart. The cable hangs in the shape of a parabola, with the lowest point 50 metres below the points of suspension.

(a)[3 points] Write an function which models the parabola mentioned in the problem. Hint: draw a picture; use symmetry.



$y = ax^2$ for some a:

$50 = a(125)^2$ div. both sides by 25:

$2 = a \cdot 625$

$a = \frac{2}{625}$

$$\boxed{y = \frac{2}{625}x^2}$$

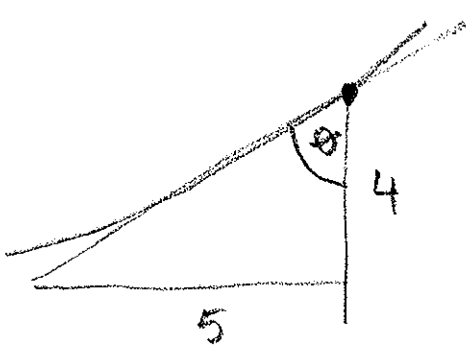
(a)[2 points] Find the slope of the cable at the point of attachment to the pillar.

The slope is given by the slope of the tangent line to $y = \frac{2}{625}x^2$ at $x = 125$.

$y' = \frac{2}{625} \cdot 2x = \frac{4}{625}x$; at $x = 125$, $y' = \frac{4 \cdot 125}{625}$

That is, $\boxed{y' = \frac{4}{5}}$

(b)[1 points] Express the angle between the pillar and the cable at the point of attachment in terms of trigonometric functions. You need not compute a numerical value.



from the picture,

$\tan\theta = \frac{5}{4}$, so

$\theta = \arctan\frac{5}{4}$

Question 10: [4 bonus points] Prove that the curves given by $5y - 2x + y^3 - x^2y = 0$ and $2y + 5x + x^4 - x^3y^2 = 0$ intersect at right angles at the origin.

Show: slope of tangents at $(0,0)$ are negative reciprocals:

Diff. implicitly:

$$5y' - 2 + 3y^2y' - (2xy + x^2y') = 0$$

$$5y' + 3y^2y' - x^2y' = 2 + 2xy$$

$$y'(5 + 3y^2 - x^2) = 2 + 2xy$$

$$y' = \frac{2 + 2xy}{5 + 3y^2 - x^2}$$

$$= \boxed{\frac{2}{5}} \text{ at } (0,0).$$

$$2y' + 5 + 4x^3 - (3x^2y^2 + x^3 2yy') = 0$$

$$2y' - 2x^3yy' = 3x^2y^2 - 5 - 4x^3$$

$$y' = \frac{3x^2y^2 - 5 - 4x^3}{2 - 2x^3y}$$

$$= \boxed{-\frac{5}{2}} \text{ at } (0,0)$$

$\frac{2}{5} \cdot -\frac{5}{2} = -1 \Rightarrow$ slopes are -ve reciprocal

This page may be used for rough work. It will not be marked.